OSU Physics Department
Comprehensive Examination #133
Solutions

Monday, January 7 and Tuesday, January 8, 2019

Winter 2019 Comprehensive Examination

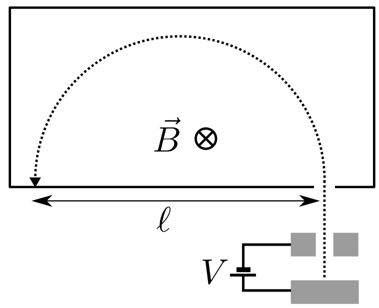
Electricity & Magnetism	9 AM-12 PM	Monday, January 7
Classical Mechanics	1 PM-4 PM	Monday, January 7
Statistical Mechanics	9 AM-12 PM	Tuesday, January 8
Quantum Mechanics	1 PM-4 PM	Tuesday, January 8

General Instructions

This Winter 2019 Comprehensive Examination consists of four separate parts of two problems each, and you have three hours to work on each part. Each problem caries equal weight (20 points). Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work on the provided pages, work each problem in its own labeled pages, and be certain that your chosen student letter (but not your name) is on the header of each page of your exam, including any unused pages. If you need additional paper for your work, use the blank pages provided. Each page of work should include the problem number, a page number, your chosen student letter, and the total number of pages actually used. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam and the collection of formulas distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please staple and return all pages of your exam—including unused pages—at the end of the exam.

Mass spectrometer Consider the following system. An ion with charge $+e$ is accelerated upward (in the figure below) from rest through a potential difference of V. Then the ion enters a region with uniform $\vec{B}$ field directed into the page with magnitude B. The ion is detected moving downward (the opposite of the direction it was moving when it entered the region with uniform magnetic field) a distance ℓ from its entry point. Neglect gravity.



(a) What is the mass of the ion?

(b) How would your answer for the mass change if it turned out that the charge on the ion was $+2e$?

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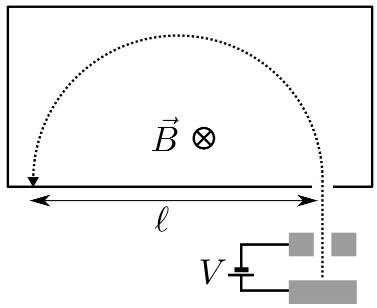


(a) What is the mass of the ion?

Solution:

The ion has two steps here: an acceleration through a potential V followed by circular motion in the magnetic field. The motion is circular because the velocity is orthogonal to $\vec{B}$ and thus the force is orthogonal to the velocity.

I will begin with the acceleration of the ion. The change in potential energy is eV, which must be equal to the final kinetic energy. Thus

$$\frac{1}{2}mv^2 = eV \tag{1.1}$$

$$v = \sqrt{\frac{2eV}{m}} \tag{1.2}$$

An incorrect approach to this would be to assume that the E-field is uniform in the ion gun and then use the equations of kinematics to solve for the speed. This would give a correct answer, but does not indicate that the answer is independent of the geometry of the ion gun.

Now we need to deal with the circular motion. Here we need to know the force acting on the ion, which is given by the Lorentz expression.

$$\vec{F} = e\vec{v} \times \vec{B} \tag{1.3}$$

Note that in Gaussian units this would be divided by the speed of light c. The force is initially to the left, and continues to be orthogonal to the velocity. The magnitude of the force is

$$|\vec{F}| = evB \tag{1.4}$$

Now we need to invoke Newton's second law, and can either remember the expression for centripetal acceleration or derive it. As graduate students, you should be able to derive the centripetal acceleration in a couple of minutes. Given that the ion is traveling in a circle with speed v and radius R, its position is given by:

$$\vec{r} = R\left(\hat{x}\cos(vt/R) + \hat{y}\sin(vt/R)\right) \tag{1.5}$$

$$\vec{v} = \frac{d\vec{r}}{dt} \tag{1.6}$$

$$= v\left(-\hat{x}\sin(vt/R) + \hat{y}\cos(vt/R)\right) \tag{1.7}$$

$$\vec{a} = \frac{d\vec{v}}{dt} \tag{1.8}$$

$$= -\frac{v^2}{R}\left(\hat{x}\cos(vt/R) + \hat{y}\sin(vt/R)\right) \tag{1.9}$$

Thus the magnitude of the acceleration is v^2/R. This tells us that

$$\vec{F} = m\vec{a} \tag{1.10}$$

$$|\vec{F}| = m|\vec{a}| \tag{1.11}$$

$$eBv = \frac{mv^2}{R} \tag{1.12}$$

$$R = \frac{mv}{eB} \tag{1.13}$$

$$= \frac{\sqrt{2meV}}{eB} \tag{1.14}$$

$$m = \frac{R^2 eB^2}{2V} \tag{1.15}$$

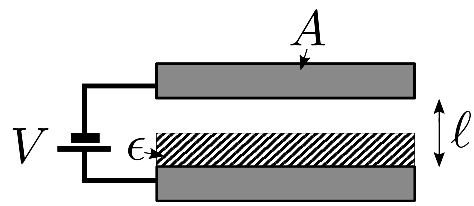
Now the radius is going to be half of ℓ, so we find that

$$m = \frac{\ell^2 eB^2}{8V} \tag{1.16}$$

(b) How would your answer for the mass change if it turned out that the charge on the ion was $+2e$?

Solution:
Since the mass is proportional to e, it must be twice as large if the charge turns out to be twice what we originally thought.

Parallel plates Consider a parallel plate capacitor with area A and distance between plates ℓ which is half full of dielectric with relative dielectric constant ϵ. The capacitor is hooked up to a battery with voltage V. The distance between the plates is small compared to the transverse dimensions of the plates.



(a) What is the force on the upper plate?

