

OSU PHYSICS DEPARTMENT
 COMPREHENSIVE EXAMINATION #128
 SOLUTIONS

Monday, April 3 and Tuesday, April 4, 2017

Spring 2017 Comprehensive Examination

Quantum Mechanics	9 AM-12 PM	Monday, April 3
Statistical Mechanics	1 PM-4 PM	Monday, April 3
Classical Mechanics	9 AM-12 PM	Tuesday, April 4
Electricity and Magnetism	1 PM-4 PM	Tuesday, April 4

General Instructions

This Spring 2017 Comprehensive Examination consists of four separate parts of two problems each. Each problem carries equal weight (20 points each) and lasts three hours. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work on the provided pages, work each problem in its own labeled pages, and be certain that your chosen student letter (but not your name) is on the header of each page of your exam, including any unused pages. If you need additional paper for your work, use the blank pages provided. Each page of work should include the problem number, a page number, your chosen student letter, and the total number of pages actually used. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam and the collection of formulas distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please staple and return all pages of your exam—including unused pages—at the end of the exam.

The first few eigenstates of the hydrogenic atom (an atom with a single electron) are given by

$$\psi_{1s}(\vec{r}) = \frac{Z^{\frac{3}{2}}}{\sqrt{\pi a_0^3}} e^{-\frac{Zr}{a_0}} \quad (1.1)$$

$$\psi_{2s}(\vec{r}) = \frac{2a_0 - Zr}{4\sqrt{2\pi a_0^5}} Z^{\frac{3}{2}} e^{-\frac{Zr}{2a_0}} \quad (1.2)$$

$$\psi_{2p_x}(\vec{r}) = \frac{x}{4\sqrt{2\pi a_0^5}} Z^{\frac{5}{2}} e^{-\frac{Zr}{2a_0}} \quad (1.3)$$

$$\psi_{2p_y}(\vec{r}) = \frac{y}{4\sqrt{2\pi a_0^5}} Z^{\frac{5}{2}} e^{-\frac{Zr}{2a_0}} \quad (1.4)$$

$$\psi_{2p_z}(\vec{r}) = \frac{z}{4\sqrt{2\pi a_0^5}} Z^{\frac{5}{2}} e^{-\frac{Zr}{2a_0}} \quad (1.5)$$

where Z is the nuclear charge and a_0 is the Bohr radius. Note that the $2p_x$ and $2p_y$ eigenstates are convenient linear combinations of the $\ell = 1$, $m = \pm 1$ eigenstates.

Consider a tritium atom (a hydrogen atom with a nucleus consisting of one proton and two neutrons) initially in its ground state. This isotope of hydrogen spontaneously turns into a Helium ion via β decay. The electron produced via the decay has around 6 keV of kinetic energy, and thus exits the picture rapidly. Assume that the only effect of the decay is to change the nuclear charge instantaneously from $Z = 1$ to $Z = 2$.

What is the probability that the resulting Helium ion will be in *its* ground state after the β decay?

You may find useful the following integral:

$$\int_0^\infty x^n e^{-x} dx = n!$$

This is an abrupt transition, which means the wave function will not have any time to change, although the potential does change. So the probability of ending up in the state $|1s\rangle_{\text{He}}$ will be given by the square of the projection of the initial state onto this final state:

$$P = |\langle 1s|_{\text{He}}|1s\rangle_{\text{H}}|^2 \quad (1.6)$$

$$= \left| \int \phi_{1s\text{He}}^*(\vec{r}) \phi_{1s\text{H}}(\vec{r}) d^3r \right|^2 \quad (1.7)$$

$$= \left| \int_0^\infty \frac{2^{\frac{3}{2}}}{\sqrt{\pi a_0^3}} e^{-\frac{2r}{a_0}} \frac{1^{\frac{3}{2}}}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}} 4\pi r^2 dr \right|^2 \quad (1.8)$$

$$= \left| \frac{8\sqrt{2}}{a_0^3} \int_0^\infty e^{-\frac{2r}{a_0}} e^{-\frac{r}{a_0}} r^2 dr \right|^2 \quad (1.9)$$

