

OSU PHYSICS DEPARTMENT
 COMPREHENSIVE EXAMINATION #132
 SOLUTIONS

Thursday, September 20 and Friday, September 21, 2018

Fall 2018 Comprehensive Examination

Statistical Mechanics	9 AM-12 PM	Thursday, September 20
Classical Mechanics	1 PM-4 PM	Thursday, September 20
Electricity and Magnetism	9 AM-12 PM	Friday, September 21
Quantum Mechanics	1 PM-4 PM	Friday, September 21

General Instructions

This Fall 2018 Comprehensive Examination consists of four separate parts of two problems each. Each problem carries equal weight (20 points each) and lasts three hours. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work on the provided pages, work each problem in its own labeled pages, and be certain that your chosen student letter (but not your name) is on the header of each page of your exam, including any unused pages. If you need additional paper for your work, use the blank pages provided. Each page of work should include the problem number, a page number, your chosen student letter, and the total number of pages actually used. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam and the collection of formulas distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please staple and return all pages of your exam—including unused pages—at the end of the exam.

Ortho- and para-hydrogen Consider a system with four energy eigenstates, three of which are degenerate. Let us call the non-degenerate ground state energy 0, and the triply degenerate excited state energy ε .

- (a) Find the entropy and internal energy as a function of temperature.
- (b) What values do the entropy and internal energy approach at high temperatures?
- (c) Explain in words why these high temperature values make sense.

Ortho- and para-hydrogen Consider a system with four energy eigenstates, three of which are degenerate. Let us call the non-degenerate ground state energy 0, and the triply degenerate excited state energy ε .

- (a) Find the entropy and internal energy as a function of temperature.

Solution:

There are two or three different approaches we can use to find the entropy and internal energy. I consider one of the simplest to be to find the Helmholtz free energy first (which is easy given the partition function) and then use that to find the entropy and finally the internal energy using standard thermodynamics. The free energy is given by:

$$F = -k_B T \ln Z \quad (1.1)$$

This requires us to remember that the partition function is given by a sum over all microstates of $e^{-\beta E_i}$ where $\beta = \frac{1}{kT}$, which means that in this case:

$$Z = e^{-\beta 0} + e^{-\beta \varepsilon} + e^{-\beta \varepsilon} + e^{-\beta \varepsilon} \quad (1.2)$$

$$= 1 + 3e^{-\beta \varepsilon} \quad (1.3)$$

Note that in the first case I summed over all four energy eigenstates. Then I grouped together those with the same energy. Thus the Helmholtz free energy is

$$F = -kT \ln(1 + 3e^{-\beta \varepsilon}) \quad (1.4)$$

Now we can remind ourselves that the Helmholtz free energy is defined as

$$F \equiv U - TS \quad (1.5)$$

which means once we solve for S we can easily find U (or vice versa). To find S we can remind ourselves of the total differential dF :

$$dF = dU - TdS - SdT \quad (1.6)$$

$$= TdS - pdV - TdS - SdT \quad (1.7)$$

$$= -SdT - pdV \quad (1.8)$$

This tells us that

$$S = - \left(\frac{\partial F}{\partial T} \right)_V \quad (1.9)$$

$$= k \ln(1 + 3e^{-\beta \varepsilon}) + kT \frac{3e^{-\beta \varepsilon}}{1 + 3e^{-\beta \varepsilon}} \left(-\varepsilon \frac{d\beta}{dT} \right) \quad (1.10)$$

$$= k \ln(1 + 3e^{-\beta \varepsilon}) + \frac{\varepsilon}{T} \frac{3e^{-\beta \varepsilon}}{1 + 3e^{-\beta \varepsilon}} \quad (1.11)$$

$$= k \ln(1 + 3e^{-\beta \varepsilon}) + \frac{\varepsilon}{T} \frac{3}{3 + e^{\beta \varepsilon}} \quad (1.12)$$

The answer need not be in this final form, but it can help to simplify a bit, to make later math easier.

To solve for U is now actually very straightforward:

$$F = U - TS \quad (1.13)$$

$$U = F + TS \quad (1.14)$$

$$= \cancel{-kT \ln(1 + 3e^{-\beta\varepsilon})} + T \left(\cancel{k \ln(1 + 3e^{-\beta\varepsilon})} + \frac{\varepsilon}{T} \frac{3}{3 + e^{\beta\varepsilon}} \right) \quad (1.15)$$

$$= \frac{3\varepsilon}{3 + e^{\beta\varepsilon}} \quad (1.16)$$

- (b) What values do the entropy and internal energy approach at high temperatures?

Solution:

I will start with the internal energy, since it looks easier. At high temperatures $\beta\varepsilon \ll 1$, which means that $e^{\beta\varepsilon} \approx 1$, thus

$$U = \frac{3\varepsilon}{3 + e^{\beta\varepsilon}} \xrightarrow{1} \quad (1.17)$$

$$\approx \frac{3\varepsilon}{3 + 1} \quad (1.18)$$

$$= \frac{3}{4}\varepsilon \quad (1.19)$$

The entropy actually works out very similar. The term that involves U has a T on the bottom and thus approaches zero.

$$S = k \ln \left(1 + 3e^{-\beta\varepsilon} \right) + \frac{\varepsilon}{T} \frac{3}{3 + e^{\beta\varepsilon}} \xrightarrow{0} \quad (1.20)$$

$$\approx k \ln 4 \quad (1.21)$$

- (c) Explain in words why these high temperature values make sense.

Solution:

Internal energy At sufficiently high temperatures, all four energy eigenstates will be equally probable. Thus the average value of the energy will be the average of the energy eigenvalues. Since three out of four eigenvalues are ε while the fourth eigenvalue is 0, the result is $\frac{3}{4}\varepsilon$.

Entropy Again, at high temperatures all eigenstates are equally probable. This suits perfectly the Boltzmann definition of entropy, which is $k \ln W$ where W is the number of microstates. Since we have four equally probable microstates, the entropy will thus be $k \ln 4$.

