OSU Physics Department
Comprehensive Examination #134
Solutions

Monday, April 1 and Tuesday, April 2, 2019

Spring 2019 Comprehensive Examination

Classical Mechanics	9 AM-12 PM	Monday, April 1
Quantum Mechanics	1 PM-4 PM	Monday, April 1
Electricity and Magnetism	9 AM-12 PM	Tuesday, April 2
Statistical Mechanics	1 PM-4 PM	Tuesday, April 2

General Instructions

This Spring 2019 Comprehensive Examination consists of four separate parts of two problems each, and you have three hours to work on each part. Each problem caries equal weight (20 points). Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work on the provided pages, work each problem in its own labeled pages, and be certain that your chosen student letter (but not your name) is on the header of each page of your exam, including any unused pages. If you need additional paper for your work, use the blank pages provided. Each page of work should include the problem number, a page number, your chosen student letter, and the total number of pages actually used. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam and the collection of formulas distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please staple and return all pages of your exam—including unused pages—at the end of the exam.

Spinning ping pong ball Consider a ping pong ball that at $t = 0$ is spinning on a surface with coefficient of kinetic friction μ. The initial angular velocity is ω_0 and the initial velocity is zero. After some period of time the ball stops slipping on the table and starts rolling without slipping. After this, assume there is no loss of energy.

(a) What fraction of the initial rotational kinetic energy is turned into translational kinetic energy?

(b) What fraction of the initial rotational kinetic energy is lost?

Reminder: the magnitude of the force due to kinetic (sliding) friction is

$$F_f = \mu N \tag{1.1}$$

where N is the magnitude of the normal force.

The moment of inertia of a hollow sphere (such as our ping pong ball) is $\frac{2}{3}MR^2$.

Spinning ping pong ball Consider a ping pong ball that at $t = 0$ is spinning on a surface with coefficient of kinetic friction μ. The initial angular velocity is ω_0 and the initial velocity is zero. After some period of time the ball stops slipping on the table and starts rolling without slipping. After this, assume there is no loss of energy.

(a) What fraction of the initial rotational kinetic energy is turned into translational kinetic energy?

Solution:
Since our ping pong ball is sliding, it feels a constant force (until it stops sliding), which is given by μN. Conveniently, the vertical forces are simple (gravity and the normal force) and the vertical acceleration is zero, so

$$F_f = \mu M g \tag{1.2}$$

where g is the acceleration due to gravity.

The torque on the ping pong ball is

$$\begin{aligned} \tau &= F_f R &(1.3)\\ &= \mu M g R &(1.4)\\ &= -I\dot{\omega} &(1.5) \end{aligned}$$

where ω is the angular velocity of the ball. We can easily integrate this to find that

$$\begin{aligned} \omega(t) &= \omega_0 - \frac{\mu M g R}{I} t &(1.6)\\ &= \omega_0 - \frac{\mu M g R}{\frac{2}{3} M R^2} t &(1.7)\\ &= \omega_0 - \frac{3}{2}\frac{\mu g}{R} t &(1.8) \end{aligned}$$

Since we know the force on the ball (nothing but friction), we can also write down and solve the translational equation of motion:

$$\begin{aligned} F_f &= M\dot{v} &(1.9)\\ v &= \frac{F_f}{M} t &(1.10)\\ &= \mu g t &(1.11) \end{aligned}$$

where I enforced that the speed start at zero. Note here that the frictional force is *accelerating* our ball, eating up its rotational energy and turning it into translational kinetic energy, but with some loss.

Now we need to find the criterion for the time t_f when it stops slipping. This is the “rolling without slipping” criterion, when the rotational velocity of the surface in its center-of-mass frame ($R\omega$) is equal in magnitude to its translational velocity v.

$$v_f = R\omega_f \tag{1.12}$$
$$\mu g t_f = R\left(\omega_0 - \frac{3}{2}\frac{\mu g}{R} t_f\right) \tag{1.13}$$
$$t_f = \frac{2}{5}\frac{R\omega_0}{\mu g} \tag{1.14}$$

The initial rotational kinetic energy is

$$K_{roti} = \frac{1}{2} I\omega_0^2 \tag{1.15}$$
$$= \frac{1}{2}\frac{2}{3} mR^2\omega_0^2 \tag{1.16}$$
$$= \frac{1}{3} mR^2\omega_0^2 \tag{1.17}$$

The final translational kinetic energy is given by

$$K_{transf} = \frac{1}{2} m v_f^2 \tag{1.18}$$
$$= \frac{1}{2} m \left(\mu g \frac{2}{5}\frac{R\omega_0}{\mu g}\right)^2 \tag{1.19}$$
$$= \frac{2}{25} mR^2\omega_0^2 \tag{1.20}$$
$$= \frac{6}{25} K_{roti} \tag{1.21}$$

So we convert $\frac{6}{25}$ of the initial kinetic energy into translational kinetic energy.

(b) What fraction of the initial rotational kinetic energy is lost?