$$= \left| \frac{8\sqrt{2}}{a_0^3} \int_0^\infty e^{-\frac{3r}{a_0}} r^2 dr \right|^2 \quad (1.10)$$

$$u = \frac{3r}{a_0} \quad (1.11)$$

$$du = \frac{3}{a_0} dr \quad (1.12)$$

$$P = \left| \frac{8\sqrt{2}}{27} \int_0^\infty e^{-u} u^2 du \right|^2 P = \left| \frac{8\sqrt{2}}{27} 2! \right|^2 \quad (1.13)$$

$$= \left(\frac{16\sqrt{2}}{27} \right)^2 \quad (1.14)$$

$$\approx 70\% \quad (1.15)$$

Consider a two-dimensional harmonic oscillator with the Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2) \quad (2.1)$$

- a) Write down the energy of this system in terms of appropriate quantum numbers.
- b) Now add to this system an anharmonic perturbation

$$V_1 = \epsilon \frac{m^2 \omega^3}{\hbar} (x^2 + y^2)^2 \quad (2.2)$$

where $\epsilon > 0$ and $\epsilon \ll 1$. Solve for the first-order correction to the energies of the three lowest-energy eigenstates.

Consider a two-dimensional harmonic oscillator with the Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2) \quad (2.3)$$

- a) Write down the energy of this system in terms of appropriate quantum numbers.

Solution:

This is a standard 2D harmonic oscillator. The problem separates in the x and y coordinates, but you aren't expected to show or prove this when asked simply to "write down" the energies. The energies are just the sum of the energies for the two (separable) coordinates:

$$E_{n_x, n_y} = \hbar\omega \left(n_x + \frac{1}{2} \right) + \hbar\omega \left(n_y + \frac{1}{2} \right) \quad (2.4)$$

$$= \hbar\omega (n_x + n_y + 1) \quad (2.5)$$

- b) Now add to this system an anharmonic perturbation

$$V_1 = \epsilon \frac{m^2\omega^3}{\hbar} (x^2 + y^2)^2 \quad (2.6)$$

where $\epsilon > 0$ and $\epsilon \ll 1$. Solve for the first-order correction to the energies of the three lowest-energy eigenstates.

Solution:

At this point we should recognize that the three lowest-energy eigenstates are $|00\rangle$, $|01\rangle$, and $|10\rangle$. The latter two are degenerate, so we will need to consider degenerate perturbation theory for them. This means we are going to need a lot of matrix elements of the perturbation, and hopefully you are already thinking that raising and lowering operators will be helpful here. Doing all these integrals by hand could be painful.

Fortunately, the formula sheet gives us all we need:

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{p}{\sqrt{2m\omega\hbar}} \quad (2.7)$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{p}{\sqrt{2m\omega\hbar}} \quad (2.8)$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad (2.9)$$

$$a|n\rangle = \sqrt{n}|n-1\rangle \quad (2.10)$$

$$(2.11)$$

We start by solving for x , and making explicit that we now have raising and lowering operators for each direction.

$$a_x + a_x^\dagger = \sqrt{\frac{2m\omega}{\hbar}}x \quad (2.12)$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_x + a_x^\dagger) \quad (2.13)$$

$$y = \sqrt{\frac{\hbar}{2m\omega}}(a_y + a_y^\dagger) \quad (2.14)$$

$$(2.15)$$

Now let us look at an expansion of V_1 . The form given in the problem helps us to see that this has rotational symmetry, but right now separating x and y is appealing.

