

OSU PHYSICS DEPARTMENT
 COMPREHENSIVE EXAMINATION #136
 SOLUTIONS

Monday, January 6 and Tuesday, January 7, 2020

Winter 2020 Comprehensive Examination

Electricity and Magnetism	9 AM-12 PM	Monday, January 6
Classical Mechanics	1 PM-4 PM	Monday, January 6
Statistical Mechanics	9 AM-12 PM	Tuesday, January 7
Quantum Mechanics	1 PM-4 PM	Tuesday, January 7

General Instructions

This Winter 2020 Comprehensive Examination consists of four separate parts of two problems each, and you have three hours to work on each part. Each problem carries equal weight (20 points). Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work on the provided pages, work each problem in its own labeled pages, and be certain that your chosen student letter (but not your name) is on the header of each page of your exam, including any unused pages. If you need additional paper for your work, use the blank pages provided. Each page of work should include the problem number, a page number, your chosen student letter, and the total number of pages actually used. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam and the collection of formulas distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please staple and return all pages of your exam—including unused pages—at the end of the exam.

Problem 1

2

Electric field Suppose there is an electric field given everywhere [OR: in a region] by

$$\vec{E}(\vec{r}) = C((3x^2z + y^2z)\hat{x} + 2xyz\hat{y} + (x^3 + xy^2)\hat{z}),$$

where C is a constant.

- (a) Can this field be derived from a potential?
- (b) Find the work required to move a charge q from the origin ($\vec{r}_0 = \vec{0}$) to an arbitrary location \vec{r}_1 .
- (c) What charge density could give rise to this electric field?
- (d) [AND/OR] Find the charge enclosed inside a cube of unit size with one corner located at the origin and the opposite corner at $x = y = z = 1$. (*Hint:* There are (at least) two possible ways to do this, employing the two different formulations of one of Maxwell's equations. One requires much less effort than the other.)

$$\vec{E}(\vec{r}) = C \{ (3x^2z + y^2z) \hat{x} + 2xy z \hat{y} + (x^3 + xy^2) \hat{z} \}$$

$$(a) \quad \nabla \times \vec{E} / C = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2z + y^2z & 2xy z & x^3 + xy^2 \end{vmatrix}$$

$$= \hat{x} (2xy - 2xy) - \hat{y} (3x^2 + y^2 - (3x^2 + y^2)) + \hat{z} (2yz - 2yz)$$

$$= \vec{0} \quad \Rightarrow \text{(yes)} //$$

$$(b) \quad W = - \int_{\vec{r}_0}^{\vec{r}_1} \vec{F} \cdot d\vec{l} = -q \int_{\vec{r}_0}^{\vec{r}_1} \vec{E} \cdot d\vec{l} = +q V(\vec{r}_1)$$

Choose straight line path $\vec{r}(t) = \vec{r}_1 \cdot t \Rightarrow d\vec{l} = \vec{r}_1 dt$

$$\Rightarrow V(\vec{r}_1) = - \int_{\vec{r}_0}^{\vec{r}_1} \vec{E} \cdot d\vec{l} = - \int_0^1 \vec{E}(t) \cdot \vec{r}_1 dt$$

$$= - \int_0^1 dt \{ x_1 E_x(t) + y_1 E_y(t) + z_1 E_z(t) \}$$

$$= -C \int_0^1 dt \{ x_1 (3(x_1 t)^2 (z_1 t) + (y_1 t)^2 (z_1 t))$$

$$+ y_1 (x_1 t) (y_1 t) (z_1 t) + z_1 ((x_1 t)^3 + (x_1 t) (y_1 t)^2) \}$$

$$= -C \{ 3x_1^3 z_1 + 3x_1 y_1^2 z_1 + x_1 y_1^2 z_1 + x_1^3 z_1 + x_1 y_1^2 z_1 \}$$

$$\times \int_0^1 dt t^3$$

$$= -C \frac{1}{4} \{ 4x_1^3 z_1 + 4x_1 y_1^2 z_1 \}$$

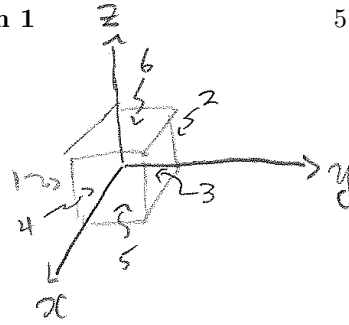
$$= -C x_1 z_1 \{ x_1^2 + y_1^2 \}$$

$$W = -C q x_1 z_1 \{ x_1^2 + y_1^2 \} //$$

$$\begin{aligned}
 (c) \quad \rho &= \epsilon_0 \nabla \cdot \vec{E} = c \epsilon_0 \left\{ \frac{\partial}{\partial x} (3x^2z + y^2z) + \frac{\partial}{\partial y} (2axyz) + \frac{\partial}{\partial z} (x^2 + x^2y^2) \right\} \\
 &= c \epsilon_0 \{ 6xz + 2xz + 0 \} = 8c \epsilon_0 xz //
 \end{aligned}$$

$$\begin{aligned}
 (d) \quad Q_{enc} &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \rho = 8c \epsilon_0 \int_0^1 dx x \int_0^1 dy \int_0^1 dz z \\
 &= 8c \epsilon_0 \frac{1}{2} \cdot 1 \cdot \frac{1}{2} = 2c \epsilon_0 //
 \end{aligned}$$