(b) Suppose the upper plate is lowered until it touches the dielectric (thus cutting the distance between the two plates in half) while still connected to the same battery. What would the new force be?

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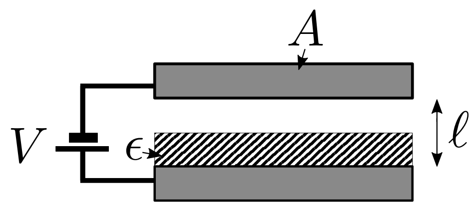

(a) What is the force on the upper plate?

Solution:
There are a couple of ways to find the force on the upper plate. We can either compute the energy stored in the capacitor and take an appropriate derivative, or we can compute the electric field felt by the upper place and the charge on it and use those to find the force. The former approach is slightly more tricky, because we have to ensure that we take the derivative with the battery disconnected (i.e. holding the charge fixed, not the potential). So instead I will demonstrate the solution where we compute the force directly from the electric field and charge. Afterwards I will explain in more detail why taking a derivative of the energy of the capacitor with the potential held fixed gives the wrong answer.

I'll begin by solving for the charge and electric field. Because the plates are close together (relative to their $\sqrt{A}$), we can assume that the electric field is always vertical within the plates, neglecting edge effects. We can use Gauss's law $\vec{\nabla} \cdot \vec{D} = \rho_f$ to solve for the D-field, and then find the E-field from the fact that the material is a linear dielectric. Naturally we'll want the integral version of Gauss's Las:

$$\oint \vec{D} \cdot d\vec{A} = Q_{\text{enc}} \tag{2.1}$$

To start, we draw a couple of Gaussian surfaces, show in Fig. 1. Each

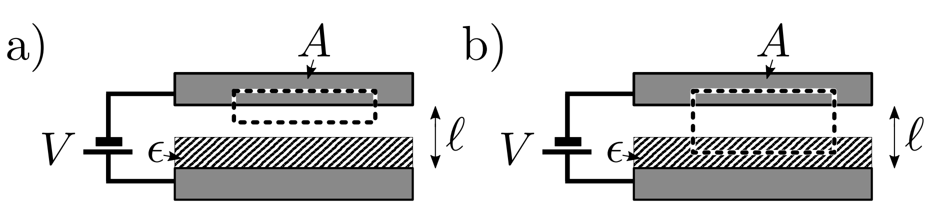

Figure 1: Gaussian surfaces for finding the electric field.

surface has its top surface in the metal where the field is zero. The side surfaces have their normal orthogonal to the field direction, so there is zero flux. Thus the total flux through the enclosed surface is just the flux through the bottom of the surface, which is given by

$$\oint \vec{D} \cdot d\vec{A} = DA_{\text{bot}} \tag{2.2}$$

where D is the magnitude of the D-field and A_{bot} is the area of the bottom of the Gaussian surface. I'm not going to worry about signs here, because they won't affect the final answer. Thus we find that

$$DA_{\text{bot}} = Q_{\text{enc}} \tag{2.3}$$

$$DA = Q \tag{2.4}$$

where in the last step I made use of the fact that the enclosed charge is proportional to the area of the Gaussian surface. You could also make the Gaussian surface as wide as the capacitor to begin with, thus skipping this step.

We now see that the D-field is uniform thoughout the capacitor, which makes the E-field have different values in the dielectric and in the vacuum.

$$E(z) = \frac{D}{\epsilon(z)} \tag{2.5}$$

$$= \begin{cases} \frac{Q}{\epsilon_0 A} & \text{in vacuum} \\ \frac{Q}{\epsilon_r \epsilon_0 A} & \text{in the dielectric} \end{cases} \tag{2.6}$$

In this answer I used ϵ_r to represent the relative dielectric constant (contrary to what was stated in the exam) to provide improved clarity. (Students who assumed ϵ was the non-relative dielectric constant were not penalized for this.)

Our problem at the moment is that we don't know the charge on the dielectric. To do that, we can integrate the electric field to find the potential difference, and set it to that of the battery.

$$|V| = \int E dz = \frac{Q}{\epsilon_0 A}\frac{\ell}{2} + \frac{Q}{\epsilon_r \epsilon_0 A}\frac{\ell}{2} \tag{2.7}$$

Thus we find that

$$Q = \frac{VA\epsilon_0}{\ell}\frac{2}{1+\frac{1}{\epsilon_r}} \tag{2.8}$$

We also see that the electric field in the vacuum is given by

$$E = \frac{Q}{\epsilon_0 A} = \frac{V}{\ell}\frac{2}{1+\frac{1}{\epsilon_r}} \tag{2.9}$$

Finding the force from $\vec{E}$ Now we need to find what the force is, given the electric field and the charge. The most common mistake on this problem was to simply multiply the two. This is incorrect, and there are a couple of explanations as to what this is the case. The charge is living at the surface. The field is zero immediately above the surface, and has the above value immediately below the surface. So it is not immediately obvious whether the proper electric field to use would be zero or $\frac{Q}{\epsilon_0}$. How to resolve this quandary?

One way to do this is to consider *why* the E-field is discontinuous at this surface. It is discontinuous at the surface because there is a surface charge here. But the net force on the surface charge due to itself must be zero (based on Newton's Third Law, if you like). So we need instead to only account for the force due to the opposite surface (the dielectric in a planar symmetry such as this exerts zero force, because it has zero net charge), and thus we only need account for the E-field due to the opposite surface. There are several approaches we could use to directly find the field due to the opposite surface, but we can also use a simple symmetry argumet to show that it must be half of the E field in the vacuum. This is because the E-fields of the two surfaces must exactly cancel inside the metal. So we get a factor of two.