If instead we went with the more general Gibbs formulation of the entropy, the probability of each eigenstate is $\frac{1}{4}$ and we would find the entropy to be

$$S = -k \sum_i P_i \ln P_i \quad (1.22)$$

$$= -k \left(\frac{1}{4} \ln \frac{1}{4} + \frac{1}{4} \ln \frac{1}{4} + \frac{1}{4} \ln \frac{1}{4} + \frac{1}{4} \ln \frac{1}{4} \right) \quad (1.23)$$

$$= k \ln 4 \quad (1.24)$$

But I would expect you to use the Boltzmann approach.

Melting ice For this problem, please use only those properties of ice/water given in Table 2. For each part, you may give an approximate numerical answer (showing work), but a purely analytic answer is probably easier. We will define p_0 to be one atmosphere of pressure, and T_0 to be 0°C .

- If I melt a gram of ice into water (at pressure and temperature p_0 and T_0), how much work is done to the ice (which becomes water)? What is Q , the energy transferred to the ice by heating?
- What is the difference in entropy between ice and liquid water at this temperature and pressure? Which has higher entropy, liquid water or ice?
- Suppose you are curious as to whether we can adiabatically compress ice to melt it into water. Given the above information, and assuming that $C_V \approx C_p$ (which is not too bad an approximation for cold liquid water) and that the heat capacity is independent of temperature (perhaps a worse approximation) what would the temperature of liquid water need to be to have the same entropy as the original ice that was at temperature T_0 ?

property	symbol	value
one atmosphere pressure	p_0	10^5 Pa
temperature 0°C	T_0	273 K
latent heat of fusion	L	334 J
heat capacity, fixed pressure	$C_{p,\text{liq}}$	$4 \frac{\text{J}}{\text{K}}$
heat capacity, fixed pressure	$C_{p,\text{ice}}$	$2 \frac{\text{J}}{\text{K}}$
ice/water coexistence slope	$\frac{dp}{dT}$	$-1.35 \times 10^7 \frac{\text{Pa}}{\text{K}}$
volume of liquid water	V_{liq}	10^{-6} m^3
volume of ice	V_{ice}	$1.09 \times 10^{-6} \text{ m}^3$

Table 1: Interesting properties of one gram of water or ice around $p_0 = 1$ atmosphere of pressure and $T_0 = 0^\circ\text{C}$, in SI units.

Possibly useful equations:

$$C_p = T \left(\frac{\partial S}{\partial T} \right)_{p,N} \quad (2.1)$$

$$C_V = T \left(\frac{\partial S}{\partial T} \right)_{V,N} = \left(\frac{\partial U}{\partial T} \right)_{V,N} \quad (2.2)$$

Melting ice For this problem, please use only those properties of ice/water given in Table 2. For each part, you may give an approximate numerical answer (showing work), but a purely analytic answer is probably easier. We will define p_0 to be one atmosphere of pressure, and T_0 to be 0°C .

- (a) If I melt a gram of ice into water (at pressure and temperature p_0 and T_0), how much work is done to the ice (which becomes water)? What is Q , the energy transferred to the ice by heating?

Solution:

6 points To find the work, we can just use the change in volume, since the pressure is fixed and $W = -\int p dV$. The work on the ice is thus

$$W = -p_0(V_{\text{liq}} - V_{\text{ice}}) \quad (2.3)$$

Note that this is a positive value, since the liquid has smaller volume than the ice.

Finding the heat requires understanding the meaning of latent heat of fusion, which is the amount of heat needed to melt the material, so

$$Q = L = 334 \text{ J} \quad (2.4)$$

- (b) What is the difference in entropy between ice and liquid water at this temperature and pressure? Which has higher entropy, liquid water or ice?

Solution:

5 points The entropy difference can be found by integrating the heat over temperature (for a quasistatic process such as melting).

$$\Delta S = \int \frac{dQ}{T} \quad (2.5)$$

The integral is easy because the temperature is not changing.

$$\Delta S = \frac{L}{T_0} \quad (2.6)$$

$$= S_{\text{liq}} - S_{\text{ice}} \quad (2.7)$$

This means that the liquid has higher entropy, which makes sense from both microscopic and macroscopic perspectives. Ice is a crystal which you'd expect to have low entropy because it's highly ordered. You need to heat ice to turn it into water, which means that water must have the higher entropy.

- (c) Suppose you are curious as to whether we can adiabatically compress ice to melt it into water. Given the above information, and assuming that $C_V \approx C_p$ (which is not too bad an approximation for cold liquid water) and that the heat capacity is independent of temperature (perhaps a worse approximation) what would the temperature of liquid water need to be to have the same entropy as the original ice that was at temperature T_0 ?

Solution:

9 points We are looking for the temperature which I will call T_1 which is the temperature of liquid water such that it has the same entropy as ice at temperature T_0 . We already established that the liquid water at temperature T_0 has entropy that is L/T_0 higher than the entropy of ice. To make the liquid water have the same entropy as the ice, we will need to cool it down. That will require an integral since the temperature is changing.

$$S_{\text{liq}, T_1} = S_{\text{ice}} + \frac{L}{T_0} + \int_{T_0}^{T_1} \frac{dQ}{T} \quad (2.8)$$

$$= S_{\text{ice}} + \frac{L}{T_0} + \int_{T_0}^{T_1} \frac{C_p dT}{T} \quad (2.9)$$

$$= S_{\text{ice}} + \frac{L}{T_0} + C_p \ln\left(\frac{T_1}{T_0}\right) \quad (2.10)$$