(d alt) The long way is Gauss' law



Side

$$\underline{1} \quad y=0 \quad d\vec{a}_1 = -\hat{y} dx dz$$

$$\underline{2} \quad x=0 \quad d\vec{a}_2 = -\hat{x} dy dz$$

$$\underline{3} \quad y=1 \quad d\vec{a}_3 = +\hat{y} dx dz$$

$$\underline{4} \quad x=1 \quad d\vec{a}_4 = +\hat{x} dy dz$$

$$\underline{5} \quad z=0 \quad d\vec{a}_5 = -\hat{z} dx dy$$

$$\underline{6} \quad z=1 \quad d\vec{a}_6 = +\hat{z} dx dy$$

$$Q_{enc} = \epsilon_0 \int_V \vec{k} \cdot d\vec{a}$$

$$Q_{enc}/\epsilon_0 = - \int_1 dx dz E_y - \int_2 dy dz E_x + \int_3 dx dz E_y + \int_4 dy dz E_x - \int_5 dx dy E_z + \int_6 dx dy E_z$$

$$= - \int_0^1 dx \int_0^1 dz \cdot 0 - \int_0^1 dy \int_0^1 dz y^2 z + \int_0^1 dx \int_0^1 dz 2xz$$

$$+ \int_0^1 dy \int_0^1 dz (3z + y^2 z) - \int_0^1 dx \int_0^1 dy (x^3 + xy^3) + \int_0^1 dx \int_0^1 dy (x^3 + xy^3)$$

$$= -\frac{1}{3} \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = -\frac{1}{6} + \frac{1}{2} + \frac{3}{2} + \frac{1}{6}$$

$$= 2$$

$$\Rightarrow Q_{enc} = 2\epsilon_0 //$$

Problem 2

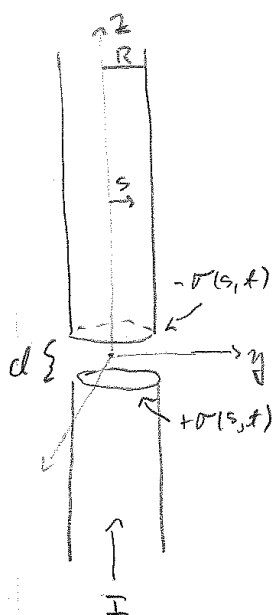
6

Solid rod Consider a long solid rod of radius R carrying a total current I up along the z -axis. The current is distributed in the rod according to $\vec{J}(s) = j_0 s^2 \hat{z}$, where s is the cylindrical radial coordinate and j_0 is a constant. There is a gap in the rod centered on the origin of width d , creating two parallel faces of the “broken” rod separated by a distance d at the origin. Because of the current in the rod, a charge density $+\sigma(s, t)$ is therefore building up on the bottom face, with an equal and opposite $-\sigma(s, t)$ building up on the top face as current flows away from the gap in the rod up the z -axis. Assume that $\sigma(s, t)$ takes the same spatial profile as $\vec{J}(s)$ on both sides of the gap and is 0 at $t = 0$. The rod is otherwise neutral.

- (a) Show that $j_0 = \frac{2I}{\pi R^4}$.
- (b) Far away from the gap, find the magnetic field inside and outside the rod. (Eliminate j_0 from your answer using the previous relation.)
- (c) If we neglect edge effects we can assume that in the gap, the electric field is uniform in z and points in the \hat{z} direction: $\vec{E} = E(s, t)\hat{z}$. Find $E(s, t)$.
- (d) Find the magnetic field in the gap (i.e. for $-d/2 < z < d/2$ and $s < R$), again neglecting edge effects.

Solution to problem 2

7



$$\vec{J} = j_0 s^2 \hat{z}$$

$$(a) I = \int_{\text{rod}} \vec{J} \cdot d\vec{a} \quad \text{w/} \quad d\vec{a} = \hat{z} 2\pi s ds$$

$$= 2\pi j_0 \int_0^R ds s^3$$

$$= \pi j_0 R^4 / 2 \quad \Rightarrow \quad j_0 = \frac{2I}{\pi R^4} //$$

$$(b) \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad (\text{Far away from gap})$$

$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}$$

Inside $I_{\text{enc}} = \int \vec{J} \cdot d\vec{a} = 2\pi j_0 \int_0^s ds s^3 = \frac{\pi j_0 s^4}{2} = I \left(\frac{s}{R}\right)^4$

Outside $I_{\text{enc}} = 2\pi j_0 \int_0^R ds s^3 = \frac{\pi j_0 R^4}{2} = I //$

so

$$B(s) \cdot 2\pi s = \mu_0 I_{\text{enc}}(s)$$

$$\vec{B} = \frac{\mu_0}{2\pi} \frac{I_{\text{enc}}(s)}{s} \hat{\phi}$$

$$= \frac{\mu_0 I}{2\pi s} \hat{\phi} \begin{cases} (s/R)^4 & s < R \\ 1 & s > R \end{cases} //$$

$$(c) \sigma(s, t) = \int_0^t dt \vec{J} \cdot \hat{z} = j_0 s^2 \cdot t = \frac{2I s^2 t}{\pi R^4}$$



Gaussian pillbox \Rightarrow
 inside gap
 (neglecting edge effects)
 $(\vec{E} = 0 \text{ inside})$
 (i.e. neutral)

$$\int \vec{E} \cdot d\vec{a} = Q_{\text{enc}} / \epsilon_0$$

$$\int E_z(s) da = \int \sigma(s) da / \epsilon_0$$

$$\Rightarrow E_z(s) = \sigma(s) / \epsilon_0$$

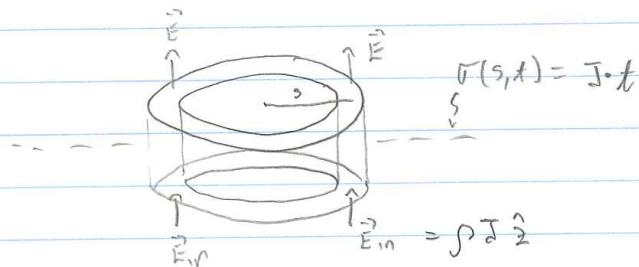
$$\vec{E}(s, t) = j_0 \frac{s^2 t}{\epsilon_0} \hat{z} = \frac{2I s^2 t}{\epsilon_0 \pi R^4} \hat{z} //$$