A second argumet we could use would be to equate the change in energy stored in the vacuum with the mechanical work done when moving the metal surface. The energy density in the vacuum has a factor of $\frac{1}{2}$ in it that gives us the same correction factor of a half.

In either case, our answer for the force is that it is

$$\boxed{F = \frac{1}{2}QE = \frac{1}{2}\frac{V^2 A\epsilon_0}{\ell^2}\left(\frac{2}{1+\frac{1}{\epsilon_r}}\right)^2} \tag{2.10}$$

with its direction downward (towards the other plate). The direction of the force we can get by simply recognizing that the two plates have opposite charges, which attract. Alternatively, you could keep track of the sign of the charge and the direction of the E-field as you do the calculation.

Finding the force from U An alternative approach to finding the force on the plate is to take a derivative of the electrostatic energy stored in the capacitor as the height of the capacitor changes. A common mistake here is to hold the voltage fixed rather than holding the charge fixed.

There is one more subtlety, which is that when we move the top capacitor, we don't change the thickness of the dielectric, so taking a simple derivative with respect to ℓ is incorrect, if we use any expression that uses $\ell/2$ as the thickness of the dielectric. Intead we need to be careful to give the thickness of the dielectric a different name than the height of the metal slab (I'll use h). Thus we will need to go back to our integral finding V to see that

$$|V| = \int E dz = \frac{Q}{\epsilon_0 A}(\ell - h) + \frac{Q}{\epsilon_r \epsilon_0 A} h \tag{2.11}$$

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{\ell - \left(1 - \frac{1}{\epsilon_r}\right) h} \tag{2.12}$$

I'll begin this explanation by working out the energy stored. There are different approaches here, but it is simplest is to simply use the equations for a capacitor (if you remember them):

$$U = \frac{1}{2} Q V = \frac{Q^2}{2C} = \frac{1}{2} C V^2 \tag{2.13}$$

$$= \frac{1}{2} \frac{\epsilon_0 A}{\ell - \left(1 - \frac{1}{\epsilon_r}\right) h} V^2 \tag{2.14}$$

$$= \frac{1}{2\epsilon_0 A} \left(\ell - \left(1 - \frac{1}{\epsilon_r}\right) h\right) Q^2 \tag{2.15}$$

We now have two ways we could take a derivative $\frac{\partial U}{\partial \ell}$: We could either evaluate $\left(\frac{\partial U}{\partial \ell}\right)_V$ or $\left(\frac{\partial U}{\partial \ell}\right)_Q$. If we want to take a derivative of the stored electrostatic energy, we need to justify which derivative will give us the force. This justification is more subtle, since it requires us to reason about how (and whether) the battery affects the force.

Moving back to the question of what to hold fixed, The battery cannot directly affect the force between the plates. The force on the plates must be purely electrostatic, and the only way the battery can cause an electrostatic force is by putting charge on the plates. Thus we must get the same force whether we disconnect the battery or leave it connected.

If we disconnect the battery, then we can see that when the plate is moved, the only energy that changes is the energy stored in the capacitor, and therefore energy conservation tells us that the change in this energy must be equal to the mechanical work done.

Conversely, if you move the plates with the battery connected, then the charges will change. That means that current will flow through the battery when you move the plate, and thus the battery will do electrochemical work. Thus the change in the energy stored in the capacitor will be equal to not just the mechanical work, but rather the sum of mechanical and electrochemical (or battery) work. We absolutely could use this approach (the work done by the battery is just $V\Delta Q$, after all), but I am not going to bother. Instead I will just conclude that

$$F = \left(\frac{\partial U}{\partial \ell}\right)_Q \tag{2.16}$$

$$= \frac{1}{2\epsilon_0}Q^2 \tag{2.17}$$

$$= \frac{1}{2}\frac{Q}{\epsilon_0}Q \tag{2.18}$$

$$= \frac{1}{2}EQ \tag{2.19}$$

which is the same answer we found above (and I won't repeat). Again, you should indicate that the force is downward, towards the other plate.

(b) Suppose the upper plate is lowered until it touches the dielectric (thus cutting the distance between the two plates in half) while still connected to the same battery. What would the new force be?

Solution:
There is an implicit assumption here that we will ignore any normal force due to the two solids touching. One approach to solving this would be to imagine an infinitesimal vacuum gap between the dielectric and the metal plate. The electric field will then be $\frac{Q}{\epsilon_0 A}$ in this gap based on the same Gaussian reasoning above. So once we solve for the charge we will be able to use the same expression above. The charge in this case is way easier. I'll just take Eq. 2.11 above and set $\ell = h$, which tells us that

$$V = \frac{Q}{\epsilon_r\epsilon_0 A}h = \frac{Q}{\epsilon_r\epsilon_0 A}\frac{\ell}{2} \tag{2.20}$$

where in the second expression I gave h the name $\ell/2$ that was specified in the problem. The force is thus

$$F = \frac{1}{2}QE \tag{2.21}$$

$$= \frac{1}{2}Q\frac{Q}{\epsilon_0 A} \tag{2.22}$$

$$= \frac{1}{2\epsilon_0 A}\frac{4\epsilon_r^2\epsilon_0^2V^2A^2}{\ell^2} \tag{2.23}$$

$$\boxed{F = \frac{1}{2}\frac{V^2\epsilon_0 A}{\ell^2}\left(4\epsilon_r^2\right)} \tag{2.24}$$

This is the same answer we would get if we had taken the solution to part (a) and removed the one from the denominator. (Note here that one thing to check is that the final answer must be proportional to the area, if that doesn't work out, then our answer wouldn't make sense, since the total force must scale with the total amount of charge, which itself must scale with the total size of the capacitor.)