$$V_1 = \epsilon \frac{m^2\omega^3}{\hbar} (x^4 + 2x^2y^2 + y^4) \quad (2.16)$$

Let's start with the ground state, looking at just one of the three terms at a time:

$$\frac{m^2\omega^3}{\hbar} \langle 00|x^4|00\rangle = \frac{\hbar\omega}{4} \langle 00|(a_x + a_x^\dagger)^4|00\rangle \quad (2.17)$$

$$= \frac{\hbar\omega}{4} \langle 00|(a_x + a_x^\dagger)^3(a_x + a_x^\dagger)|00\rangle \quad (2.18)$$

$$= \frac{\hbar\omega}{4} \langle 00|(a_x + a_x^\dagger)^3|10\rangle \quad (2.19)$$

$$= \frac{\hbar\omega}{4} \langle 10|(a_x + a_x^\dagger)^2|10\rangle \quad (2.20)$$

$$= \frac{\hbar\omega}{4} \langle 10|(a_x + a_x^\dagger)(a_x + a_x^\dagger)|10\rangle \quad (2.21)$$

$$= \frac{\hbar\omega}{4} (\sqrt{2}\langle 20| + \langle 10|) (\sqrt{2}|20\rangle + |10\rangle) \quad (2.22)$$

$$= \frac{\hbar\omega}{4} (2 + 1) \quad (2.23)$$

$$= \frac{3}{4}\hbar\omega \quad (2.24)$$

$$\frac{m^2\omega^3}{\hbar} \langle 00|y^4|00\rangle = \frac{3}{4}\hbar\omega \quad (2.25)$$

Just one matrix element left and we'll have the ground state shift.

$$\langle 00|x^2y^2|00\rangle = \langle 0|x^2|0\rangle \langle 0|y^2|0\rangle \quad (2.26)$$

by which I mean that we can do the x^2y^2 matrix element as two separate matrix elements in each of the two directions.

$$\langle 0|x^2|0\rangle = \frac{\hbar}{2m\omega} (\langle 0|a + a^\dagger)(a + a^\dagger|0\rangle) \quad (2.27)$$

$$= \frac{\hbar}{2m\omega} \langle 1|1\rangle \quad (2.28)$$

$$= \frac{\hbar}{2m\omega} \quad (2.29)$$

$$\frac{m^2\omega^3}{\hbar}\langle 00|x^2y^2|00\rangle = \frac{1}{4}\hbar\omega \quad (2.30)$$

Thus we find that

$$\langle 00|V_1|00\rangle = \epsilon \left(\frac{3}{4} + \frac{3}{4} + 2\frac{1}{4} \right) \hbar\omega \quad (2.31)$$

The shift of the ground state is thus:

$$\boxed{\Delta E_{00} = 2\epsilon\hbar\omega} \quad (2.32)$$

Looking at the excited states is, of course, more work. I will take the approach of applying x^2 and y^2 respectively to $|10\rangle$, from which (using symmetry) I can get all the matrix elements using inner products such as $(\langle 10|x^2\rangle)(y^2|10\rangle)$. The first we have already done the work for:

$$y^2|10\rangle = \frac{\hbar}{2m\omega}(a_y + a_y^\dagger)^2|10\rangle \quad (2.33)$$

$$= \frac{\hbar}{2m\omega} \left(\sqrt{2}|12\rangle + |10\rangle \right) \quad (2.34)$$

Now for the second:

$$x^2|10\rangle = \frac{\hbar}{2m\omega}(a_x + a_x^\dagger)^2|10\rangle \quad (2.35)$$

$$= \frac{\hbar}{2m\omega}(a_x + a_x^\dagger) \left(\sqrt{2}|20\rangle + |00\rangle \right) \quad (2.36)$$

$$= \frac{\hbar}{2m\omega} \left(\sqrt{6}|30\rangle + 3|10\rangle \right) \quad (2.37)$$

Now we just have to mash these together in some inner products:

$$\frac{m^2\omega^3}{\hbar}\langle 10|y^4|10\rangle = \frac{3}{4}\hbar\omega \quad (2.38)$$

$$\frac{m^2\omega^3}{\hbar}\langle 10|x^4|10\rangle = \frac{m^2\omega^3}{\hbar} (\langle 10|x^2\rangle) (x^2|10\rangle) \quad (2.39)$$

$$= \frac{\hbar\omega}{4} \left(\sqrt{6}\langle 30| + 3\langle 10| \right) \left(\sqrt{6}|30\rangle + 3|10\rangle \right) \quad (2.40)$$