Solution:

Grading note *It turned out that this question was confusing to a number of students. who interpreted it to mean what was the fractional decrease in the rotational kinetic energy. Since the question was confusing in this way (and didn't require any serious extra effort with either interpretation), I did not assign any points to this question specifically.*

The easy way to find this is by finding the final rotational kinetic energy and doing a subtraction.

$$K_{rotf} = \frac{1}{2} I\omega_f^2 \tag{1.22}$$

$$= \frac{\omega_f^2}{\omega_0^2} K_{roti} \tag{1.23}$$

$$= \frac{\left(\omega_0 - \frac{3}{2}\frac{\mu g}{R} t_f\right)^2}{\omega_0^2} K_{roti} \tag{1.24}$$

$$= \frac{\left(\omega_0 - \frac{3}{2}\frac{\mu g}{R}\frac{2}{5}\frac{R\omega_0}{\mu g}\right)^2}{\omega_0^2} K_{roti} \tag{1.25}$$

$$= \frac{9}{25} K_{roti} \tag{1.26}$$

Putting these together we can find that the energy lost is

$$E_{lost} = K_{roti} - K_{rotf} - K_{transf} \tag{1.27}$$

$$= \left(1 - \frac{9}{25} - \frac{6}{25}\right) K_{roti} \tag{1.28}$$

$$= \frac{2}{5} K_{roti} \tag{1.29}$$

So we lose $\frac{2}{5}$ of the initial energy due to friction.

Reminder: the magnitude of the force due to kinetic (sliding) friction is

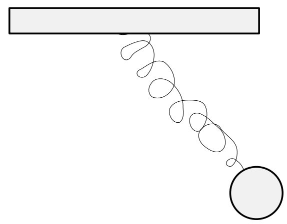
$$F_f = \mu N \tag{1.30}$$

where N is the magnitude of the normal force.

The moment of inertia of a hollow sphere (such as our ping pong ball) is $\frac{2}{3}MR^2$.

Problem 2

Spring pendulum Consider a mass m that is hanging from a spring with spring constant k and an equilibrium length L (that is to say, if there is no external force on the spring its length will be L).



(a) How many normal modes do you expect this system to have?

(b) Identify and describe the normal modes of this system.

(c) Solve for the frequencies of all of the normal modes.

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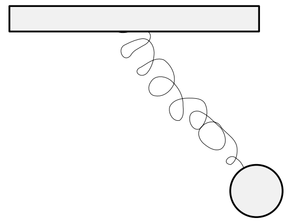


(a) How many normal modes do you expect this system to have?

Solution:
There are three normal modes, because the mass can move in three possible directions.

(b) Identify and describe the normal modes of this system.

Solution:
This can be solved by symmetry alone.

There must be one up-and-down mode, since if the displacement is vertical the force will always be vertical also.

The other two normal modes would involve displacements in the two orthogonal horizontal directions. These would be degenerate, so we could take a linear combination of the two modes to find uniform circular motion that is clockwise or counterclockwise as the two modes.

(c) Solve for the frequencies of all of the normal modes.

Solution:
The force on the object is given by

$$\vec{F} = m\vec{g} - k(|\vec{r}| - L)\hat{r} \tag{2.1}$$

where I measure $\vec{r}$ from the mounting point of the spring and $\vec{g}$ is the acceleration due to gravity (which I'll put in the $+\hat{z}$ direction). If we consider the vertical mode, we can see that this equation reduces down to

$$\begin{aligned} F_z &= mg - k(z - L) && (2.2)\\ &= mg + kL - kz && (2.3) \end{aligned}$$

and we can pretty easily see that the frequency is the very familiar $\sqrt{k/m}$. We'll start by finding the equilibrium z. Then we will rewrite things in terms of displacement from equilibrium.

$$0 = mg - k(z_0 - L) \tag{2.4}$$
$$= mg + kL - kz_0 \tag{2.5}$$
$$z_0 = L + \frac{mg}{k} \tag{2.6}$$

Actually, doing this computation in Cartesian coordinates is pretty inconvenient, because of having to deal with this $|\vec{r}| - L$ distance. It's way easier if we use spherical coordinates:

$$V = -mgr\cos\theta + \frac{1}{2}k(r-L)^2 \tag{2.7}$$

The kinetic energy in this case is of course slightly more complicated. There are several ways to solve for it in spherical coordinates. Some involve tedious use of trig identities to simplify the result. The easiest way is to begin by writing the displacement vector in spherical coordinates and taking a derivative:

$$\vec{r} = r\hat{r} \tag{2.8}$$
$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}} \tag{2.9}$$

The second term shows up because the direction $\hat{r}$ is itself time dependent. Then we need to know what $\dot{\hat{r}}$ is, which is most easily done geometrically. If you change theta, $\hat{r}$ moves in the $\hat{\theta}$ direction, and if you change ϕ then $\hat{r}$ moves in the $\hat{\phi}$ direction, but the distance it moves needs to be scaled down by $\sin\theta$ because as $\sin\theta \to 0$ the value of ϕ becomes less important.

$$\dot{\hat{r}} = \frac{\partial\hat{r}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{r}}{\partial\phi}\dot{\phi} \tag{2.10}$$
$$= \dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi} \tag{2.11}$$

Taken together, we can see that the speed is given by

$$\left|\dot{\vec{r}}\right|^2 = \dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) \tag{2.12}$$

Note that you could do any subportion of this in Cartesian coordinates if you wanted to do so.