Another way to approach this (and similarly on the previous part) would be to solve for the electric field due to everything but the top plate. This would give you the same answer for the final force, but your electric field due to the bottom plate + dielectric would be $E_{\text{everything but top plate}} = \frac{Q}{2\epsilon_0 A}$ and the factor of $\frac{1}{2}$ not be included in the $F = QE_{\text{everything but top plate}}$ equation.

Problem 3

Two identical, uniform rods of mass m and length ℓ, connected by a string, are hung from the ceiling via another string, as shown in Figure 2. Both strings have negligible mass and length. The system forms a physical double pendulum in the plane of the paper. g is the gravitational constant near the earth and its direction is indicated by the arrow in the drawing.

Note: The moment of inertia of each rod about its own center of mass is $I_{cm} = \frac{1}{12} m\ell^2$.

(a) For small oscillations around the equilibrium, find the frequencies of the normal modes of the system.

(b) For each of the normal modes, describe the motion of the system.

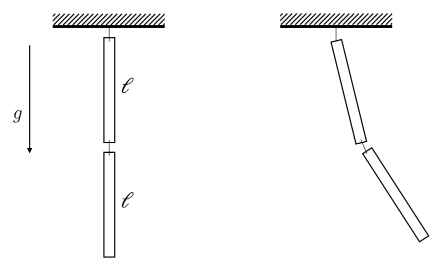

Figure 2: The double pendulum.

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Note: The moment of inertia of each rod about its own center of mass is $I_{cm} = \frac{1}{12} m\ell^2$.

(a) For small oscillations around the equilibrium, find the frequencies of the normal modes of the system.

Solution:

Let θ and ϕ be the angles of the top and bottom rods with respect to the vertical, respectively.

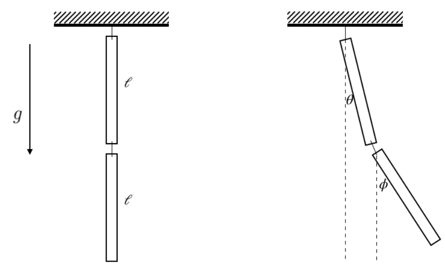

Figure 3: The double pendulum solution.

The kinetic energy of the system is

$$T = \frac{1}{2}\left[m\left(\frac{\ell}{2}\dot{\theta}\right)^2 + \frac{1}{12}m\ell^2\dot{\theta}^2 + m\left(\ell^2\dot{\theta}^2 + \ell^2\dot{\theta}\dot{\phi}\cos(\theta-\phi) + \frac{\ell^2}{4}\dot{\phi}^2\right) + \frac{1}{12}m\ell^2\dot{\phi}^2\right].$$

The potential energy of the system is

$$V = mg\frac{\ell}{2}(1-\cos\theta) + mg\left[\frac{3}{2}\ell - \left(\ell\cos\theta + \frac{\ell}{2}\cos\phi\right)\right].$$

For small oscillations around the equilibrium, we have the following approximations

$$1-\cos\theta \approx \frac{1}{2}\theta^2, \quad 1-\cos\phi \approx \frac{1}{2}\phi^2, \quad \cos(\theta-\phi)\approx 1.$$

Thus, for small oscillations around the equilibrium, the Lagrangian of the system is

$$L = T - V = \frac{4}{6}m\ell^2\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\theta}\dot{\phi} + \frac{1}{6}m\ell^2\dot{\phi}^2 - \frac{1}{4}mg\ell(3\theta^2+\phi^2).$$

Lagrange's equations are

$$\begin{cases} \frac{1}{2}\left(\frac{8}{3}\ell\ddot{\theta} + \ell\ddot{\phi} + 3g\theta\right) = 0 \\ \frac{1}{2}\left(\ell\ddot{\theta} + \frac{2}{3}\ell\ddot{\phi} + g\phi\right) = 0, \end{cases}$$

or

$$\begin{cases} \frac{1}{2}\left(\frac{8}{3}\ddot{\theta} + \ddot{\phi} + 3\omega_0^2\theta\right) = 0 \\ \frac{1}{2}\left(\ddot{\theta} + \frac{2}{3}\ddot{\phi} + \omega_0^2\phi\right) = 0, \end{cases}$$

where $\omega_0 = \sqrt{\frac{g}{\ell}}$.

To look for normal modes, we assume the following forms for θ and ϕ

$$\theta = A\cos\omega t, \quad \phi = B\cos\omega t.$$

The above Lagrangian's equations become

$$\begin{cases} \left(\frac{3}{2}\omega_0^2 - \frac{4}{3}\omega^2\right)A - \frac{1}{2}\omega^2 B = 0 \\ -\frac{1}{2}\omega^2 A + \left(\frac{1}{2}\omega_0^2 - \frac{1}{3}\omega^2\right)B = 0. \end{cases}$$

This yields normal mode frequencies of

$$\omega^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\omega_0^2.$$

(b) For each of the normal modes, describe the motion of the system.

Solution:

For each of the normal modes, describe the motion of the system.

For the $\omega^2 = \left(3 + \frac{6}{\sqrt{7}}\right)\omega_0^2$ frequency, $B = \left(-\frac{2\sqrt{7}}{3} - \frac{1}{3}\right)A$.

For the $\omega^2 = \left(3 - \frac{6}{\sqrt{7}}\right)\omega_0^2$ frequency, $B = \left(\frac{2\sqrt{7}}{3} - \frac{1}{3}\right)A$.

Problem 4

Consider the motion of a particle of mass m with a potential energy

$$V(\vec{r}) = -\frac{k}{r} - \frac{\alpha}{2r^2},$$

where $k > 0$, $\vec{r}$ is the position vector of the particle from the origin and r is the distance from the particle to the origin. At $t = 0$, the particle has angular momentum $\vec{L}$, the magnitude of which is L.

(a) Show that the angular momentum $\vec{L}$ is conserved.