do it in thin rings to get s-dependence

Why can we take the field inside the rod to be zero? The problem statement should have said the rod was highly conductive i.e. has low resistivity ρ . Then since $\vec{J} = \vec{E}_{in} / \rho$ inside the rod,

$$\begin{aligned}\vec{E}_{in} &= \rho \vec{J} \\ &= \rho j_0 \delta^2 \hat{z}\end{aligned}$$

cylindrical
A Gaussian pill box across the bottom surface of the gap gives



$$\oint \vec{E} \cdot d\vec{a} = q_{enc} / \epsilon_0$$

$$\int_{top} E_z da - \int_{bottom} E_{in} da = \int \sigma(s,t) da / \epsilon_0 \quad \text{Taking the cylinder very thin,}$$

$$E_z - E_{in} = \sigma(s,t) / \epsilon_0$$

$$E_z = \frac{\sigma}{\epsilon_0} + E_{in}$$

$$= \frac{J \cdot t}{\epsilon_0} + \rho J$$

negligible after $t > 0$

$$= J \left(\frac{t}{\epsilon_0} + \rho \right) \quad \Rightarrow \quad \frac{E_{in}}{E_z} = \frac{\rho J}{J \left(\frac{t}{\epsilon_0} + \rho \right)} = \frac{1}{(1 + t / \rho \epsilon_0)}$$

$$\approx \frac{J t}{\epsilon_0}$$

$$= \frac{\sigma}{\epsilon_0} \quad (\text{as above})$$

$$\begin{aligned}
 \text{(d)} \quad \nabla \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\
 &= \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\frac{V(s,t)}{\epsilon_0} \hat{z} \right) \\
 &= \mu_0 \vec{J}(s) \quad [\text{see part (c)}]
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \oint \vec{B} \cdot d\vec{l} &= \mu_0 \int \vec{J} \cdot d\vec{a} \\
 2\pi s B(s) &= \mu_0 \frac{\pi j_0 s^2}{2} = \mu_0 I \left(\frac{s}{R} \right)^2 \\
 \vec{B}(s) &= \frac{\mu_0 I}{2\pi s} \left(\frac{s}{R} \right)^2 \hat{\varphi} //
 \end{aligned}$$

(notice it agrees with the field inside the rod)

Simple pendulum in an accelerating frame A pendulum of length l and mass m is suspended from the ceiling of a railroad car which moves with constant acceleration a in the horizontal x -direction. The acceleration of the pendulum due to gravity is g . If y is the vertical axis, the pendulum moves in the x - y plane. Find

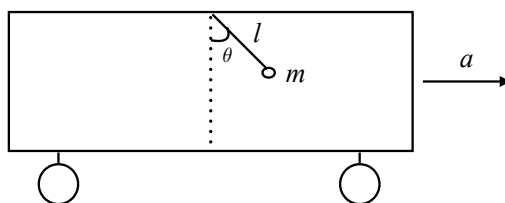
- (a) the equation of motion for the angle the pendulum makes with the vertical y -direction,
- (b) the equilibrium angle of the pendulum,
- (c) the frequency of small oscillations about the equilibrium point, and,
- (d) the limit for the frequency of small oscillations as $a \rightarrow 0$.

Simple pendulum in an accelerating frame A pendulum of length l and mass m is suspended from the ceiling of a railroad car which moves with constant acceleration a in the horizontal x -direction. The acceleration of the pendulum due to gravity is g . If y is the vertical axis, the pendulum moves in the x - y plane. Find

- (a) the equation of motion for the angle the pendulum makes with the vertical y -direction,

Solution:

Figure 1: Schematic of accelerating train.



Setting the origin at the point where the pendulum is suspended at $t = 0$ and assuming the train starts accelerating from rest gives

$$x = \frac{1}{2}at^2 + l \sin \theta,$$

$$y = -l \cos \theta$$

which means $\dot{x} = at + \dot{\theta}l \cos \theta$, $\dot{y} = \dot{\theta}l \sin \theta$. Since the potential energy $U = mgy$ and the kinetic energy is $K = 1/2m[\dot{x}^2 + \dot{y}^2]$, the Lagrangian is

$$L = K - U = \frac{1}{2}m \left[(at + \dot{\theta}l \cos \theta)^2 + \dot{\theta}^2 l^2 \sin^2 \theta \right] + mgl \cos \theta.$$

Lagrange's equation for θ is

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}.$$

Now,

$$\frac{\partial L}{\partial \dot{\theta}} = m \left[atl \cos \theta + \dot{\theta}l^2 \right]$$

and therefore

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = m \left[al \cos \theta - atl\dot{\theta} \sin \theta + \ddot{\theta}l^2 \right].$$

Also,

$$\frac{\partial L}{\partial \theta} = m \left[-atl\dot{\theta} \sin \theta \right] - mgl \sin \theta,$$

so that the equation of motion for θ is

$$\ddot{\theta} = -\frac{a}{l} \cos \theta - \frac{g}{l} \sin \theta.$$

(b) the equilibrium angle of the pendulum,

Solution:

Setting $\ddot{\theta} = 0 = -a/l \cos \theta_0 - g/l \sin \theta_0$ and solving for θ_0 gives

$$\tan \theta_0 = -\frac{a}{g}.$$

(c) the frequency of small oscillations about the equilibrium point, and,

Solution:

Letting $\theta = \theta_0 + \delta$, and using trigonometric relations for the sine and cosine of the sum of two angles yields

$$\ddot{\delta} = -\frac{a}{l} [\cos \theta_0 - \delta \sin \theta_0] - \frac{g}{l} [\sin \theta_0 + \delta \cos \theta_0].$$

The first term in the square brackets vanishes because $0 = -a/l \cos \theta_0 - g/l \sin \theta_0$, which leaves,

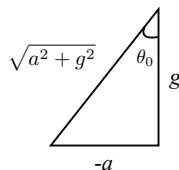
$$\ddot{\delta} = \frac{\delta}{l} [a \sin \theta_0 - g \cos \theta_0].$$

Since $\tan \theta_0 = -a/g$, we can construct the triangle in the figure below which means

$$\sin \theta_0 = -\frac{a}{\sqrt{a^2 + g^2}}, \quad \cos \theta_0 = \frac{g}{\sqrt{a^2 + g^2}}.$$

We can then write the equation as,

Figure 2: Triangle with sides g and a , used to solve for $\sin \theta_0$ and $\cos \theta_0$



$$\ddot{\delta} = -\frac{\sqrt{a^2 + g^2}}{l}\delta,$$

so that the oscillation frequency is

$$\omega^2 = \frac{\sqrt{a^2 + g^2}}{l}.$$

(d) the limit for the frequency of small oscillations as $a \rightarrow 0$.

Solution:

$$\omega^2 = g/l.$$

Problem 4

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Central forces The Lagrangian for two masses m_1 and m_2 , which interact with one another through a spherically symmetric force directed along the line connecting the two objects with potential $U(r)$, can be written as

$$L = \frac{1}{2}\mu \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) - U(r),$$

where r is the separation between the two objects and $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass.

- (a) Show that there is a conserved quantity $l = \mu r^2 \dot{\phi}$.
- (b) Use this result to show that the total energy of the system can be written as

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r)$$

- (c) The terms that do not depend on \dot{r} in the total energy can be taken to form an effective potential

$$V(r) = \frac{l^2}{2\mu r^2} + U(r)$$

for the two particle system. Show that if the force that gives rise to the potential $U(r)$ is $F(r) = -k/r^n$ there are no stable circular orbits for $n \geq 3$.

Central forces The Lagrangian for two masses m_1 and m_2 , which interact with one another through a spherically symmetric force directed along the line connecting the two objects with potential $U(r)$, can be written as

$$L = \frac{1}{2}\mu \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) - U(r),$$

where r is the separation between the two objects and $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass.

- (a) Show that there is a conserved quantity $l = \mu r^2 \dot{\phi}$.

Solution:

The Lagrangian does not depend on ϕ , only on $\dot{\phi}$, so

$$\frac{\partial L}{\partial \phi} = 0,$$

and

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = 0,$$

and the following is a conserved quantity

$$\frac{\partial L}{\partial \dot{\phi}} = \text{CONSTANT} = \mu r^2 \dot{\phi} = l$$

- (b) Use this result to show that the total energy of the system can be written as

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r)$$

Solution:

The total energy is

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\phi}^2 + U(r).$$

If we solve for $\dot{\phi}$ in part a. we get $\dot{\phi} = l/(\mu r^2)$ which we can put back into the equation for the energy to get our result

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r).$$

- (c) The terms that do not depend on \dot{r} in the total energy can be taken to form an effective potential

$$V(r) = \frac{l^2}{2\mu r^2} + U(r)$$

for the two particle system. Show that if the force that gives rise to the potential $U(r)$ is $F(r) = -k/r^n$ there are no stable circular orbits for $n \geq 3$.

Solution:

For this force $F(r) = -k/r^n = -\frac{\partial}{\partial r}U(r)$ the potential is

$$U(r) = -\frac{k}{n-1} \frac{1}{r^{n-1}},$$

so that the effective potential is

$$V(r) = \frac{l^2}{2\mu r^2} - \frac{k}{n-1} \frac{1}{r^{n-1}}.$$

For a stable orbit we need to find a minimum for this potential, i.e. we need to find an extremum $dV/dr|_{r=r_0} = 0$, which gives

$$r_0 = \left(\frac{\mu k}{l^2} \right)^{1/(n-3)}$$

and the condition on the curvature at this point is $d^2V/dr^2 > 0$ so that it forms a stable minimum. This gives

$$(3-n) \frac{l^2}{\mu} > 0$$

which means $n < 3$.

	0 molecules	1 molecule
A	ϵ_{A0}	ϵ_{A1}
B	ϵ_{B0}	ϵ_{B1}

Table 1: Energy of a single binding site in each of the four states.

Binding to a surface Consider a surface with N_S binding sites. Each site has two possible internal states (A and B) and in either of those states a single gas molecule can bind to the site. A single site thus has four possible states, with the energies shown in Table 2.

- Solve for the mean number of molecules bound to the surface at temperature T , when the chemical potential is μ .
- Solve for the entropy of a single binding site at temperature T in the limit of high chemical potential μ . *This corresponds to a maximum number of molecules being bound.*
- Show that when $\epsilon_{B1} = \epsilon_{A1}$ the entropy of a single binding site at high chemical potential is $k_{B1} \ln 2$.¹

Reminder: The probability of a given microstate in the grand canonical ensemble is proportional to $e^{-\beta(E_i - \mu N_i)}$ where E_i is the energy of the microstate and N_i is the number of molecules in the microstate.

¹In the exam as given to students, there was a typo in which this read $\epsilon_B = \epsilon_A$.

	0 molecules	1 molecule
A	ϵ_{A0}	ϵ_{A1}
B	ϵ_{B0}	ϵ_{B1}

Table 2: Energy of a single binding site in each of the four states.

Binding to a surface Consider a surface with N_S binding sites. Each site has two possible internal states (A and B) and in either of those states a single gas molecule can bind to the site. A single site thus has four possible states, with the energies shown in Table 2.