$$= \hbar\omega \frac{6+9}{4} \quad (2.41)$$

$$= \frac{15}{4}\hbar\omega \quad (2.42)$$

$$\frac{m^2\omega^3}{\hbar}\langle 10|x^2y^2|10\rangle = \frac{m^2\omega^3}{\hbar} (\langle 10|x^2\rangle) (y^2|10\rangle) \quad (2.43)$$

$$= \frac{\hbar\omega}{4} \left(\sqrt{6}\langle 30| + 3\langle 10| \right) \left(\sqrt{2}|12\rangle + |10\rangle \right) \quad (2.44)$$

$$= \frac{3}{4}\hbar\omega \quad (2.45)$$

Putting these together, we can find the diagonal elements of the Hamiltonian matrix within our degenerate subspace.

$$\langle 10|V_1|10\rangle = \epsilon\hbar\omega \left(\frac{3}{4} + \frac{15}{4} + 2\frac{3}{4} \right) \quad (2.46)$$

$$= 6\epsilon\hbar\omega \quad (2.47)$$

$$\langle 01|V_1|01\rangle = 6\epsilon\hbar\omega \quad (2.48)$$

Now we just need the off-diagonal term. Almost there! Fortunately, we can reuse our work from before, using symmetry to translate to $|01\rangle$.

$$\frac{m^2\omega^3}{\hbar} \langle 01|y^4|10\rangle = 0 \quad (2.49)$$

$$\frac{m^2\omega^3}{\hbar} \langle 01|x^4|10\rangle = 0 \quad (2.50)$$

These ones were easy, because the y^4 doesn't affect the x portion of the state, which is orthogonal. And vice versa for the x^4 .

$$\frac{m^2\omega^3}{\hbar} \langle 01|y^2x^2|10\rangle = \frac{m^2\omega^3}{\hbar} (\langle 01|x^2\rangle (y^2|10\rangle)) \quad (2.51)$$

$$= (stuff) \left(\sqrt{2}\langle 21| + \langle 01| \right) \left(\sqrt{2}|12\rangle + |10\rangle \right) \quad (2.52)$$

$$= 0 \quad (2.53)$$

So yay, the off-diagonal matrix elements are zero, so we don't need to do anything special here. This makes sense because this is a rotationally symmetric potential, so there is no way it could couple the x and y directions. So our result is now easy:

$$\boxed{\Delta E_{10} = \Delta E_{01} = 6\epsilon\hbar\omega} \quad (2.54)$$

The equation of state and the internal energy of an ideal gas are given as

$$PV = Nk_B T \text{ and } U = \frac{3}{2}Nk_B T,$$

where N is the number of particles, and P , V and T are the pressure, volume, temperature of the gas.

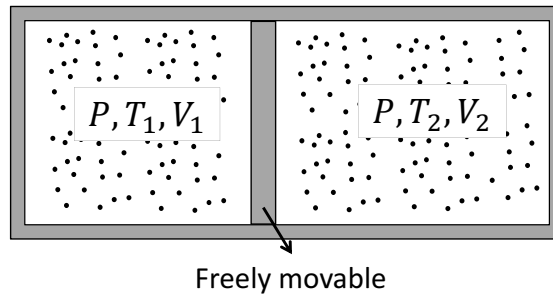
- a) The heat absorbed in an infinitesimal process is given by the first law as

$$dQ = TdS = dU + PdV.$$

Find the heat capacities at constant volume and at constant pressure of an ideal gas,

$$C_V = \left(\frac{\partial Q}{\partial T} \right)_V = T \left(\frac{\partial S}{\partial T} \right)_V \text{ and } C_P = \left(\frac{\partial Q}{\partial T} \right)_P = T \left(\frac{\partial S}{\partial T} \right)_P$$

- b) Two ideal gases with the same pressure P and the same number of particles N , but with different temperatures T_1 and T_2 , are confined in two compartments of volume V_1 and V_2 . The two chambers are separated by a freely movable wall through which heat can be exchanged. Find the final temperature and the change in entropy after the system has reached equilibrium.