In the second step there I used the assumption that $C_p = C_V$. For the two entropies to be equal as requested, we want:

$$\frac{L}{T_0} = -C_p \ln\left(\frac{T_1}{T_0}\right) \quad (2.11)$$

$$= C_p \quad (2.12)$$

$$\ln\left(\frac{T_0}{T_1}\right) = \frac{L}{C_p T_0} \quad (2.13)$$

$$\frac{T_0}{T_1} = e^{\frac{L}{C_p T_0}} \quad (2.14)$$

$$T_1 = T_0 e^{-\frac{L}{C_p T_0}} \quad (2.15)$$

$$\approx e^{-\frac{334}{4 \times 273}} \times 275 \text{ K} \quad (2.16)$$

$$\approx 275 \text{ K} \times e^{-\frac{1}{3}} \quad (2.17)$$

$$\approx 275 \text{ K} \times \left(1 - \frac{1}{3}\right) \quad (2.18)$$

$$\approx 200 \text{ K} \approx -100^\circ\text{C} \quad (2.19)$$

That's a crazy low temperature. Liquid water cannot coexist with solid ice (in equilibrium) below -20°C . There are numerous reasons why the old myth about ice skates being slippery due to the pressure melting the ice must be false.

Possibly useful equations:

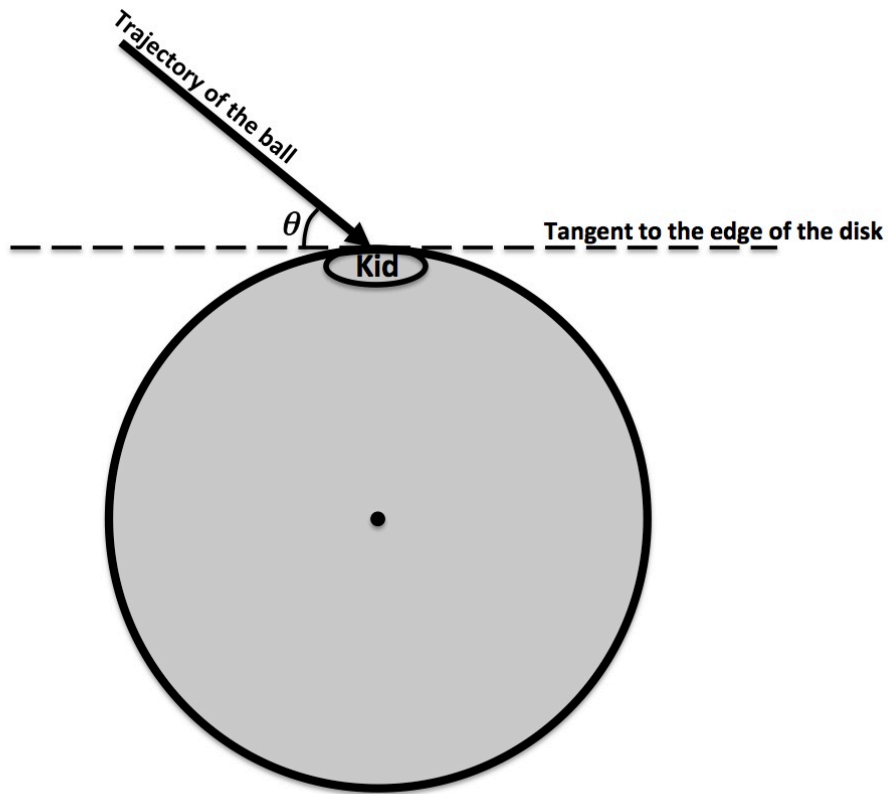
$$C_p = T \left(\frac{\partial S}{\partial T}\right)_{p,N} \quad (2.20)$$

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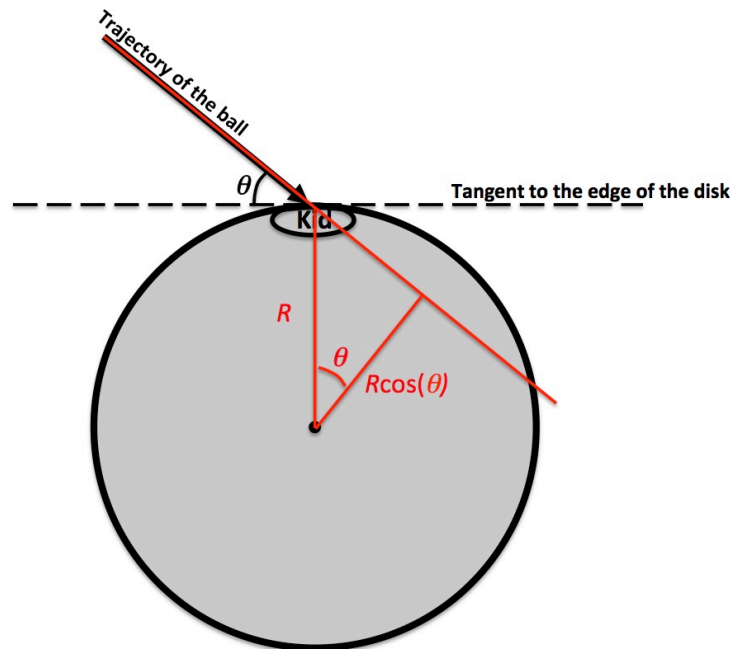
Table 2: Interesting properties of one gram of water or ice around $p_0 = 1$ atmosphere of pressure and $T_0 = 0^\circ\text{C}$, in SI units.

A kid of mass m_1 stands on the edge of a frictionless merry-go-round made of a uniform disk of mass m_2 and radius R that spins without friction around its center. The merry-go-round is initially at rest. A friend throws a ball of mass m_3 with velocity v that the kid on the merry-go-round catches in her arms. What is the angular velocity ω of the merry-go-round after the ball is caught, as a function of the angle θ of the ball trajectory (see figure)? Remember that the moment of inertia of a solid disk of mass m about its axis is $I=mr^2/2$.