- (a) Solve for the mean number of molecules bound to the surface at temperature T , when the chemical potential is μ .

Solution:

We can find the mean number of molecules bound to a single site by

$$\langle N \rangle_1 = \sum_i N_i P_i \quad (5.1)$$

$$= \sum_i N_i \frac{e^{-\beta(E_i - \mu N_i)}}{\mathcal{Z}} \quad (5.2)$$

where $\beta = \frac{1}{k_B T}$ and \mathcal{Z} is the Gibbs sum, which is the sum of the Boltzmann factor with the chemical potential term in it. N_i is always zero or one, so we just get:

$$\langle N \rangle_1 = \frac{e^{-\beta(\epsilon_{A1} - \mu)} + e^{-\beta(\epsilon_{B1} - \mu)}}{e^{-\beta\epsilon_{A0}} + e^{-\beta\epsilon_{B0}} + e^{-\beta(\epsilon_{A1} - \mu)} + e^{-\beta(\epsilon_{B1} - \mu)}} \quad (5.3)$$

To find the total number of molecules bound, we just need to multiply by the number of binding sites N_S .

$$\langle N \rangle = N_S \langle N \rangle_1 \quad (5.4)$$

$$= N_S \frac{e^{-\beta(\epsilon_{A1} - \mu)} + e^{-\beta(\epsilon_{B1} - \mu)}}{e^{-\beta\epsilon_{A0}} + e^{-\beta\epsilon_{B0}} + e^{-\beta(\epsilon_{A1} - \mu)} + e^{-\beta(\epsilon_{B1} - \mu)}} \quad (5.5)$$

- (b) Solve for the entropy of a single binding site at temperature T in the limit of high chemical potential μ . *This corresponds to a maximum number of molecules being bound.*

Solution:

We can begin by asking what the probabilities will be in the high μ limit. This causes $e^{-\beta(\epsilon_A - \mu)}$ to get very large, while 1 and $e^{-\beta\epsilon_0}$ stay smallish. Which means that the probability of having no molecule bound becomes

zero. We can make this mathematically explicit, starting by writing down the probabilities before taking a limit as $\beta\mu \gg 1$.

$$P_{A0} = \frac{e^{-\beta\epsilon_{A0}}}{e^{-\beta\epsilon_{A0}} + e^{-\beta\epsilon_{B0}} + e^{-\beta(\epsilon_{A1}-\mu)} + e^{-\beta(\epsilon_{B1}-\mu)}} \rightarrow 0 \quad (5.6)$$

$$P_{B0} = \frac{e^{-\beta\epsilon_{B0}}}{e^{-\beta\epsilon_{A0}} + e^{-\beta\epsilon_{B0}} + e^{-\beta(\epsilon_{A1}-\mu)} + e^{-\beta(\epsilon_{B1}-\mu)}} \rightarrow 0 \quad (5.7)$$

$$P_{A1} = \frac{e^{-\beta(\epsilon_{A1}-\mu)}}{e^{-\beta\epsilon_{A0}} + e^{-\beta\epsilon_{B0}} + e^{-\beta(\epsilon_{A1}-\mu)} + e^{-\beta(\epsilon_{B1}-\mu)}} \quad (5.8)$$

$$\rightarrow \frac{e^{-\beta\epsilon_{A1}}}{e^{-\beta\epsilon_{A1}} + e^{-\beta\epsilon_{B1}}} \quad (5.9)$$

$$P_{B1} = \frac{e^{-\beta(\epsilon_{B1}-\mu)}}{e^{-\beta\epsilon_{A0}} + e^{-\beta\epsilon_{B0}} + e^{-\beta(\epsilon_{A1}-\mu)} + e^{-\beta(\epsilon_{B1}-\mu)}} \quad (5.10)$$

$$\rightarrow \frac{e^{-\beta\epsilon_{B1}}}{e^{-\beta\epsilon_{A1}} + e^{-\beta\epsilon_{B1}}} \quad (5.11)$$

In the cases with zero gas bound, the bottom becomes infinite while the top remains finite, so it goes to zero. In the other two cases (A1 and B1) the infinities cancel giving us an answer that is independent of the chemical potential (provided it is large enough).

The entropy is (always) given by

$$S = -k_{B1} \sum_i P_i \ln P_i \quad (5.12)$$

$$= -k_{B1} (P_{A1} \ln P_{A1} + P_{B1} \ln P_{B1}) \quad (5.13)$$

$$= -k_{B1} \left(\frac{e^{-\beta\epsilon_{A1}}}{e^{-\beta\epsilon_{A1}} + e^{-\beta\epsilon_{B1}}} \ln \left(\frac{e^{-\beta\epsilon_{A1}}}{e^{-\beta\epsilon_{A1}} + e^{-\beta\epsilon_{B1}}} \right) + \frac{e^{-\beta\epsilon_{B1}}}{e^{-\beta\epsilon_{A1}} + e^{-\beta\epsilon_{B1}}} \ln \left(\frac{e^{-\beta\epsilon_{B1}}}{e^{-\beta\epsilon_{A1}} + e^{-\beta\epsilon_{B1}}} \right) \right) \quad (5.14)$$

$$= \frac{k_{B1}}{\mathcal{Z}} \left(e^{-\beta\epsilon_{A1}} \left(\cancel{\beta\epsilon_{A1}} \ln \mathcal{Z} \right) + e^{-\beta\epsilon_{B1}} \left(\cancel{\beta\epsilon_{B1}} \ln \mathcal{Z} \right) \right) \quad (5.15)$$

$$= k_{B1} \frac{1}{\mathcal{Z}} \ln \mathcal{Z} + k_{B1} \beta \frac{\epsilon_{A1} e^{-\beta\epsilon_{A1}} + \epsilon_{B1} e^{-\beta\epsilon_{B1}}}{\mathcal{Z}} \quad (5.16)$$

$$= k_{B1} \ln \mathcal{Z} + \frac{1}{T} \frac{\epsilon_{A1} e^{-\beta\epsilon_{A1}} + \epsilon_{B1} e^{-\beta\epsilon_{B1}}}{\mathcal{Z}} \quad (5.17)$$