(b) Determine the value of α for which the particle moves in a circular orbit of radius r_0? Express α in terms of k, L, m, and r_0.

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(a) Show that the angular momentum $\vec{L}$ is conserved.

Solution:
Because the potential energy of the particle is only a function of r, the force on the particle $\vec{F}$ is

$$\vec{F} = -\frac{d}{dr}\left(-\frac{k}{r} - \frac{\alpha}{2r^2}\right)\hat{r} = \left(-\frac{k}{r^2} - \frac{\alpha}{r^3}\right)\hat{r}.$$

The torque on the particle with respect to the origin is

$$\tau = \vec{r} \times \vec{F} = r\hat{r} \times \left(-\frac{k}{r^2} - \frac{\alpha}{r^3}\right)\hat{r} = 0.$$

From $\dfrac{d\vec{L}}{dt} = \tau$, we have $\dfrac{d\vec{L}}{dt} = 0$.

Thus, the angular momentum $\vec{L}$ of the particle is conserved.

(b) Determine the value of α for which the particle moves in a circular orbit of radius r_0? Express α in terms of k, L, m, and r_0.

Solution:
The effective potential of the particle is

$$V_{\text{eff}}(\vec{r}) = V(\vec{r}) + \frac{L^2}{2mr^2} = -\frac{k}{r} - \frac{\alpha}{2r^2} + \frac{L^2}{2mr^2}.$$

For the particle to move in a circular orbit of radius r_0, the effective potential $V_{\text{eff}}(\vec{r})$ satisfies

$$\frac{d}{dr}V_{\text{eff}}(\vec{r}) = \frac{d}{dr}\left(-\frac{k}{r} - \frac{\alpha}{2r^2} + \frac{L^2}{2mr^2}\right)\Bigg|_{r=r_0} = 0,$$

which yields

$$\frac{k}{r_0^2} + \frac{\alpha}{r_0^3} - \frac{L^2}{mr_0^3} = 0.$$

Thus

$$\alpha = \frac{L^2}{m} - kr_0.$$

Alternative solution to (b) Another solution to part (b) would be to use the centripetal acceleration and Newton's second law. If the particle is moving in a circular orbit, the magnitude of its accelreation must be

$$a = \frac{v^2}{r_0} \tag{4.1}$$
$$F = ma \tag{4.2}$$
$$= \frac{mv^2}{r_0} \tag{4.3}$$

The speed may be computed from the given angular momentum

$$\vec{L} = \vec{r} \times \vec{p} \tag{4.4}$$
$$L = r_0 m v \tag{4.5}$$
$$v = \frac{L}{r_0 m} \tag{4.6}$$

Putting this into an expression for the force give us

$$F = \frac{m\left(\frac{L}{r_0 m}\right)^2}{r_0} \tag{4.7}$$
$$= \frac{L^2}{m r_0^3} \tag{4.8}$$
$$= \frac{k}{r_0^2} + \frac{\alpha}{r_0^3} \tag{4.9}$$
$$\alpha = \frac{L^2}{m} - k r_0 \tag{4.10}$$

This approach would be more suitable if you are unfamiliar with effective potentials for central forces.

1. An experiment is carried out on a thermodynamic system made of N atoms, each of which can assume three possible states with energies E_0, E_1, and E_2. The N atoms are independent and do not interact with each other. The experiment reveals that at temperature T the fraction of atoms in the ground level is α_0, while the fraction of atoms in the first excited state is α_1. Find equations to express the value of E_1, and E_2, assuming $E_0=0$.

1. **An experiment is carried out on a thermodynamic system made of *N* atoms, each of which can assume three possible states with energies E_0, E_1, and E_2. The *N* atoms are independent and do not interact with each other. The experiment reveals that at temperature *T* the fraction of atoms in the ground level is α_0, while the fraction of atoms in the first excited state is α_1. Find equations to express the value of E_1, and E_2, assuming $E_0=0$.**

Using statistical mechanics, we know that the fraction of atoms in a given state reads:

$$P_i = \frac{e^{-\frac{E_i}{kT}}}{Z}$$

Where *Z* is the partition function of the system. The probability is also given by:

$$P_i = \frac{N_i}{N} = \alpha_i$$

The ratio of the number of atoms in the *i-th* excited state to those in the ground state is therefore given by:

$$\frac{P_i}{P_o} = \frac{\frac{e^{-\frac{E_i}{kT}}}{Z}}{\frac{e^{-\frac{E_0}{kT}}}{Z}} = \frac{e^{-\frac{E_i}{kT}}}{e^{-\frac{E_0}{kT}}} = e^{-\frac{(E_i-E_0)}{kT}}$$

This is also equal to:

$$\frac{P_i}{P_o} = \frac{N_i}{N_0} = \frac{\alpha_i}{\alpha_0}$$

Combining these two:

$$\frac{\alpha_i}{\alpha_0} = e^{-\frac{(E_i-E_0)}{kT}}$$

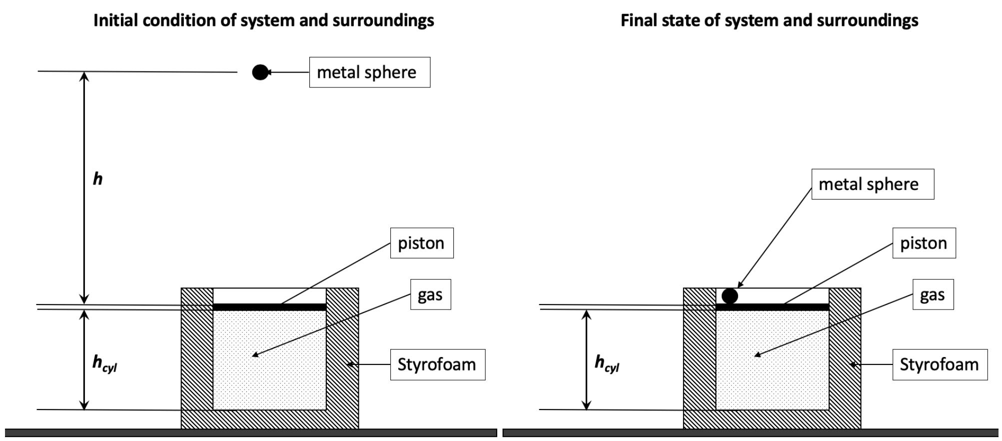
And, solving for E_i:

$$E_i = E_0 + kTln\left(\frac{\alpha_0}{\alpha_i}\right)$$

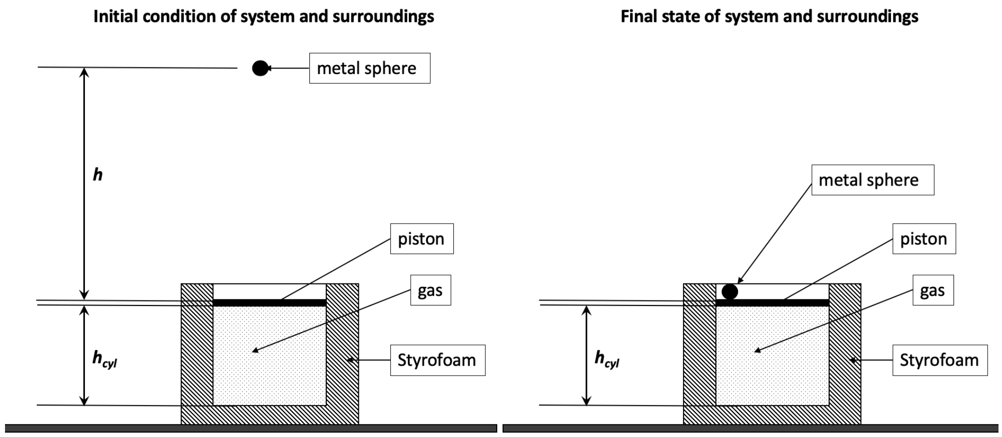
From which, remembering that $E_0=0$:

$$E_1 = kTln\left(\frac{\alpha_0}{\alpha_1}\right); \qquad E_2 = kTln\left(\frac{\alpha_0}{1-\alpha_0-\alpha_1}\right);$$

2. A metal sphere (e.g. a ball bearing) of mass *m* falls on a gas cylinder with a piston, isolated with Styrofoam, from an initial height *h* (see figure below). After a few bounces, the ball comes to rest on top of the piston (final state figure). Assume that the surroundings (air, cylinder, piston, sphere) do not absorb any energy in the process, and adopt ideal gas laws ($pV = NkT; U = \frac{3}{2}NkT$). Find the height *h* for which the volume of the gas remains unchanged after the gas has reached a steady-state configuration.



2. **A metal sphere (e.g. a ball bearing) of mass *m* falls on a gas cylinder with a piston, isolated with Styrofoam, from an initial height *h* (see figure below). After a few bounces, the ball comes to rest on top of the piston (final state figure). Assume that the surroundings (air, cylinder, piston, sphere) do not absorb any energy in the process, and adopt ideal gas laws ($pV = NkT; U = \frac{3}{2}NkT$). Find the height *h* for which the volume of the gas remains unchanged after the gas has reached a steady-state configuration.**



Since the volume remains constant, there is no work involved. This implies that the change in pressure of the gas is equal to the pressure added by the weight of the sphere. To find the pressure change, we consider the change in internal energy due to the transfer of the potential energy of the sphere to the gas. We have:

$$\Delta U = \delta Q = mgh$$

Since it is an ideal gas, the change in internal energy is directly connected to the change in temperature:

$$\Delta U = \frac{3}{2}Nk\Delta T$$

We now consider the equation of state $pV = NkT$ which, combined to the equation above gives:

$$\Delta U = \frac{3}{2}\Delta(pV) = \frac{3}{2}(p\Delta V + V\Delta p) = \frac{3}{2}V\Delta p$$

Combining this with the first equation we have:

$$\frac{3}{2}V\Delta p = mgh$$

As a last step we notice that $\Delta p = mg/A$, giving:

$$\frac{3}{2}V\Delta p = \frac{3}{2}V\frac{mg}{A} = \frac{3}{2}mgh_{cyl} = mgh$$

Simplifying out the mass ang gravity acceleration we find that the volume of the gas does not change if:

$$h = \frac{3}{2}h_{cyl}$$

Problem 7

Measurements of the energy of an harmonic oscillator system yield the results $\hbar\omega/2$ and $3\hbar\omega/2$ with equal probability. Measurements of the position of this system yield the expectation value $\langle x\rangle = -\sqrt{\hbar/2m\omega}\,\sin\omega t$. Calculate the expectation value of the momentum as a function of time.