We can now write down our Lagrangian in sphrical coordinates:

$$L = T - V \tag{2.13}$$
$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)\right) + mgr\cos\theta - \frac{1}{2}k(r-L)^2 \tag{2.14}$$

and we can either find our equations of motion first and then do a power series expansion second, or the opposite.

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\dot{r}} \qquad \frac{\partial L}{\partial\theta} = \frac{d}{dt}\frac{\partial L}{\dot{\theta}} \qquad \frac{\partial L}{\partial\phi} = \frac{d}{dt}\frac{\partial L}{\dot{\phi}} \tag{2.15}$$

Let's start with the radial version

$$mr\left(\dot\theta^2+\sin^2\theta\dot\phi^2\right)+mg\cos\theta-k(r-L)=m\ddot r \tag{2.16}$$

$$-m\sin\theta\cos\theta\dot\phi^2-mgr\sin\theta=mr^2\ddot\theta+mr\dot r\dot\theta \tag{2.17}$$

$$0=mr^2\sin^2\theta\ddot\phi+mr\dot r\sin^2\theta+mr^2\sin\theta\cos\theta\dot\theta\dot\phi \tag{2.18}$$

At this point, we can see that there is a conserved angular momentum and all sorts of other things, but it looks pretty nasty. That's because I decided to expand about equilibrium *after* finding the equations of motion. So we can simplify things by expanding under the assumption that $\theta\approx 0$ and $r\approx L+\frac{mg}{k}$. Then keeping only the terms, the first two equations of motion turn into

$$-k\left(r-L-\frac{mg}{k}\right)=m\ddot r \tag{2.19}$$

$$-mg\left(L+\frac{mg}{k}\right)\theta=m\left(L+\frac{mg}{k}\right)^2\ddot\theta \tag{2.20}$$

The third equation has no linear term, but that's all right because we can construct our three normal modes such that $\dot\phi=0$ for each of them.

Based on Eq. 2.19, we can see that the normal mode in which r varies has frequency

$$\omega_z=\sqrt{k/m}, \tag{2.21}$$

since Eq. 2.19 is just the equation for an ordinary spring with spring constant k and mass m.

For the sideways modes, we need to consider Eq. 2.20, which simplifies to

$$\ddot\theta=-\frac{g}{L+\frac{mg}{k}}\theta \tag{2.22}$$

This has a solution with a frequency

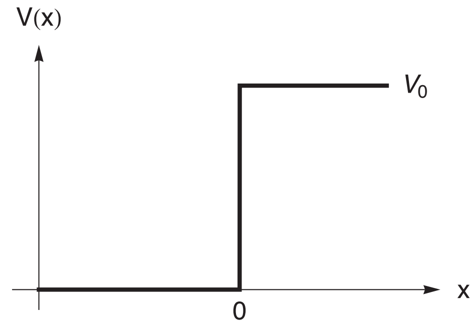
$$\omega_{xy}=\sqrt{\frac{g}{L+\frac{mg}{k}}} \tag{2.23}$$

Thus we have three normal modes, one moving up and down with frequency ω_z above, and two swinging sideways with the same frequency ω_{xy}.

We can see that the vertical mode just has the native frequency of the spring. The horizontal modes have a frequency which is identical to the ordinary pendulum frequency for a pendulum with length equal to that of the spring in its equilibrium stretched by gravity.

We can also check a couple of intersting limiting cases. One is the case where the spring constant k is very large, such that $L\gg\frac{mg}{k}$. In this case, we gain the usual pendulum frequency $\sqrt{\frac{g}{L}}$. The other limiting case is where the equilibrium length is very short (or equivalently, the spring is very soft) so $L\ll\frac{mg}{k}$. In this limit, we find that the frequency of the horizontal modes becomes equal to that of the vertical modes, and is just the usual spring frequency $\sqrt{\frac{k}{m}}$. These cases can give us confidence that our frequency is correct.

Consider a potential energy step as shown below with a beam of particles incident from the left.



a) Calculate the probability of reflection for the case where the energy of the incident particles is less than V_0.

b) Calculate the probability of reflection for the case where the energy of the incident particles is greater than V_0.

c) Sketch your results as a function of the incident energy and comment on the energy dependence.