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Solution:

The heat capacity at constant volume is

$$C_V = T \left(\frac{\partial S}{\partial T} \right)_V = \left(\frac{\partial U}{\partial T} \right)_V = \frac{3}{2}Nk_B$$

With a constant pressure,

$$PdV = d(PV) = d(Nk_B T) = Nk_B dT,$$

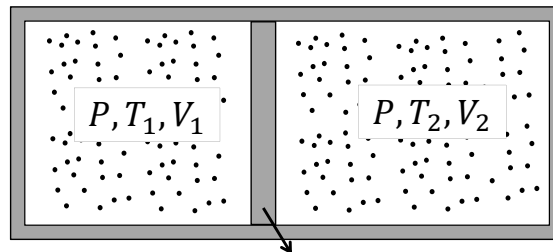
and hence

$$dQ = dU + PdV = dU + Nk_B dT$$

Therefore, the heat capacity at constant pressure is

$$C_P = \left(\frac{\partial Q}{\partial T} \right)_P = \left(\frac{\partial U}{\partial T} \right)_P + Nk_B = \frac{5}{2}Nk_B$$

- b) Two ideal gases with the same pressure P and the same number of particles N , but with different temperatures T_1 and T_2 , are confined in two compartments of volume V_1 and V_2 . The two chambers are separated by a freely movable wall through which heat can be exchanged. Find the final temperature and the change in entropy after the system has reached equilibrium.



Freely movable

Solution:

Since the energy must be conserved,

$$U = \frac{3}{2}k_B T_1 + \frac{3}{2}Nk_B T_2 = \frac{3}{2}2Nk_B T_f,$$

the final temperature is

$$T_f = \frac{T_1 + T_2}{2}$$

The heat exchange between the gases occurs at the same pressure,

$$dQ = TdS = C_P dT$$

Thus

$$\Delta S_1 = \int \frac{dQ}{T} = C_P \int_{T_1}^{T_f} \frac{dT}{T} = C_P \ln \frac{T_f}{T_1}.$$

Similarly,

$$\Delta S_2 = C_P \ln \frac{T_f}{T_2}.$$

Therefore, the total change in entropy is

$$\Delta S = \Delta S_1 + \Delta S_2 = C_P \ln \frac{T_f^2}{T_1 T_2} = \frac{5}{2}Nk_B \ln \frac{(T_1 + T_2)^2}{4T_1 T_2}$$

Consider a one-dimensional harmonic oscillator which is in equilibrium with a heat reservoir at absolute temperature T . The energy of such an oscillator is given by

$$E = \frac{p^2}{2m} + \frac{1}{2}\kappa x^2$$

where the first term on the right is the kinetic energy involving the momentum p and mass m , and the second term on the right is the potential energy involving the position coordinate x and spring constant κ .

- a) The equipartition theorem states that the mean value of each independent quadratic term in the energy is $\frac{1}{2}k_B T$, which is valid only in the *classical* statistical mechanics, i.e, the Plank constant $\hbar \rightarrow 0$ in the *classical* limit. What is the mean total energy of the harmonic oscillator according to the equipartition theorem?
- b) According to quantum mechanics the possible energy levels of the harmonic oscillator are given by

$$E_n = \left(n + \frac{1}{2}\right) \hbar \omega$$

where the quantum number $n = 0, 1, 2, 3, \dots$ and the angular frequency of oscillation $\omega = \sqrt{\kappa/m}$.

- (i) Find the mean total energy of the quantum harmonic oscillator.
- (ii) Show that the mean total energy is in agreement with the classical result at high temperature, $k_B T \gg \hbar \omega$.

You may find useful the geometric series:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \dots = \frac{1}{1-r}$$

Consider a one-dimensional harmonic oscillator which is in equilibrium with a heat reservoir at absolute temperature T . The energy of such an oscillator is given by

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Solution:

The mean total energy of the harmonic oscillator is

$$\langle E \rangle = \left\langle \frac{p^2}{2m} \right\rangle + \left\langle \frac{1}{2}\kappa x^2 \right\rangle$$

According to the equipartition theorem,

$$\left\langle \frac{p^2}{2m} \right\rangle = \left\langle \frac{1}{2}\kappa x^2 \right\rangle = \frac{1}{2}k_B T$$