- (c) Show that when $\epsilon_{B1} = \epsilon_{A1}$ the entropy of a single binding site at high chemical potential is $k_{B1} \ln 2$.²

Solution:

When $\epsilon_{A1} = \epsilon_{B1}$ we can solve this mathematically by taking this limit.

$$\mathcal{Z} = e^{-\beta\epsilon_{A1}} + e^{-\beta\epsilon_{B1}} = 2e^{-\beta\epsilon_{A1}}, \quad (5.18)$$

²In the exam as given to students, there was a typo in which this read $\epsilon_B = \epsilon_A$.

and our entropy comes out to

$$S = k_{B1} \ln(2e^{-\beta\epsilon_{A1}}) + \frac{\epsilon_{A1}}{T} \quad (5.19)$$

$$= k_{B1} \ln 2 - k_{B1}\beta\epsilon_{A1} + \frac{\epsilon_{A1}}{T} \quad (5.20)$$

$$= k_{B1} \ln 2 \quad (5.21)$$

You can also get to this answer with a little less in-between math by noticing in Eq 5.11 and Eq ?? that there are just two states with non-zero probabilities and those two probabilities are equal when the energies are equal. Then you can use Boltzmann's entropy to see that $S = k_{B1} \ln 2$.

Reminder: The probability of a given microstate in the grand canonical ensemble is proportional to $e^{-\beta(E_i - \mu N_i)}$ where E_i is the energy of the microstate and N_i is the number of molecules in the microstate.

Leaky bottle in a bottle Consider a bottle filled with N_0 molecules of an ideal gas (which you may *not* assume to be monatomic) at temperature T_0 . The bottle has volume V_0 , and is placed in a larger bottle with volume $10V_0$ that initially has vacuum in it (before the smaller bottle is placed in it). The big bottle is in a room that is also at temperature T_0 .

The small bottle has thin walls, so you may neglect the volume occupied by the walls and cap of the bottle. The smaller (inside) bottle has a slow leak that lets gas escape into the outside bottle.

- (a) When the system has reached equilibrium, how much gas will remain in the small bottle?
- (b) How much energy is transferred to or from the room by heating during this process?
- (c) What is the change in entropy of the entire system?

Reminder: the internal energy of an ideal gas depends on its temperature and number of molecules, but not on its volume. **Note: always explain your reasoning when writing your answers.**

Leaky bottle in a bottle Consider a bottle filled with N_0 molecules of an ideal gas (which you may *not* assume to be monatomic) at temperature T_0 . The bottle has volume V_0 , and is placed in a larger bottle with volume $10V_0$ that initially has vacuum in it (before the smaller bottle is placed in it). The big bottle is in a room that is also at temperature T_0 .

The small bottle has thin walls, so you may neglect the volume occupied by the walls and cap of the bottle. The smaller (inside) bottle has a slow leak that lets gas escape into the outside bottle.

- (a) When the system has reached equilibrium, how much gas will remain in the small bottle?

Solution:

At equilibrium the chemical potential in the small bottle must equal the chemical potential in the large bottle, and their temperatures must also be equal (and must be equal to the temperature of the room, T_0). This means that the density in each bottle must be equal. Thus the ratio of numbers of molecules must be equal to the ratio of volumes.

Since the smaller bottle holds 1/10 of the total volume, it will hold 1/10 of the molecules. Thus $N_B = \frac{1}{10}N_0$.

- (b) How much energy is transferred to or from the room by heating during this process?

Solution:

As the gas leaks out, there is no work being done since the bottles are neither moving nor expanding. The internal energy also doesn't change, because the final and initial temperatures of the gas are equal, and the internal energy of an ideal gas only depends on number and temperature (neither of which change).

Since the internal energy does not change, and no work is done, by the First Law no heating may have happened. So the room was neither cooled nor heated. Therefore $Q = 0$ for this process.

- (c) What is the change in entropy of the entire system?

Solution:

To find the change in entropy of the entire system, we have to imagine a reversible path between the initial and final state. In this case, since the temperature does not change, we need to imagine a slow isothermal expansion to ten times the initial volume. We can write

$$\Delta S = \int \frac{dQ}{T} \tag{6.1}$$

To find the heat, we invoke the First Law, recognizing that since the internal energy doesn't change (since the temperature doesn't change), the heat must be equal in magnitude to the work.

$$dU = dQ + dQ = 0 \quad (6.2)$$

$$dQ = -dW \quad (6.3)$$

$$= pdV \quad (6.4)$$

Putting this into our entropy, we find that

$$\Delta S = \int \frac{p}{T} dV \quad (6.5)$$

$$= \int_{V_0}^{10V_0} \frac{Nk_B}{V} dV \quad (6.6)$$

$$= Nk_B \ln V \Big|_{V_0}^{10V_0} \quad (6.7)$$

$$= Nk_B \ln 10 \quad (6.8)$$

And there is our answer, the entropy increases by $Nk_B \ln 10$. We are relieved that the answer is positive, because otherwise we would have violated the Second Law with our leaky bottle.

Reminder: the internal energy of an ideal gas depends on its temperature and number of molecules, but not on its volume. Note: always explain your reasoning when writing your answers.

Problem 7

24

Consider a particle bound in the harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$. A perturbation $H' = \gamma x^3$ is applied to the system.