Solution

a) When the energy of the incident particles is less than the height of the potential energy step, the wave function on the right side is a decaying exponential:

$$\varphi_E(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x < 0 \\ Ce^{-qx}, & x > 0 \end{cases}$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$
$$q = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

The boundary conditions at the step are

$$\varphi(0) : A + B = C$$
$$\left.\frac{d\varphi(x)}{dx}\right|_{x=0} : ikA - ikB = qC$$

Substitute the first equation into the second equation and solve for the ratio of the reflected amplitude to the incident amplitude

$$ikA - ikB = q(A + B)$$
$$ikA - qA = ikB + qB$$
$$\frac{B}{A} = \frac{ik + q}{ik - q}$$

The absolute square of this gives the reflection coefficient

$$R = \frac{|B|^2}{|A|^2} = \frac{ik + q}{ik - q} = \frac{k^2 + q^2}{k^2 + q^2} = 1$$

So 100% of the particles are reflected and there is no probability of transmission. There is some penetration of the wave function into the step, but the wave function decays to zero and never reaches infinity (where your detector is).

b) When the energy of the incident particles is greater than the height of the potential energy step, the wave function on the right side is a complex exponential:

$$\varphi_E(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x}, & x < 0 \\ Ce^{ik_2x}, & x > 0 \end{cases}$$

where

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$
$$k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$

The boundary conditions at the step are

$$\varphi(0) : A + B = C$$
$$\left.\frac{d\varphi(x)}{dx}\right|_{x=0} : ik_1 A - ik_1 B = ik_2 C$$

Substitute the first equation into the second equation and solve for the ratio of the reflected amplitude to the incident amplitude

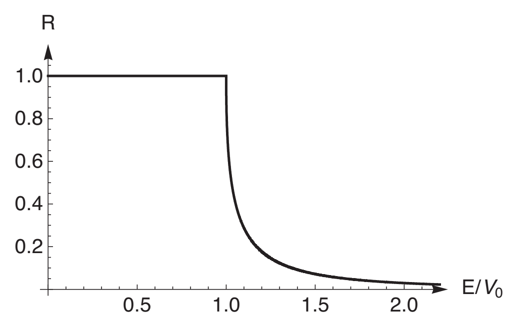
$$ik_1 A - ik_1 B = ik_2 (A + B)$$
$$ik_1 A - ik_2 A = ik_1 B + ik_2 B$$
$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2}$$

The absolute square of this gives the reflection coefficient

$$R = \frac{|B|^2}{|A|^2} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} = \left(\frac{\sqrt{E} - \sqrt{E - V_0}}{\sqrt{E} + \sqrt{E - V_0}}\right)^2$$

So less than 100% of the particles are reflected and there is some probability of transmission

c) Plot:



The reflection probability is unity until the energy exceeds the step height, after which the reflection decreases monotonically.

Consider an infinite square well potential, as shown below, with walls at $x = 0$ and $x = L$; that is, $V(x) = 0$ for $0 < x < L$; $V(x) = \infty$ otherwise.

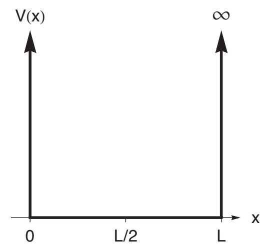


Figure 1: Infinite square well

a) Now impose a small perturbation on this potential of the form $H' = L\ V_0\ \delta(x - L/2)$, where $\delta(x)$ is the Dirac delta function. Calculate the first-order correction to the energy of the nth state of the infinite well.

b) Give some physical explanation of why your answer for (a) is different for even and odd values of n.

Now consider the case where we impose a different small perturbation on the infinite square well potential, as shown below, with ε a small number.

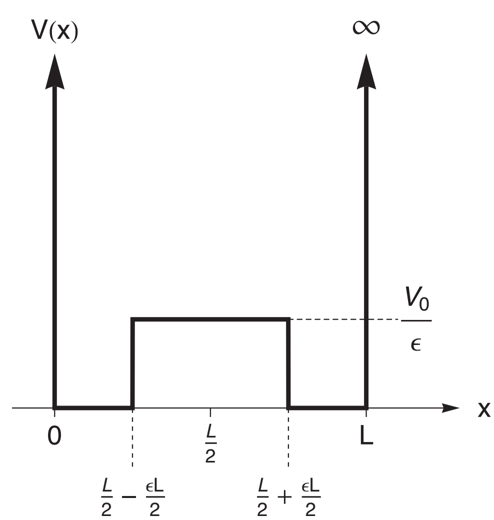


Figure 2: Perturbed infinite square well (part(c)).

c) Calculate the first-order perturbation correction to the energy of the ground state of the infinite well.

d) In the limit where ε goes to zero, compare your answer to (c) with the answer to (a). Discuss.