Therefore,

$$\langle E \rangle = \frac{1}{2}k_B T + \frac{1}{2}k_B T = k_B T$$

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- (i) Find the mean total energy of the quantum harmonic oscillator.
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Solution:

- (i) The mean total energy of the quantum harmonic oscillator is

$$\langle E \rangle = \frac{\sum_n E_n e^{-\beta E_n}}{\sum_n e^{-\beta E_n}} = -\frac{1}{\sum_n e^{-\beta E_n}} \cdot \frac{\partial}{\partial \beta} \left(\sum_n e^{-\beta E_n} \right) = -\frac{1}{Z} \frac{\partial Z}{\partial \beta}$$

where $\beta = \frac{1}{k_B T}$ and $Z = \sum_n e^{-\beta E_n}$. Since the partition function Z is

$$Z = e^{-\frac{1}{2}\beta\hbar\omega} \sum_n e^{-n\beta\hbar\omega} = \frac{e^{-\frac{1}{2}\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} = \frac{1}{2 \sinh\left(\frac{1}{2}\beta\hbar\omega\right)},$$

the mean total energy is

$$\langle E \rangle = -\sinh\left(\frac{1}{2}\beta\hbar\omega\right) \cdot \left(-\frac{\frac{1}{2}\hbar\omega \cosh\left(\frac{1}{2}\beta\hbar\omega\right)}{\sinh^2\left(\frac{1}{2}\beta\hbar\omega\right)} \right) = \frac{1}{2}\hbar\omega \coth\left(\frac{1}{2}\beta\hbar\omega\right)$$

(ii) At high temperature, $k_B T \gg \hbar\omega \rightarrow \beta\hbar\omega \ll 1$, and hence the mean total energy is approximately

$$\langle E \rangle = \frac{1}{2}\hbar\omega \coth\left(\frac{1}{2}\beta\hbar\omega\right) \cong \frac{1}{2}\hbar\omega \frac{1}{\frac{1}{2}\beta\hbar\omega} = \frac{1}{\beta} = k_B T$$

The mean total energy at high temperature is in agreement with the classical result, $k_B T$.

You may find useful the geometric series:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \dots = \frac{1}{1-r}$$

A mass m on an ideal spring oscillates with frequency ω_0 when no other forces other than the spring restoring force are present. Now suppose the mass oscillates in a viscous fluid that lightly damps the motion, with the damping force proportional to the velocity of the oscillator (call the proportionality constant β). The system is driven with a sinusoidal force at frequency ω . After many time constants, m/β , the lightly damped system settles into a steady-state oscillatory response. The motion is 1-dimensional, in the x direction.

- (c) Find the equation of motion of the mass under steady state conditions, explicitly including the amplitude and phase (relative to the driving force) in terms of the given parameters of the system.
- (c) Make a large, clearly drawn and clearly labeled sketches of the amplitude and phase of the displacement as a function of the driving frequency. Discuss the behavior of the amplitude and phase of the displacement as the driving frequency becomes equal to, much larger than, and much smaller than the frequency of maximum response.
- (c) Give a physical discussion of the system at high frequencies including why the displacement and driving force are opposite in phase.

$$(a) \text{ Newton } m\ddot{x} = \sum F = \underbrace{-kx}_{\text{spring}} - \underbrace{\beta\dot{x}}_{\text{damping}} + \underbrace{F_0 e^{i\omega t}}_{\text{driving}} \quad (1)$$

$$\text{Simplify } \ddot{x} = -\frac{k}{m}x - \frac{\beta}{m}\dot{x} + \frac{F_0}{m}e^{i\omega t} \quad (2)$$

At steady state under conditions of light damping, the displacement must oscillate at the frequency of the driving force ω . Define the amplitude and phase this way with x_0 and ϕ real quantities. ϕ is the phase of the displacement relative to the driving force. $\pi > \phi > 0$ means the displacement leads the driving force.