Solution:

The two measured energies are the $n = 0, 1$ states, so the results tells us that:

$$\mathcal{P}_{E_n} = \left|\langle n|\psi(t)\rangle\right|^2 = \tfrac{1}{2}\left(\delta_{n0} + \delta_{n1}\right)$$

Energy measurements are time independent, so we have no information on the time dependence of the amplitudes, but we know something about them from the Schrödinger time evolution. Thus the original state and the time-evolved state are:

$$\begin{aligned}
&|\psi(0)\rangle = c_0|0\rangle + c_1|1\rangle \quad \Rightarrow \quad |c_0|^2 = |c_1|^2 = \tfrac{1}{2} \\
&|\psi(0)\rangle = \tfrac{1}{\sqrt{2}}\left[e^{i\theta_0}|0\rangle + e^{i\theta_1}|1\rangle\right] \\
&|\psi(t)\rangle = e^{-i\frac{\omega t}{2}}\tfrac{1}{\sqrt{2}}\left[e^{i\theta_0}|0\rangle + e^{i\theta_1}e^{-i\omega t}|1\rangle\right]
\end{aligned}$$

Now use this to find the expectation value of the position:

$$\begin{aligned}
\langle x\rangle &= \langle\psi(t)|x|\psi(t)\rangle = \sqrt{\tfrac{\hbar}{2m\omega}}\langle\psi(t)|a^\dagger + a|\psi(t)\rangle \\
&= \sqrt{\tfrac{\hbar}{2m\omega}}e^{+i\frac{\omega t}{2}}\tfrac{1}{\sqrt{2}}\left(e^{-i\theta_0}\langle 0| + e^{-i\theta_1}\langle 1|e^{+i\omega t}\right)\left(a^\dagger + a\right)e^{-i\frac{\omega t}{2}}\tfrac{1}{\sqrt{2}}\left(e^{i\theta_0}|0\rangle + e^{i\theta_1}|1\rangle e^{-i\omega t}\right) \\
&= \sqrt{\tfrac{\hbar}{2m\omega}}\tfrac{1}{2}\left[e^{-i\theta_1}e^{+i\omega t}e^{i\theta_0}\langle 1|a^\dagger|0\rangle + e^{-i\theta_0}e^{i\theta_1}e^{-i\omega t}\langle 0|a|1\rangle\right] \\
&= \sqrt{\tfrac{\hbar}{2m\omega}}\tfrac{1}{2}\left[e^{+i\omega t + i\theta_0 - i\theta_1} + e^{-i\omega t - i\theta_0 + i\theta_1}\right] = \sqrt{\tfrac{\hbar}{2m\omega}}\cos\left(\omega t + \theta_0 - \theta_1\right)
\end{aligned}$$

Hence $\langle x\rangle = -\sqrt{\hbar/2m\omega}\sin\omega t$ implies that $\left(\theta_0 - \theta_1\right) = \pi/2$. The overall phase is unknown but doesn't matter (cannot be measured). Now use this to find the expectation value of the momentum:

$$\begin{aligned}
\langle p\rangle &= \langle\psi(t)|p|\psi(t)\rangle = i\sqrt{\tfrac{m\omega\hbar}{2}}\langle\psi(t)|a^\dagger - a|\psi(t)\rangle \\
&= \sqrt{\tfrac{m\omega\hbar}{2}}e^{+i\frac{\omega t}{2}}\tfrac{i}{\sqrt{2}}\left(e^{-i\theta_0}\langle 0| + e^{-i\theta_1}\langle 1|e^{+i\omega t}\right)\left(a^\dagger - a\right)e^{-i\frac{\omega t}{2}}\tfrac{1}{\sqrt{2}}\left(e^{i\theta_0}|0\rangle + e^{i\theta_1}|1\rangle e^{-i\omega t}\right) \\
&= \sqrt{\tfrac{m\omega\hbar}{2}}\tfrac{i}{2}\left(e^{-i\theta_1}e^{+i\omega t}e^{i\theta_0}\langle 1|a^\dagger|0\rangle - e^{-i\theta_0}e^{i\theta_1}e^{-i\omega t}\langle 0|a|1\rangle\right) \\
&= \sqrt{\tfrac{m\omega\hbar}{2}}\tfrac{i}{2}\left(e^{+i\omega t + i\theta_0 - i\theta_1} - e^{-i\omega t - i\theta_0 + i\theta_1}\right) = -\sqrt{\tfrac{m\omega\hbar}{2}}\sin\left(\omega t + \theta_0 - \theta_1\right) = -\sqrt{\tfrac{m\omega\hbar}{2}}\sin\left(\omega t + \pi/2\right) \\
&= -\sqrt{\tfrac{m\omega\hbar}{2}}\cos\omega t
\end{aligned}$$

Consider a quantum mechanical system with a three-dimensional state space. In the basis defined by three orthonormal kets $|1\rangle$, $|2\rangle$, and $|3\rangle$, an observable A is represented by the matrix

$$A \doteq a\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix},$$

where a is a real, positive constant. The Hamiltonian H of the system has three distinct eigenvalues $E_1 = b$, $E_2 = 2b$, and $E_3 = 3b$ (b is a real, positive constant) with corresponding eigenstates $|E_1\rangle = |1\rangle$, $|E_2\rangle = |2\rangle$, and $|E_3\rangle = |3\rangle$. At time $t = 0$, the state of the system is

$$|\psi(0)\rangle = \frac{1}{\sqrt{6}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle + \frac{1}{\sqrt{3}}|3\rangle .$$

a) What is the matrix representation of the Hamiltonian H in the basis defined by $|1\rangle$, $|2\rangle$, and $|3\rangle$?

b) Find the expectation value $\langle H \rangle$ and the r.m.s. deviation ΔH at time $t = 0$.

c) At time t_0 (where $t_0 > 0$) the observable A is measured. What are the possible results of that measurement?

d) What is the probability that the measurement of the observable A at time t_0 yields a negative value?

e) For the particular case where the result of the above measurement of A does yield a negative value, find the state of the system for times $t > t_0$.