Solution

a) The first-order energy correction to the nth state is:

$$E_n^{(1)} = \langle n^{(0)} | H' | n^{(0)} \rangle$$

With $H' = LV_0\delta(x - L/2)$ and $\varphi_n^{(0)} = \sqrt{2/L}\sin(n\pi x/L)$, we find

$$E_n^{(1)} = \int_0^L \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right) LV_0\delta\left(x - \frac{L}{2}\right)dx = 2V_0\sin^2\left(\frac{n\pi}{2}\right)$$

For odd values of n, the correction is $2V_0$, while for even values of n, it is zero:

$$\boxed{E_n^{(1)} = \begin{cases} 2V_0 & ;\, n \text{ odd} \\ 0 & ;\, n \text{ even} \end{cases}}$$

b) The wave function for a state with an even value of n is zero at the location of the delta function, so it does not "sample" the perturbation, and the energy is therefore unaffected. Not so for states with odd values of n, where the energy levels are indeed shifted.

c) For this square bump the first-order perturbation is

$$E_n^{(1)} = \langle n^{(0)} | \hat{H}' | n^{(0)} \rangle = \int_{L/2-\varepsilon L/2}^{L/2+\varepsilon L/2} \varphi_n^*(x)\frac{V_0}{\varepsilon}\varphi_n(x)\,dx = \int_{L/2-\varepsilon L/2}^{L/2+\varepsilon L/2}\left[\varphi_1^*(x)\frac{V_0}{\varepsilon}\varphi_1(x)\right]dx$$

$$= \frac{V_0}{\varepsilon}\int_{L/2-\varepsilon L/2}^{L/2+\varepsilon L/2}\left[\frac{2}{L}\sin^2\left(\frac{\pi x}{L}\right)\right]dx = \frac{V_0}{\varepsilon}\frac{2}{L}\int_{L/2-\varepsilon L/2}^{L/2+\varepsilon L/2}\left(\frac{1}{2}\left[1 - \cos\left(\frac{2\pi x}{L}\right)\right]\right)dx$$

$$= \frac{V_0}{\varepsilon L}\left[x - \left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi x}{L}\right)\right]_{L/2-\varepsilon L/2}^{L/2+\varepsilon L/2}$$

$$= \frac{V_0}{\varepsilon L}\left[\frac{L}{2} + \varepsilon\frac{L}{2} - \left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi}{L}\left(\frac{L}{2} + \varepsilon\frac{L}{2}\right)\right) - \left(\frac{L}{2} - \varepsilon\frac{L}{2}\right) + \left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi}{L}\left(\frac{L}{2} - \varepsilon\frac{L}{2}\right)\right)\right]$$

$$= \frac{V_0}{\varepsilon L}\left[\varepsilon L - \left(\frac{L}{2\pi}\right)\sin(\pi + \varepsilon\pi) + \left(\frac{L}{2\pi}\right)\sin(\pi - \varepsilon\pi)\right]$$

$$= \frac{V_0}{\varepsilon L}\left[\varepsilon L + \left(\frac{L}{2\pi}\right)\sin(\varepsilon\pi) + \left(\frac{L}{2\pi}\right)\sin(\varepsilon\pi)\right]$$

$$\boxed{E_1^{(1)} = V_0\left[1 + \frac{\sin(\varepsilon\pi)}{\varepsilon\pi}\right]}$$

d) In the limit of small ε, we get

$$E_1^{(1)} \cong V_0\left[1 + \frac{1}{\varepsilon\pi}\varepsilon\pi\right] = 2V_0$$

$$E_1^{(1)} \cong 2V_0$$

just as we got in part (a). This is to be expected because in the limit of ε -> 0, the square bump looks like a delta function, and we arranged its parameters at the beginning so that the area of the bump $\left((V_0/\varepsilon)\varepsilon L = LV_0\right)$ is the same as the area of the delta function.

Problem 5

A conducting sphere is grounded and is placed in an otherwise uniform electric field $\mathbf{E}_0 = E_0\hat{z}$.

(a) Calculate the potential outside of the sphere.

(b) Calculate the charge density on the surface of the sphere.

A conducting sphere is grounded and is placed in an otherwise uniform electric field $\mathbf{E}_0 = E_0\hat{z}$.

(a) Calculate the potential outside of the sphere.

Solution:
Choosing the z axis to be along the symmetric axis of the sphere the solution $\phi(r, \theta)$ can be expressed using Legendre expansion as

$$\begin{aligned} \Phi = \sum_{l=0}^{\infty}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(\cos\theta) \\ \Phi \to -E_0 r cos\theta, \text{for } r \gg a \\ \Phi(r = a) = 0 \end{aligned} \tag{5.1}$$

By matching the boundary conditions (and apply the uniqueness theorem) we find the solution is

$$\Phi = (A_1 r + \frac{B_1}{r^2})\cos\theta \tag{5.2}$$

where $A_1 = -E_0$, $B_1 = E_0 a^3$.

(b) Calculate the charge density on the surface of the sphere.