Solution:

This problem can be solved by conservation of angular momentum with respect to the center of the merry-go-round. Linear momentum is not conserved because the pivot point can apply a force to the center of mass of the merry-go-round. To compute the angular momentum of the ball before it is caught, we use the geometry sketched below:



We find that the angular momentum of the system is (equal to the angular momentum of the ball):

$$q_1 = m_3 v R \cos \theta$$

The total angular momentum of the system after the ball is caught is the sum of the ball's angular momentum, the kid's angular momentum, and the merry-go-round angular momentum. We can write them as a function of the angular velocity as:

$$q_{ball} = m_3 \omega R^2$$

$$q_{kid} = m_1 \omega R^2$$

$$q_{merry-go-round} = \frac{m_2 \omega R^2}{2}$$

$$q_2 = q_{ball} + q_{kid} + q_{merry-go-round} = m_3 \omega R^2 + m_1 \omega R^2 + \frac{m_2 \omega R^2}{2} = \left(m_3 + m_1 + \frac{m_2}{2} \right) \omega R^2$$

Enforcing conservation of angular momentum yields:

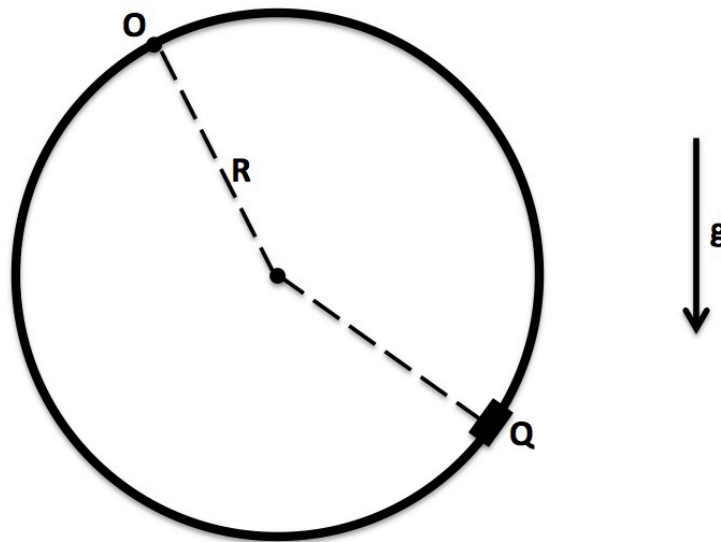
$$q_1 = q_2$$

$$m_3 v R \cos \theta = \left(m_3 + m_1 + \frac{m_2}{2} \right) \omega R^2$$

$$m_3 v \cos \theta = \left(m_3 + m_1 + \frac{m_2}{2} \right) \omega R$$

$$\omega = \frac{m_3 v}{\left(m_3 + m_1 + \frac{m_2}{2} \right) R} \cos \theta$$

A thin wire hoop of radius R and mass m oscillates in its own plane hanging from a single fixed point (point O in the figure). A bead of mass m (same as the hoop) is constrained to move on the hoop without friction (point Q in the figure). The moment of inertia of the hoop for rotation around a point on the hoop is $I=2mR^2$.



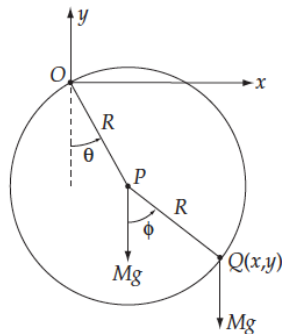
- a) Consider only small oscillations, and show that the eigenfrequencies are:

$$\omega_1 = \sqrt{\frac{2g}{R}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{g}{2R}}$$

- b) Find the eigenmodes associated to the two eigenfrequencies

Solution:Part a)

We consider the angles θ and ϕ in the figure as our generalized coordinates:



The coordinate transformation from the x and y position of the bead to the angles θ and ϕ is:

$$\begin{cases} x = R \sin \theta + R \sin \phi \\ y = -R \cos \theta - R \cos \phi \end{cases}$$

and their derivatives:

$$\begin{cases} \dot{x} = R(\dot{\theta} \cos \theta + \dot{\phi} \cos \phi) \\ \dot{y} = R(\dot{\theta} \sin \theta + \dot{\phi} \sin \phi) \end{cases}$$

we write the kinetic energy as:

$$\begin{aligned} T &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\theta}^2 = \\ &= \frac{1}{2} m R^2 (\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi} \cos(\theta - \phi)) + m R^2 \dot{\theta}^2 = \\ &\equiv \frac{1}{2} m R^2 (3\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}) \end{aligned}$$

Where we have made the small angle approximation $\cos(\theta - \phi) \approx 1$ between the second and third line. The potential energy is the sum of the potential energy of the hoop and of the bead and can be written, for small angles:

$$\begin{aligned}
 U &= U_{hoop} + U_{bead} = \\
 &= -mgR\cos\theta - mgR(\cos\theta + \cos\phi) = \\
 &= -mgR[2\cos\theta + \cos\phi] \cong \\
 &= -mgR\left[3 - \theta^2 - \frac{\phi^2}{2}\right] \\
 U &= mgR\left(\theta^2 + \frac{\phi^2}{2}\right)
 \end{aligned}$$

Here we have made the small angle approximation $\cos\theta \approx 1 - \theta^2/2$ and $\cos\phi \approx 1 - \phi^2/2$ (between lines 2 and 3) and eventually neglected the constant value $-3mgR$.