Solution:

a) Because the states $|1\rangle$, $|2\rangle$, and $|3\rangle$ are the eigenstates of the Hamiltonian, the matrix representation of H in that basis is diagonal, with the corresponding eigenvalues along the diagonal:

$$H \doteq \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} = \begin{pmatrix} b & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 3b \end{pmatrix}.$$

b) The expectation value $\langle H \rangle$ is

$$\langle H \rangle = \langle \psi | H | \psi \rangle = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} b & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 3b \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{b}{\sqrt{6}} \\ \frac{2b}{\sqrt{2}} \\ \frac{3b}{\sqrt{3}} \end{pmatrix} = \frac{b}{6} + \frac{2b}{2} + \frac{3b}{3} = \frac{13}{6} b \cong 2.2b$$

The r.m.s. deviation ΔH is

$$\Delta H = \sqrt{\left\langle \left(H - \langle H \rangle \right)^2 \right\rangle} = \sqrt{\langle H^2 \rangle - \langle H \rangle^2},$$

so first we need to find the expectation value $\langle H^2 \rangle$:

$$\langle H^2 \rangle = \langle \psi | H^2 | \psi \rangle = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} b^2 & 0 & 0 \\ 0 & 4b^2 & 0 \\ 0 & 0 & 9b^2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{b^2}{\sqrt{6}} \\ \frac{4b^2}{\sqrt{2}} \\ \frac{9b^2}{\sqrt{3}} \end{pmatrix} = \frac{b^2}{6} + \frac{4b^2}{2} + \frac{9b^2}{3} = \frac{31}{6} b^2$$

Thus we get:

$$\Delta H = \sqrt{\langle H^2 \rangle - \langle H \rangle^2} = \sqrt{\tfrac{31}{6} b^2 - \left(\tfrac{13}{6} b \right)^2} = \tfrac{\sqrt{17}}{6} b \cong 0.7b .$$

c) To find the possible results of a measurement of A, we need to know its eigenvalues, so diagonalize A:

$$\begin{vmatrix} 3a-\lambda & 0 & 0 \\ 0 & -\lambda & 2ia \\ 0 & -2ia & -\lambda \end{vmatrix} = 0 \;\Rightarrow\; (3a-\lambda)\left(\lambda^2-4a^2\right)=0$$

$$\Rightarrow\; \lambda = 3a, 2a, -2a$$

Hence the possible results of the measurement of the observable A are

$$a_1 = 3a, \quad a_2 = 2a, \quad a_3 = -2a$$

d) We now need to know the eigenvector of the negative eigenvalue:

$$a\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = -2a\begin{pmatrix} u \\ v \\ w \end{pmatrix} \;\Rightarrow\; \begin{matrix} 3u=-2u \\ 2iw=-2v \\ -2iv=-2w \end{matrix} \;\Rightarrow\; v=-iw, u=0$$

$$|u|^2+|v|^2+|w|^2=1 \;\Rightarrow\; |v|^2+|w|^2=1 \;\Rightarrow\; u=0, v=\tfrac{1}{\sqrt{2}}, w=\tfrac{i}{\sqrt{2}} \;\Rightarrow\; |a_3\rangle \doteq \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$

The time-evolved state vector is

$$\begin{aligned} |\psi(t)\rangle &= \tfrac{1}{\sqrt{6}} e^{-i\frac{E_1 t}{\hbar}}|1\rangle + \tfrac{1}{\sqrt{2}} e^{-i\frac{E_2 t}{\hbar}}|2\rangle + \tfrac{1}{\sqrt{3}} e^{-i\frac{E_3 t}{\hbar}}|3\rangle \\ &= \tfrac{1}{\sqrt{6}} e^{-i\frac{bt}{\hbar}}|1\rangle + \tfrac{1}{\sqrt{2}} e^{-i\frac{2bt}{\hbar}}|2\rangle + \tfrac{1}{\sqrt{3}} e^{-i\frac{3bt}{\hbar}}|3\rangle \end{aligned} .$$

The probability of measuring the negative eigenvalue of A at time t_0 is

$$\mathcal{P}_{a_3} = \left|\langle a_3|\psi(t_0)\rangle\right|^2 = \left|\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{6}} e^{-i\frac{bt_0}{\hbar}} \\ \frac{1}{\sqrt{2}} e^{-i\frac{2bt_0}{\hbar}} \\ \frac{1}{\sqrt{3}} e^{-i\frac{3bt_0}{\hbar}} \end{pmatrix}\right|^2 = \left|\tfrac{1}{\sqrt{4}} e^{-i\frac{2bt_0}{\hbar}} - \tfrac{i}{\sqrt{6}} e^{-i\frac{3bt_0}{\hbar}}\right|^2$$

$$= \tfrac{1}{6}\left|\tfrac{\sqrt{3}}{\sqrt{2}} - ie^{-i\frac{bt_0}{\hbar}}\right|^2 = \tfrac{1}{6}\left(\tfrac{3}{2}+1-i\tfrac{\sqrt{3}}{\sqrt{2}}e^{-i\frac{bt_0}{\hbar}}+i\tfrac{\sqrt{3}}{\sqrt{2}}e^{+i\frac{bt_0}{\hbar}}\right) = \tfrac{1}{6}\left(\tfrac{5}{2}-\tfrac{2\sqrt{3}}{\sqrt{2}}\sin\left(\frac{bt_0}{\hbar}\right)\right)$$

$$= \tfrac{5}{12} - \tfrac{1}{\sqrt{6}}\sin\left(\frac{bt_0}{\hbar}\right)$$

e) The state vector immediately after the measurement of the negative eigenvalue is the corresponding eigenstate:

$$\left|\psi\left(t_0\right)\right\rangle = \left|a_3\right\rangle \doteq \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$

This state will time evolve to

$$\left|\psi\left(t > t_0\right)\right\rangle \doteq \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} e^{-i\frac{E_2\left(t-t_0\right)}{\hbar}} \\ \frac{i}{\sqrt{2}} e^{-i\frac{E_3\left(t-t_0\right)}{\hbar}} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} e^{-i\frac{2b\left(t-t_0\right)}{\hbar}} \\ \frac{i}{\sqrt{2}} e^{-i\frac{3b\left(t-t_0\right)}{\hbar}} \end{pmatrix}$$