Solution:
In order to find the charge density, notice that the electric field inside the sphere is zero, therefore the surface charge density is

$$\sigma = \epsilon_0 E_r|_{r=a} = 3\epsilon_0 E_0 \cos\theta.$$

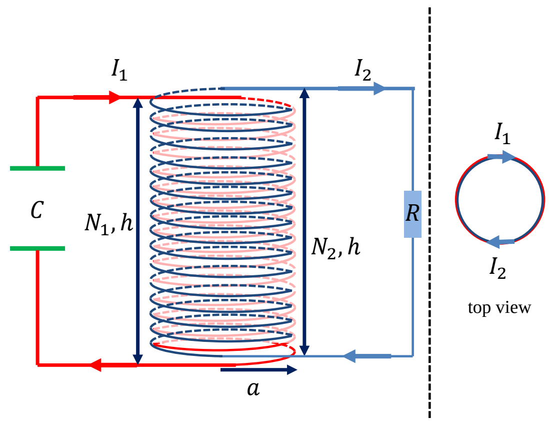


Figure 1:

Two ideal solenoids (tightly wrapped, neglecting boundary effects) are coaxially coupled. Both solenoids have a circular cross-section of radius a, and length of $h \gg a$. The solenoids are placed at equal heights so their first and last rounds are overlapping respectively. If looked down from the top, the currents I_1 and I_2 shown in the figure run clockwise in both of the solenoids.

The first solenoid has total number of N_1 rounds. This solenoid is connected to a capacitor whose capacitance is C, therefore making a L-C circuit.

The second solenoid has total number of N_2 rounds. This solenoid is connected to a resistor of resistance R, therefore making an L-R circuit.

(a) If at certain moment we observe the currents running through each circuits are I_1 and I_2 respectively (as shown in the figure), what is the magnetic field in the solenoids.

(b) Show that the potential across the capacitor and resistor are proportional to each other. This means the two coupled solenoids form a transformer.

(c) Derive the equation that governs the time dependence of $Q(t)$ – the charge of the capacitor. Show that $Q(t)$ can be described as a damped harmonic oscillator.

(d) Consider the limiting case where $R \to \infty$. If at $t = 0$ we have $Q(t = 0) = Q_0$ and $I_1(t = 0) = 0$, solve for $Q(t)$ at later times.

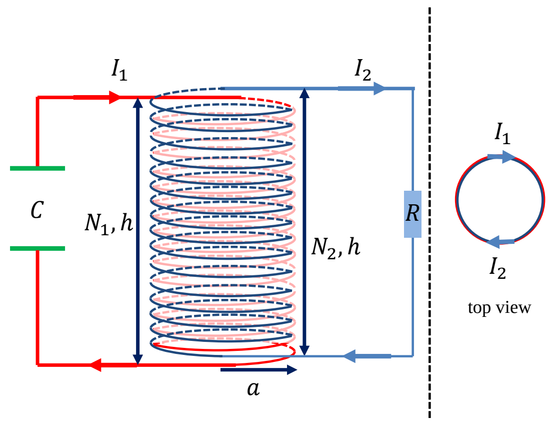


Figure 2:

Two ideal solenoids (tightly wrapped, neglecting boundary effects) are coaxially coupled. Both solenoids have a circular cross-section of radius a, and length of $h \gg a$. The solenoids are placed at equal heights so their first and last rounds are overlapping respectively. If looked down from the top, the currents I_1 and I_2 shown in the figure run clockwise in both of the solenoids.

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(a) If at certain moment we observe the currents running through each circuits are I_1 and I_2 respectively (as shown in the figure), what is the magnetic field in the solenoids.

Solution:
As for ideal solenoids, the magnetic field inside is a constant. Let's call it $B(t)$ along the axis of the coupled solenoids, where positive direction of B is up. From the familiar magnetic field inside a solenoid (which can be derived using Ampere loop), we can derive

$$B = -\mu_0 I_1 \frac{N_1}{h} - \mu_0 I_2 \frac{N_2}{h}, \tag{6.1}$$

where positive direction of I_1 and I_2 are indicated in the figure, such that

$$I_1 = -\frac{dQ}{dt}, \tag{6.2}$$

where Q is the charge in the upper plate of the capacitor.

(b) Show that the potential across the capacitor and resistor are proportional to each other. This means the two coupled solenoids form a transformer.

Solution:
Assuming the electromotive potential across the two solenoids are ϵ_1 and ϵ_2. (top minus bottom)

$$\begin{aligned}\epsilon_1 &= N_1(\frac{d}{dt})B\pi a^2 \\ \epsilon_2 &= N_2(-\frac{d}{dt})B\pi a^2\end{aligned} \tag{6.3}$$

(c) Derive the equation that governs the time dependence of $Q(t)$ – the charge of the capacitor. Show that $Q(t)$ can be described as a damped harmonic oscillator.

Solution:
Additionally we have the Kirchhoff's law

$$\begin{aligned}Q/C + \epsilon_1 &= 0 \\ I_2 R + \epsilon_2 &= 0\end{aligned} \tag{6.4}$$

Together, we can derive the equation of Q

$$\begin{aligned}L_1\frac{d^2Q}{dt^2} + \frac{L_2}{RC}\frac{dQ}{dt} + \frac{Q}{C} &= 0 \\ L_1 &= \frac{\mu_0 N_1^2 \pi a^2}{h} \\ L_2 &= \frac{\mu_0 N_2^2 \pi a^2}{h}\end{aligned} \tag{6.5}$$

This is identical to the equation for a damped oscillator.