We write the Lagrangian:

$$L = T - U = \frac{1}{2}mR^2(3\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}) - mgR\left(\theta^2 + \frac{\phi^2}{2}\right)$$

The equation of motion for θ reads:

$$\begin{aligned}
 \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} &= \frac{\partial L}{\partial \theta} \\
 \frac{d}{dt} (MR^2(3\dot{\theta} + \dot{\phi})) &= -2mgR\theta \\
 R(3\ddot{\theta} + \ddot{\phi}) &= -2g\theta
 \end{aligned}$$

The equation of motion for ϕ reads

$$\begin{aligned}
 \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} &= \frac{\partial L}{\partial \phi} \\
 \frac{d}{dt} (mR^2(\dot{\phi} + \dot{\theta})) &= -mgR\phi \\
 R(\ddot{\phi} + \ddot{\theta}) &= -g\phi
 \end{aligned}$$

We notice that the two equations can be written with the formalism

$$M\ddot{\bar{x}} = -K\bar{x} \quad (1)$$

with

$$\bar{x} = \begin{bmatrix} \theta \\ \phi \end{bmatrix}, \quad M = \begin{bmatrix} 3R & R \\ R & R \end{bmatrix}, \quad \text{and} \quad K = \begin{bmatrix} 2g & 0 \\ 0 & g \end{bmatrix}$$

The eigenfrequencies are therefore found by:

$$\det(K - \omega^2 M) = 0$$

$$\det \begin{bmatrix} 2g - 3\omega^2 R & -\omega^2 R \\ -\omega^2 R & g - \omega^2 R \end{bmatrix} = 0$$

$$(2g - 3\omega^2 R)(g - \omega^2 R) - \omega^4 R^2 = 0$$

$$2g^2 - 2g\omega^2 R - 3g\omega^2 R + 3\omega^4 R^2 - \omega^4 R^2 = 0$$

$$2\omega^4 R^2 - 5g\omega^2 R + 2g^2 = 0$$

$$\omega^2 = \frac{5gR \pm \sqrt{25g^2 R^2 - 16g^2 R^2}}{4R^2} = \frac{5g \pm g\sqrt{25-16}}{4R} = \frac{5g \pm 3g}{4R} = \begin{cases} \frac{2g}{R} \\ \frac{g}{2R} \end{cases}$$

Part b)

To find the eigenmodes we substitute the eigenfrequencies one by one in the equation $(K - \omega^2 M)\vec{x} = 0$. Starting from the first solution:

$$\begin{aligned} (K - \omega^2 M)\vec{x} &= 0 \\ \left(\begin{bmatrix} 2g & 0 \\ 0 & g \end{bmatrix} - \frac{2g}{R} \begin{bmatrix} 3R & R \\ R & R \end{bmatrix} \right) \vec{x} &= 0 \\ \begin{bmatrix} 2g - 6g & -2g \\ -2g & -g \end{bmatrix} \vec{x} &= 0 \\ \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} g\vec{x} &= 0 \\ 2\theta + \phi &= 0 \\ \theta &= -\frac{\phi}{2} \end{aligned}$$

Analogously for the second eigenvalue:

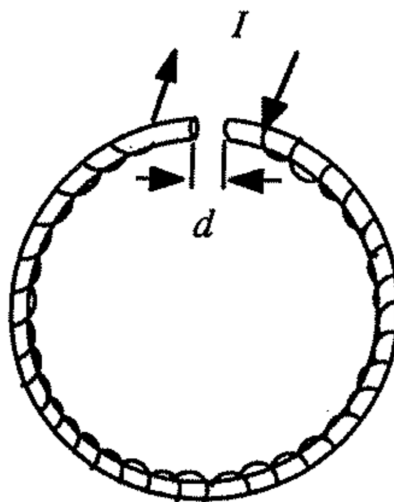
$$\begin{aligned} (K - \omega^2 M)\vec{x} &= 0 \\ \left(\begin{bmatrix} 2g & 0 \\ 0 & g \end{bmatrix} - \frac{g}{2R} \begin{bmatrix} 3R & R \\ R & R \end{bmatrix} \right) \vec{x} &= 0 \\ \begin{bmatrix} 2g - \frac{3}{2}g & -\frac{g}{2} \\ -\frac{g}{2} & \frac{g}{2} \end{bmatrix} \vec{x} &= 0 \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{g}{2} \vec{x} &= 0 \\ \theta - \phi &= 0 \\ \theta &= \phi \end{aligned}$$

After normalization, the two eigenvectors can be written as:

$$x_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 2 \\ -\frac{1}{\sqrt{5}} \end{bmatrix}, \omega_1 = \sqrt{\frac{2g}{R}}$$

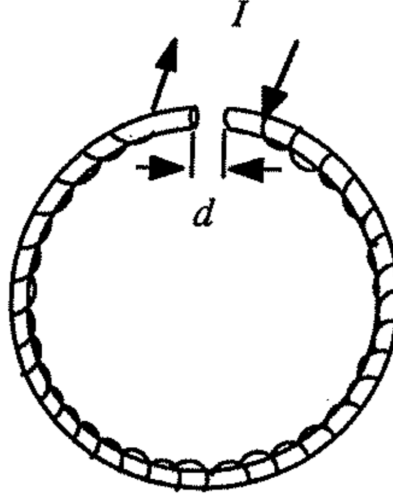
$$x_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \omega_2 = \sqrt{\frac{g}{2R}}$$

A ferromagnetic rod of length L and negligible diameter is bent into a circular shape leaving a small gap of width d between the ends of the rod. A current carrying coil (current I) is wrapped around the rod for a total of N turns as shown in the figure. Assume $\mu_r \gg 1$ for the relative permeability of the ferromagnetic rod and that $d \ll L$.



- Find the magnetic field \mathbf{H} and the magnetic induction \mathbf{B} in the gap and in the material of the rod.
- Calculate all fields from part a) in the limit $\mu_r \rightarrow \infty$ and verify that $\mathbf{H} \rightarrow \mathbf{0}$ in the rod in this limit.

A ferromagnetic rod of length L and negligible diameter is bent into a circular shape leaving a small gap of width d between the ends of the rod. A current carrying coil (current I) is wrapped around the rod for a total of N turns as shown in the figure. Assume $\mu_r \gg 1$ for the relative permeability of the ferromagnetic rod and that $d \ll L$.



- (a) Find the magnetic field \mathbf{H} and the magnetic induction \mathbf{B} in the gap and in the material of the rod.