- a) Calculate the first-order corrections to the energies of the two lowest energy eigenstates.
- b) Calculate the second-order energy corrections to these two states.
- c) Find the first-order corrections to the eigenstates of these two states.

a) The first-order correction to the energy is zero because the perturbation x^3 is odd and the energy eigenstates are either even or odd so that their squares are even. This is true for all states.

$$E_n^{(1)} = \langle n^{(0)} | \hat{H}' | n^{(0)} \rangle = \langle n^{(0)} | \gamma x^3 | n^{(0)} \rangle = \int_{-\infty}^{\infty} \varphi_n^{(0)*}(x) \gamma x^3 \varphi_n^{(0)}(x) dx = \gamma \int_{-\infty}^{\infty} x^3 |\varphi_n^{(0)}(x)|^2 dx = 0$$

b) The second-order correction to the energy is

$$E_n^{(2)} = \sum_{k \neq n} \frac{|\langle n^{(0)} | \hat{H}' | k^{(0)} \rangle|^2}{E_n^{(0)} - E_k^{(0)}}$$

Use the ladder operators to find the required matrix elements:

$$\begin{aligned} x &= \sqrt{\frac{\hbar}{2m\omega}} (a^\dagger + a) \\ x^3 &= \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} (a^\dagger + a)^3 \\ &= \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} (a^\dagger a^\dagger a^\dagger + a^\dagger a^\dagger a + a^\dagger a a^\dagger + a^\dagger a a + a a^\dagger a^\dagger + a a^\dagger a + a a a^\dagger + a a a) \end{aligned}$$

This combination of ladder operators means that matrix elements of the x^3 operator are zero unless the two states differ in n by ± 1 or ± 3 . Hence the energy shifts are

$$\begin{aligned} E_0^{(2)} &= \sum_{k \neq 0} \frac{|\langle 0^{(0)} | \hat{H}' | k^{(0)} \rangle|^2}{E_0^{(0)} - E_k^{(0)}} = \frac{|\langle 0^{(0)} | \gamma x^3 | 1^{(0)} \rangle|^2}{E_0^{(0)} - E_1^{(0)}} + \frac{|\langle 0^{(0)} | \gamma x^3 | 3^{(0)} \rangle|^2}{E_0^{(0)} - E_3^{(0)}} \\ E_1^{(2)} &= \sum_{k \neq 1} \frac{|\langle 1^{(0)} | \hat{H}' | k^{(0)} \rangle|^2}{E_1^{(0)} - E_k^{(0)}} = \frac{|\langle 1^{(0)} | \gamma x^3 | 0^{(0)} \rangle|^2}{E_1^{(0)} - E_0^{(0)}} + \frac{|\langle 1^{(0)} | \gamma x^3 | 2^{(0)} \rangle|^2}{E_1^{(0)} - E_2^{(0)}} + \frac{|\langle 1^{(0)} | \gamma x^3 | 4^{(0)} \rangle|^2}{E_1^{(0)} - E_4^{(0)}} \end{aligned}$$

The required matrix elements are

$$\begin{aligned} \langle 0^{(0)} | \hat{H}' | 1^{(0)} \rangle &= \gamma \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} \langle 0^{(0)} | (a^\dagger a^\dagger a^\dagger + a^\dagger a^\dagger a + a^\dagger a a^\dagger + a^\dagger a a + a a^\dagger a^\dagger + \boxed{a a^\dagger a + a a a^\dagger} + a a a) | 1^{(0)} \rangle \\ &= \gamma \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} (\sqrt{1}\sqrt{1}\sqrt{1} + \sqrt{1}\sqrt{2}\sqrt{2}) = \gamma \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} 3 \\ \langle 0^{(0)} | \hat{H}' | 3^{(0)} \rangle &= \gamma \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} \langle 0^{(0)} | (a a a) | 3^{(0)} \rangle = \gamma \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} (\sqrt{1}\sqrt{2}\sqrt{3}) = \gamma \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} \sqrt{6} \\ \langle 1^{(0)} | \hat{H}' | 0^{(0)} \rangle &= \gamma \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} \langle 1^{(0)} | (a^\dagger a a^\dagger + a a^\dagger a^\dagger) | 0^{(0)} \rangle = \gamma \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} (\sqrt{1}\sqrt{1}\sqrt{1} + \sqrt{2}\sqrt{2}\sqrt{1}) = \gamma \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} 3 \\ \langle 1^{(0)} | \hat{H}' | 2^{(0)} \rangle &= \gamma \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} \langle 1^{(0)} | (a^\dagger a a + a a^\dagger a + a a a^\dagger) | 2^{(0)} \rangle \\ &= \gamma \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} (\sqrt{1}\sqrt{1}\sqrt{2} + \sqrt{2}\sqrt{2}\sqrt{2} + \sqrt{2}\sqrt{3}\sqrt{3}) = \gamma \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} 6\sqrt{2} \\ \langle 1^{(0)} | \hat{H}' | 4^{(0)} \rangle &= \gamma \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} \langle 1^{(0)} | (a a a) | 4^{(0)} \rangle = \gamma \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} (\sqrt{2}\sqrt{3}\sqrt{4}) = \gamma \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} 2\sqrt{6} \end{aligned}$$