(d) Consider the limiting case where $R \to \infty$. If at $t = 0$ we have $Q(t = 0) = Q_0$ and $I_1(t = 0) = 0$, solve for $Q(t)$ at later times.

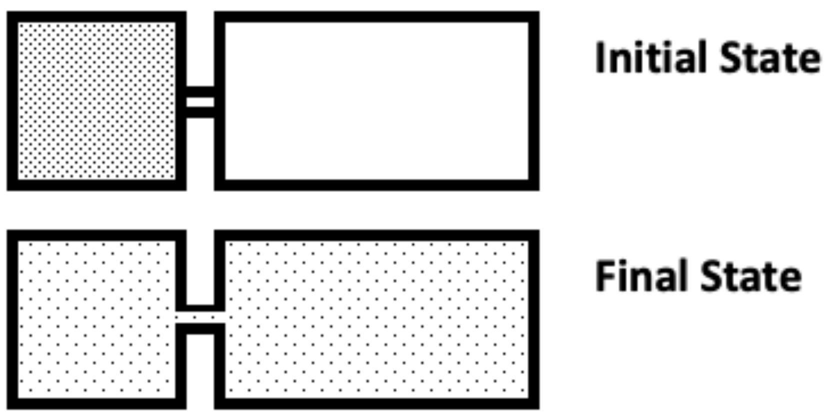
Solution:
In the limit of large R, this becomes a harmonic oscillator with frequency

$$\omega = \sqrt{\frac{1}{L_1 C}} \tag{6.6}$$

using the initial conditions we can find $Q = Q_0 \cos \omega t$.

Consider an experiment in which an ideal monoatomic gas ($pV = NkT; U = \frac{3}{2}NkT$) is initially confined in a container that is connected to an empty container (see, e.g., the figure below). The connection is initially closed and is very small. The two containers are thermally isolated from the surroundings (e.g., wrapped in Styrofoam). At a certain time, the connection between the two containers is opened and the gas slowly leaks into the empty container. After sufficient time has elapsed, the system goes to equilibrium.

a. Define a set of state variables that you need to describe the system.
b. Compute the change of temperature of the system between the initial and final states.
c. Compute the change of entropy of the system.



SOLUTION

For part a), we define the initial state of the system with $p_{initial}$, the initial pressure of the gas in the left container, $V_{initial}$, the volume of the left container, and $T_{initial}$, the temperature of the gas in the left container. We define the final state with p_{final}, V_{final}, and T_{final}. Note that only two of these are independent, so the three variables are redundant.

For part b), we consider that the gas from the left container is in free expansion in the big one. As a consequence, the work done by the gas is null, since it is not working against an external pressure. Since the system is isolated, the heat exchange is also null. We therefore have:

$$\Delta U = Q - W = 0 - 0 = 0$$

In a monoatomic ideal gas, the internal energy is directly proportional to temperature

$$U = \frac{3}{4} NkT$$

and therefore constant internal energy implies constant temperature. We finally have:

$$\Delta T = 0$$

For part c), we remember that S is a state variable and therefore does not depend on the process that links the initial state to the final state.
To compute the change in entropy, we need to perform a quasi-static process. One way is to consider first an adiabatic expansion to the new volume and then an isochoric process to raise the temperature back to the initial value. In an adiabatic process the product pV^{γ} is constant, where $\gamma = C_p/C_V$ is the ratio of the heat capacity of the gas at constant pressure over the heat capacity at constant volume. We can use this to compute the temperature of the gas at the end of the expansion. We have:

$$p_{final} V_{final}^{\gamma} = p_{initial} V_{initial}^{\gamma}$$
$$p_{final} V_{final} V_{final}^{\gamma-1} = p_{initial} V_{initial} V_{initial}^{\gamma-1}$$
$$T_{final} V_{final}^{\gamma-1} = T_{initial} V_{initial}^{\gamma-1}$$
$$T_{final} = T_{initial} \left(\frac{V_{initial}}{V_{final}} \right)^{\gamma-1}$$

This adiabatic process has no entropy change. Now we need to heat the gas back to its initial temperature. This is done at constant volume:

$$dU = TdS = Q = C_V dT$$
$$dS = C_V \frac{dT}{T}$$
$$\Delta S = C_V \int_{T_{final}}^{T_{initial}} \frac{dT}{T} = C_V log \frac{T_{initial}}{T_{final}}$$

Using the result of the adiabatic expansion

$$\Delta S = C_V log\left(\frac{V_{final}}{V_{initial}}\right)^{\gamma-1} = C_V(\gamma-1)log\left(\frac{V_{final}}{V_{initial}}\right) = \left(C_p - C_V\right)log\left(\frac{V_{final}}{V_{initial}}\right)$$

An alternative method is to consider that the internal energy of the initial and final states is the same. We can therefore chose a quasi-static process that has *dU*=0 at all times. This gives us:

$$TdS = pdV$$
$$dS = \frac{p}{T}dV = Nk\frac{dV}{V}$$

Where we have used the equation of state in the last step. Integrating we have:

$$\Delta S = \int_{V_{initial}}^{V_{final}} Nk\frac{dV}{V} = Nk\ log\left(\frac{V_{final}}{V_{initial}}\right)$$

These two solutions are equivalent since it can be shown that $Nk = \left(C_p - C_V\right)$.