Solution:

$$\oint \vec{H} \cdot d\vec{l} = I_{\text{total}}, \quad (5.1)$$

with $I_{\text{total}} = NI$ being the total enclosed free current. Label the fields in the gap region and in the rod as H_G, B_G and H_R, B_R respectively:

$$H_R L + H_G d = NI. \quad (5.2)$$

The normal component of \vec{B} is continuous at the interface and together with the constituent relations in the gap, $B_G = \mu_0 H_G$, and in the rod, $B_R = \mu_r \mu_0 H_R$, we obtain

$$H_R = \frac{NI}{L + \mu_r d}, \quad (5.3)$$

$$H_G = \mu_r H_R = \mu_r \frac{NI}{L + \mu_r d}, \quad (5.4)$$

$$B_G = \mu_0 H_G = \mu_0 \mu_r \frac{NI}{L + \mu_r d}, \quad (5.5)$$

$$B_R = B_G = \mu_0 \mu_r \frac{NI}{L + \mu_r d}. \quad (5.6)$$

- (b) Calculate all fields from part a) in the limit $\mu_r \rightarrow \infty$ and verify that $\mathbf{H} \rightarrow \mathbf{0}$ in the rod in this limit.

Solution:

Clearly, from (3)-(6)

$$\lim_{\mu_r \rightarrow \infty} H_R = 0 \quad \text{and} \quad \lim_{\mu_r \rightarrow \infty} H_G = \frac{NI}{d}, \text{ etc.}$$

Consider a hollow conducting sphere of radius a and negligible thickness that is grounded. A point charge Q is placed inside the sphere at a distance b ($b < a$) from the center of the sphere.

Calculate the force between this point charge and the sphere.

Solution:

This problem can be solved using method of images. By placing a charge Q' outside the sphere a distance c from the center of the sphere such that the conducting sphere is equipotential. Symmetry dictates that the center of the sphere, the actual charge Q and the image charge Q' lie on a straight line. To determine the 2 unknowns (Q' and c), it is sufficient to consider 2 points on the sphere and require $V = 0$ for both points. We can choose these 2 points to lie on the axis defined by the 2 charges. $V=0$ yields 2 conditions:

$$\frac{Q}{a-b} = \frac{Q'}{c-a}$$

and

$$\frac{Q}{a+b} = \frac{Q'}{c+a},$$

or

$$c = \frac{a^2}{b}$$

and

$$Q' = -Q \frac{a}{b}.$$

We can verify that $V = 0$ for a general point on the sphere ($r = a, \phi, \theta$):

$$V(r = a, \phi, \theta) = \frac{Q}{4\pi\epsilon_0 d_1} + \frac{Q'}{4\pi\epsilon_0 d_2},$$

where $d_1^2 = a^2 + b^2 - 2ab \cos \theta$ and $d_2^2 = a^2 + c^2 - 2ac \cos \theta$. $V = 0$ requires $Q'/Q = -d_2/d_1$. Using our first condition

$$c = \frac{a^2}{b}$$

we get $d_1/d_2 = b/a$ and

$$Q'/Q = -d_2/d_1 = -a/b,$$

which is our second condition.

Finally, the electric field inside the sphere due to the charge Q and the surface charge on the sphere is identical to the field due to the charge Q and the image charge Q' . The force between the charge and sphere is attractive and its magnitude is

$$|F| = \frac{|QQ'|}{4\pi\epsilon_0(c-b)^2}.$$

Consider a quantum mechanical harmonic oscillator. The zeroth-order Hamiltonian is

$$H_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2,$$

where ω is the natural frequency of the system. Now consider adding a perturbation to the Hamiltonian of the form

$$H' = \varepsilon \frac{1}{2}m\omega^2\hat{x}^2,$$

where ε is a small dimensionless term parameterizing the strength of the perturbation.

- a) Use perturbation theory to find the first-order correction to the energy of each level.
- b) Use perturbation theory to find the second-order correction to the energy of each level.
- c) For this perturbation, the exact energy levels can be found by simplifying the total Hamiltonian and relating it to a problem for which you know the solution. Find the exact answer and compare it to the above perturbation results to the same order in the perturbation parameter ε .

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a) The energy levels of the harmonic oscillator are nondegenerate, so we use nondegenerate perturbation theory. The first-order correction to the energy is the expectation value

$$E_n^{(1)} = \langle n^{(0)} | H' | n^{(0)} \rangle$$

To calculate this, it is most convenient to express the perturbation Hamiltonian using ladder operators

$$H' = \varepsilon \frac{1}{2}m\omega^2 \left(\frac{\hbar}{2m\omega} \right) (a^\dagger + a)^2$$

$$H' = \varepsilon \frac{1}{4}\hbar\omega (a^\dagger a^\dagger + a^\dagger a + a a^\dagger + a a)$$

The expectation value of the perturbation is

$$E_n^{(1)} = \varepsilon \frac{1}{4}\hbar\omega \langle n^{(0)} | (a^\dagger a^\dagger + a^\dagger a + a a^\dagger + a a) | n^{(0)} \rangle$$

The operators $a^\dagger a^\dagger$ and $a a$ contribute zero because they raise or lower the state $|n^{(0)}\rangle$ twice and produce a new state that is orthogonal to $|n^{(0)}\rangle$. The remaining terms are calculated using $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$:

$$\begin{aligned}
E_n^{(1)} &= \varepsilon \frac{1}{4} \hbar \omega \langle n^{(0)} | (a^\dagger a + a a^\dagger) | n^{(0)} \rangle \\
&= \varepsilon \frac{1}{4} \hbar \omega \langle n^{(0)} | (\sqrt{n} \sqrt{n} + \sqrt{n+1} \sqrt{n+1}) | n^{(0)} \rangle \\
&= \varepsilon \frac{1}{4} \hbar \omega (n + n + 1) \\
&= \varepsilon \frac{1}{2} \hbar \omega \left(n + \frac{1}{2} \right)
\end{aligned}$$