The energy shifts are

$$E_0^{(2)} = \gamma^2 \left(\frac{\hbar}{2m\omega}\right)^3 \left(\frac{9}{-\hbar\omega} + \frac{6}{-3\hbar\omega}\right) = \gamma^2 \left(\frac{\hbar}{2m\omega}\right)^3 \left(\frac{-11}{\hbar\omega}\right)$$

$$E_1^{(2)} = \gamma^2 \left(\frac{\hbar}{2m\omega}\right)^3 \left(\frac{9}{+\hbar\omega} + \frac{72}{-\hbar\omega} + \frac{24}{-3\hbar\omega}\right) = \gamma^2 \left(\frac{\hbar}{2m\omega}\right)^3 \left(\frac{-71}{\hbar\omega}\right)$$

c) The first-order corrections to the eigenstates are

$$|n^{(1)}\rangle = \sum_{k \neq n} c_{nk} |n^{(0)}\rangle$$

where the expansion coefficients use the same matrix elements from above (note that they are all real)

$$c_{nk} = \frac{\langle k^{(0)} | H' | n^{(0)} \rangle}{E_n^{(0)} - E_k^{(0)}}$$

Thus

$$|0^{(1)}\rangle = \frac{\langle 1^{(0)} | H' | 0^{(0)} \rangle}{E_0^{(0)} - E_1^{(0)}} |1^{(0)}\rangle + \frac{\langle 3^{(0)} | H' | 0^{(0)} \rangle}{E_0^{(0)} - E_3^{(0)}} |3^{(0)}\rangle$$

$$= \gamma \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\frac{1}{\hbar\omega}\right) \left(-3|1^{(0)}\rangle - \sqrt{\frac{2}{3}}|3^{(0)}\rangle\right)$$

$$|1^{(1)}\rangle = \frac{\langle 0^{(0)} | H' | 1^{(0)} \rangle}{E_1^{(0)} - E_0^{(0)}} |0^{(0)}\rangle + \frac{\langle 2^{(0)} | H' | 1^{(0)} \rangle}{E_1^{(0)} - E_2^{(0)}} |2^{(0)}\rangle + \frac{\langle 4^{(0)} | H' | 1^{(0)} \rangle}{E_1^{(0)} - E_4^{(0)}} |4^{(0)}\rangle$$

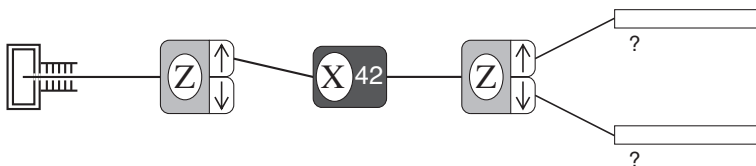
$$= \gamma \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\frac{1}{\hbar\omega}\right) \left(+3|0^{(0)}\rangle - 6\sqrt{2}|2^{(0)}\rangle - 2\sqrt{\frac{2}{3}}|4^{(0)}\rangle\right)$$

Problem 8

27

A beam of identical neutral particles with spin $\frac{1}{2}$ and a magnetic moment $\boldsymbol{\mu} = -(e/m_e)\mathbf{S}$ travels along the y -axis. The beam passes through a series of two Stern-Gerlach spin analyzing magnets, each of which is designed to analyze the spin component along the z -axis. The first Stern-Gerlach analyzer only allows particles with spin **up** (along the z -axis) to pass through. The second Stern-Gerlach analyzer only allows particles with spin **down** (along the z -axis) to pass through. The particles travel at speed v between the two analyzers, which are separated by a region of length d in which there is a uniform magnetic field B_0 pointing in the x -direction. Determine the smallest value of d such that only 25% of the particles transmitted by the first analyzer are transmitted by the second analyzer.

Experiment schematic:



The measurement at the first analyzer collapses the state to

$$|\psi(0)\rangle = |+\rangle$$

In the field aligned along the x -axis, the energy eigenstates are $|\pm\rangle_x$ and the energy eigenvalues are $\pm \hbar\omega_0/2$ with $\omega_0 = eB_0/m_e$. The initial state vector written in the energy basis is

$$\begin{aligned} |\psi(0)\rangle = |+\rangle &= (|+\rangle_x \langle +| + |-\rangle_x \langle -|) |+\rangle = ({}_x\langle +| + \rangle) |+\rangle_x + ({}_x\langle -| + \rangle) |-\rangle_x \\ &= \frac{1}{\sqrt{2}} |+\rangle_x + \frac{1}{\sqrt{2}} |-\rangle_x \end{aligned}$$

The time evolved state is

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-iE_+t/\hbar} |+\rangle_x + \frac{1}{\sqrt{2}} e^{-iE_-t/\hbar} |-\rangle_x = \frac{1}{\sqrt{2}} e^{-i\omega_0 t/2} |+\rangle_x + \frac{1}{\sqrt{2}} e^{+i\omega_0 t/2} |-\rangle_x$$

The probability of measuring S_z to be $-\hbar/2$ is

$$\begin{aligned} \mathcal{P}_- &= |\langle - | \psi(t) \rangle|^2 = \left| \langle - | \left(\frac{1}{\sqrt{2}} e^{-i\omega_0 t/2} |+\rangle_x + \frac{1}{\sqrt{2}} e^{+i\omega_0 t/2} |-\rangle_x \right) \right|^2 \\ &= \left| \frac{1}{\sqrt{2}} e^{-i\omega_0 t/2} \langle - | + \rangle_x + \frac{1}{\sqrt{2}} e^{+i\omega_0 t/2} \langle - | - \rangle_x \right|^2 = \left| \frac{1}{2} e^{-i\omega_0 t/2} - \frac{1}{2} e^{+i\omega_0 t/2} \right|^2 = \sin^2 \frac{\omega_0 t}{2} \end{aligned}$$

To have this probability equal to 25% requires

$$\mathcal{P}_- = \sin^2 \frac{eB_0 t}{2m_e} = \frac{1}{4} \Rightarrow \frac{eB_0 t}{2m_e} = \frac{\pi}{6}$$

The time to traverse the distance d is $t = d/v$, yielding

$$\frac{eB_0 d}{2m_e v} = \frac{\pi}{6} \Rightarrow d = \frac{\pi m_e v}{3eB_0}$$