Consider a system of *N* independent (uncoupled) harmonic oscillators with the same angular frequency ω (this could represent, e.g., an Einstein model for a crystal). For each oscillator, the energy of state *n* is given by:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

a. Find the internal energy of the system at temperature *T* (the solution may be left in the form of a sum of an infinite number of terms).
b. Find the temperature dependence of the heat capacity of the system in the limit of low temperatures (be careful to keep at least the lowest order of temperature dependence).

SOLUTION

Part (a): The internal energy of the system is given by the sum of the internal energy of each state time the probability of occupation of such state. We have therefore:

$$U = \sum_{n=0}^{\infty} p_n E_n$$

Since the N oscillators are independent, the internal energy of the system will simply be the average energy of one oscillator times the number of oscillators:

$$U = N \sum_{n=0}^{\infty} p_n \hbar\omega \left(n + \frac{1}{2}\right)$$

Since the states are not degenerate, we simply have for the probabilities:

$$p_n = \frac{e^{-\frac{E_n}{kT}}}{\sum_{m=0}^{\infty} e^{-\frac{E_m}{kT}}} = \frac{e^{-\frac{\hbar\omega\left(n+\frac{1}{2}\right)}{kT}}}{\sum_{m=0}^{\infty} e^{-\frac{\hbar\omega\left(m+\frac{1}{2}\right)}{kT}}} = \frac{e^{-\frac{\hbar\omega}{2kT}} e^{-\frac{n\hbar\omega}{kT}}}{e^{-\frac{\hbar\omega}{2kT}} \sum_{m=0}^{\infty} e^{-\frac{m\hbar\omega}{kT}}} = \frac{e^{-\frac{n\hbar\omega}{kT}}}{\sum_{m=0}^{\infty} e^{-\frac{m\hbar\omega}{kT}}}$$

Plugging these into the equation above yields:

$$U = \frac{N}{\sum_{m=0}^{\infty} e^{-\frac{m\hbar\omega}{kT}}} \sum_{n=0}^{\infty} e^{-\frac{n\hbar\omega}{kT}} \hbar\omega \left(n + \frac{1}{2}\right) = \frac{N}{\sum_{m=0}^{\infty} e^{-\frac{m\hbar\omega}{kT}}} \left(\frac{1}{2} \sum_{n=0}^{\infty} e^{-\frac{n\hbar\omega}{kT}} \hbar\omega + \sum_{n=0}^{\infty} n e^{-\frac{n\hbar\omega}{kT}} \hbar\omega\right) =$$

$$= \frac{1}{2} \frac{N\hbar\omega \sum_{n=0}^{\infty} e^{-\frac{n\hbar\omega}{kT}}}{\sum_{m=0}^{\infty} e^{-\frac{m\hbar\omega}{kT}}} + \frac{N\hbar\omega \sum_{n=0}^{\infty} n e^{-\frac{n\hbar\omega}{kT}}}{\sum_{m=0}^{\infty} e^{-\frac{m\hbar\omega}{kT}}} = N\hbar\omega \left(\frac{1}{2} + \frac{\sum_{n=0}^{\infty} n e^{-\frac{n\hbar\omega}{kT}}}{\sum_{m=0}^{\infty} e^{-\frac{m\hbar\omega}{kT}}}\right)$$

Part (b): The heat capacity is the derivative of the internal energy with respect to the temperature. We first approximate the solution above for low temperatures. We note that the exponential term $e^{-\frac{n\hbar\omega}{kT}}$ becomes increasingly small for increasing *n*. Therefore, we keep only the first term of the sums. We have:

$$\frac{\sum_{n=0}^{\infty} n e^{-\frac{n\hbar\omega}{kT}}}{\sum_{m=0}^{\infty} e^{-\frac{m\hbar\omega}{kT}}} \sim \frac{e^{-\frac{\hbar\omega}{kT}}}{1} = e^{-\frac{\hbar\omega}{kT}}$$

This yields:

$$\frac{\partial U}{\partial T} = \frac{\partial}{\partial T}\left(N\hbar\omega e^{-\frac{\hbar\omega}{kT}}\right) = N \frac{\hbar^2 \omega^2}{kT^2} e^{-\frac{\hbar\omega}{kT}}$$