The resultant energy of level n to first order in the perturbation is

$$\begin{aligned}
E_n &= E_n^{(0)} + E_n^{(1)} \\
&= \hbar \omega \left(n + \frac{1}{2} \right) + \varepsilon \frac{1}{2} \hbar \omega \left(n + \frac{1}{2} \right) \\
&= \hbar \omega \left(n + \frac{1}{2} \right) \left(1 + \frac{\varepsilon}{2} \right)
\end{aligned}$$

Each state is shifted upwards, with the shift larger for larger states.

b) Now consider the second-order energy correction

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle n^{(0)} | H' | m^{(0)} \rangle|^2}{E_n^{(0)} - E_m^{(0)}}.$$

This looks like an infinite sum, which would be problematic, but we plow ahead and find that the sum is reduced for the harmonic oscillator case. The matrix elements are

$$\begin{aligned}
\langle n^{(0)} | H' | m^{(0)} \rangle &= \varepsilon \frac{1}{4} \hbar \omega \langle n^{(0)} | (a^\dagger a^\dagger + a^\dagger a + a a^\dagger + a a) | m^{(0)} \rangle \\
&= \varepsilon \frac{1}{4} \hbar \omega \left[\begin{array}{l} \sqrt{m+1} \sqrt{m+2} \delta_{n,m+2} + \sqrt{m} \sqrt{m} \delta_{n,m} \\ + \sqrt{m+1} \sqrt{m+1} \delta_{n,m} + \sqrt{m} \sqrt{m-1} \delta_{n,m-2} \end{array} \right]
\end{aligned}$$

For a given energy level n , only two terms in the sum ($m \neq n$) contribute, yielding

$$\begin{aligned}
E_n^{(2)} &= \frac{\left[\frac{\varepsilon}{4} \hbar \omega \sqrt{n-1} \sqrt{n} \right]^2}{E_n^{(0)} - E_{n-2}^{(0)}} + \frac{\left[\frac{\varepsilon}{4} \hbar \omega \sqrt{n+2} \sqrt{n+1} \right]^2}{E_n^{(0)} - E_{n+2}^{(0)}} \\
&= \left(\frac{\varepsilon}{4} \hbar \omega \right)^2 \left[\frac{n(n-1)}{2\hbar\omega} + \frac{(n+1)(n+2)}{-2\hbar\omega} \right] \\
&= \frac{\varepsilon^2}{32} \hbar \omega \left[n^2 - n - (n^2 + 3n + 2) \right] \\
&= -\frac{\varepsilon^2}{8} \hbar \omega \left(n + \frac{1}{2} \right)
\end{aligned}$$

Note that the second-order contribution is negative. Only the two levels $m = n + 2$ and $m = n - 2$ contribute to the energy correction. They each have the same magnitude energy denominators, but the matrix element is larger for the $m = n + 2$ state above the state of interest, so the level is pushed down. The resultant energy of level n to second order in the perturbation is

$$\begin{aligned} E_n &= E_n^{(0)} + E_n^{(1)} + E_n^{(2)} \\ &= \hbar\omega \left(n + \frac{1}{2} \right) \left(1 + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} \right) \end{aligned}$$

c) The exact Hamiltonian is

$$\begin{aligned} H &= H_0 + H' \\ &= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \varepsilon \frac{1}{2}m\omega^2 x^2 \\ &= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 (1 + \varepsilon) \\ &= \frac{p^2}{2m} + \frac{1}{2}m\omega_p^2 x^2 \end{aligned}$$

where we have defined a new perturbed harmonic frequency

$$\omega_p = \omega\sqrt{1 + \varepsilon}$$

This new Hamiltonian has the same form as the original harmonic oscillator problem we have already solved, but with a new characteristic frequency. Hence, we know the energy eigenvalues exactly. They are

$$E_n = \left(n + \frac{1}{2} \right) \hbar\omega_p = \left(n + \frac{1}{2} \right) \hbar\omega\sqrt{1 + \varepsilon}$$

The perturbation theory result above was obtained to second order in the perturbation parameter ε , so we must compare it to the exact result at this same order. Expanding the exact result in powers of ε gives

$$\begin{aligned} E_n &= \left(n + \frac{1}{2} \right) \hbar\omega (1 + \varepsilon)^{1/2} \\ &= \left(n + \frac{1}{2} \right) \hbar\omega \left(1 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \dots \right) \end{aligned}$$

Thus we see that the two results agree, at least to second order.

Consider a spin-1/2 particle with a magnetic moment

$$\boldsymbol{\mu} = -\frac{e}{m_e} \mathbf{S}$$

- a) At time $t = 0$, the observable S_x is measured, with the result $+\hbar/2$. What is the state vector $|\Psi(t=0)\rangle$ immediately after the measurement?
- b) Immediately after the measurement, a magnetic field $\vec{\mathbf{B}} = B_0 \hat{\mathbf{z}}$ is applied and the particle is allowed to evolve for a time T . What is the state vector $|\Psi(t=T)\rangle$ of the system at time $t = T$?
- c) At $t = T$, the magnetic field is very rapidly changed to $\vec{\mathbf{B}} = B_0 \hat{\mathbf{y}}$. After another time interval T , a measurement of S_x is carried out once more. What is the probability that a value $+\hbar/2$ is found?

Consider a spin-1/2 particle with a magnetic moment

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- At time $t = 0$, the observable S_x is measured, with the result $\hbar/2$. What is the state vector $|\psi(t=0)\rangle$ immediately after the measurement?
- Immediately after the measurement, a magnetic field $\vec{\mathbf{B}} = B_0 \hat{\mathbf{z}}$ is applied and the particle is allowed to evolve for a time T . What is the state of the system at time $t = T$?
- At $t = T$, the magnetic field is very rapidly changed to $\vec{\mathbf{B}} = B_0 \hat{\mathbf{y}}$. After another time interval T , a measurement of S_x is carried out once more. What is the probability that a value $\hbar/2$ is found?

(a) The measurement collapses the state to

$$|\psi(0)\rangle = |+\rangle_x$$

(b) In a field aligned along the z -axis, the energy eigenstates are $|\pm\rangle$ and the energy eigenvalues are $\pm\hbar\omega_0/2$ with $\omega_0 = eB_0/m_e$. The initial state vector written in the energy basis is

$$\begin{aligned} |\psi(0)\rangle &= |+\rangle_x = (|+\rangle \langle +| + |-\rangle \langle -|)|+\rangle_x = (\langle +|+\rangle_x)|+\rangle + (\langle -|+\rangle_x)|-\rangle \\ &= \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle \end{aligned}$$

The time evolved state is

$$|\psi(T)\rangle = \frac{1}{\sqrt{2}} e^{-iE_+T/\hbar} |+\rangle + \frac{1}{\sqrt{2}} e^{-iE_-T/\hbar} |-\rangle = \frac{1}{\sqrt{2}} e^{-i\omega_0 T/2} |+\rangle + \frac{1}{\sqrt{2}} e^{+i\omega_0 T/2} |-\rangle$$

(c) In this new field the energy eigenstates are $|\pm\rangle_y$ and the energy eigenvalues are $\pm\hbar\omega_0/2$ with $\omega_0 = eB_0/m_e$. We start the clock over for this new time evolution. The initial state vector written in this new energy basis is

$$\begin{aligned} |\psi(0)\rangle &= \frac{1}{\sqrt{2}} e^{-i\omega_0 T/2} |+\rangle + \frac{1}{\sqrt{2}} e^{+i\omega_0 T/2} |-\rangle = (|+\rangle_y \langle +| + |-\rangle_y \langle -|) \left(\frac{1}{\sqrt{2}} e^{-i\omega_0 T/2} |+\rangle + \frac{1}{\sqrt{2}} e^{+i\omega_0 T/2} |-\rangle \right) \\ &= \left(\frac{1}{\sqrt{2}} e^{-i\omega_0 T/2} \langle +|+\rangle_y + \frac{1}{\sqrt{2}} e^{+i\omega_0 T/2} \langle +|-\rangle_y \right) |+\rangle_y + \left(\frac{1}{\sqrt{2}} e^{-i\omega_0 T/2} \langle -|+\rangle_y + \frac{1}{\sqrt{2}} e^{+i\omega_0 T/2} \langle -|-\rangle_y \right) |-\rangle_y \\ &= \frac{1}{2} \left(e^{-i\omega_0 T/2} - i e^{+i\omega_0 T/2} \right) |+\rangle_y + \frac{1}{2} \left(e^{-i\omega_0 T/2} + i e^{+i\omega_0 T/2} \right) |-\rangle_y \end{aligned}$$

The time evolution of this state is

$$\begin{aligned}
|\psi(T)\rangle &= \frac{1}{2} \begin{pmatrix} e^{-i\omega_0 T/2} & -i e^{+i\omega_0 T/2} \end{pmatrix} e^{-iE_+ t/\hbar} |+\rangle_y + \frac{1}{2} \begin{pmatrix} e^{-i\omega_0 T/2} & +i e^{+i\omega_0 T/2} \end{pmatrix} e^{-iE_- t/\hbar} |-\rangle_y \\
&= \frac{1}{2} \begin{pmatrix} e^{-i\omega_0 T/2} & -i e^{+i\omega_0 T/2} \end{pmatrix} e^{-i\omega_0 T/2} |+\rangle_y + \frac{1}{2} \begin{pmatrix} e^{-i\omega_0 T/2} & +i e^{+i\omega_0 T/2} \end{pmatrix} e^{+i\omega_0 T/2} |-\rangle_y \\
&= \frac{1}{2} \begin{pmatrix} e^{-i\omega_0 T} & -i \end{pmatrix} |+\rangle_y + \frac{1}{2} \begin{pmatrix} 1 & +i e^{+i\omega_0 T} \end{pmatrix} |-\rangle_y
\end{aligned}$$

The probability of measuring S_x to be $+\hbar/2$ is

$$\mathcal{P}_{+x} = \left| \langle + | \psi(T) \rangle \right|^2 = \left| \langle + | \left(\frac{1}{2} \begin{pmatrix} e^{-i\omega_0 T} & -i \end{pmatrix} |+\rangle_y + \frac{1}{2} \begin{pmatrix} 1 & +i e^{+i\omega_0 T} \end{pmatrix} |-\rangle_y \right) \right|^2$$

Using matrix notation, we have

$$\begin{aligned}
\mathcal{P}_{+x} &= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \left\{ \frac{1}{2} \begin{pmatrix} e^{-i\omega_0 T} & -i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & +i e^{+i\omega_0 T} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\} \right|^2 \\
&= \frac{1}{16} \left| \begin{pmatrix} e^{-i\omega_0 T} & -i \end{pmatrix} \begin{pmatrix} 1+i \\ 1-i \end{pmatrix} + \begin{pmatrix} 1 & +i e^{+i\omega_0 T} \end{pmatrix} \begin{pmatrix} 1-i \\ 1+i \end{pmatrix} \right|^2 \\
&= \frac{1}{16} \left| e^{-i\omega_0 T} + i e^{-i\omega_0 T} - i + 1 + 1 - i + i e^{+i\omega_0 T} + e^{+i\omega_0 T} \right|^2 \\
&= \frac{1}{4} \left| 1 - i + \cos \omega_0 T + i \cos \omega_0 T \right|^2 = \frac{1}{2} \left(1 + \cos^2 \omega_0 T \right) = \frac{1}{4} \left(3 + \cos 2\omega_0 T \right)
\end{aligned}$$