$$\text{Displacement: } x(t) = x_0 e^{i(\omega t + \phi)} \quad (3)$$

$$\text{Velocity: } \dot{x} = i\omega x_0 e^{i(\omega t + \phi)} = i\omega x \quad (= \omega x e^{i\pi/2}) \quad (4)$$

$$\text{Acceleration: } \ddot{x} = -\omega^2 x_0 e^{i(\omega t + \phi)} = -\omega^2 x; \quad (= \omega^2 x e^{i\pi}) \quad (5)$$

Find x_0 and ϕ in terms of parameters of system. Substitute:

$$\begin{aligned} \ddot{x} &= -\omega^2 x - \gamma \dot{x} + \frac{F_0}{m} e^{i\omega t} \\ -\omega^2 x + \omega_0^2 x + i\omega \gamma x &= \frac{F_0}{m} e^{i\omega t} \\ x_0 e^{i(\omega t + \phi)} &= \frac{F_0 / m}{-\omega^2 + \omega_0^2 + i\omega \gamma} e^{i\omega t} \\ x_0 e^{i\phi} &= \frac{F_0 / m}{-\omega^2 + \omega_0^2 + i\omega \gamma} \end{aligned} \quad (6)$$

RHS is complex number in the form $a + ib$. Recast in polar form $Ae^{i\theta} = \sqrt{a^2 + b^2} e^{i \arctan(b/a)}$, but need to move i to the numerator.

$$\begin{aligned} \frac{F_0 / m}{-\omega^2 + \omega_0^2 + i\omega \gamma} &= \frac{F_0 / m (\omega_0^2 - \omega^2 - i\omega \gamma)}{(\omega_0^2 - \omega^2 + i\omega \gamma)(\omega_0^2 - \omega^2 - i\omega \gamma)} = \frac{F_0}{m} \frac{\overbrace{\omega_0^2 - \omega^2}^a \overbrace{-i\omega \gamma}^{ib}}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2} \\ &= \frac{F_0}{m} \frac{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2} e^{i \arctan\left(\frac{-\omega \gamma}{\omega_0^2 - \omega^2}\right)}}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2} = \frac{F_0 / m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} e^{i \arctan\left(\frac{-\omega \gamma}{\omega_0^2 - \omega^2}\right)} \end{aligned} \quad (7)$$

Easy to identify **amplitude and phase of displacement**:

$$x_0 = \frac{F_0 / m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}}; \quad \phi = \arctan\left(\frac{-\omega \gamma}{\omega_0^2 - \omega^2}\right) \quad (8)$$

Not asked for, but time derivatives give amplitude and phase of velocity and acceleration:

$$v_0 = \omega x_0 = \frac{F_0 \omega / m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}}; \quad \phi_v = \phi + \frac{\pi}{2} = \arctan\left(\frac{-\omega \gamma}{\omega_0^2 - \omega^2}\right) + \frac{\pi}{2}$$

$$a_0 = \omega^2 x_0 = \frac{F_0 \omega^2 / m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}}; \quad \phi_a = \phi + \pi = \arctan\left(\frac{-\omega \gamma}{\omega_0^2 - \omega^2}\right) + \pi$$

(b) Draw sketches: (label ω_0 , π etc)

Low frequency:

Displacement **amplitude** is finite at zero frequency; displacement **phase** is zero as $\omega \rightarrow 0$

$$x_0(\omega = 0) = \frac{F_0 / m}{\sqrt{(\omega_0^2 - 0^2)^2 + 0^2 \gamma^2}} = \frac{F_0}{m \omega_0^2} \quad \phi = \arctan\left(\frac{-0 \gamma}{\omega_0^2 - 0}\right) = \arctan(0^-) \Rightarrow \phi \rightarrow 0^-$$

and note that phase angle is negative as ω increases from 0.

Resonant frequency:

Displacement **amplitude** is maximal at ω_0 where denominator is smallest (resonant frequency)

$$x_0(\omega = \omega_0) = \frac{F_0}{m \omega_0 \gamma}$$

Displacement **phase** is $-\pi/2$.

$$\phi(\omega = \omega_0) = \arctan\left(\frac{-\omega_0 \gamma}{\omega_0^2 - \omega_0^2}\right) = \arctan(\infty^-) \Rightarrow \phi \rightarrow -\frac{\pi}{2}$$

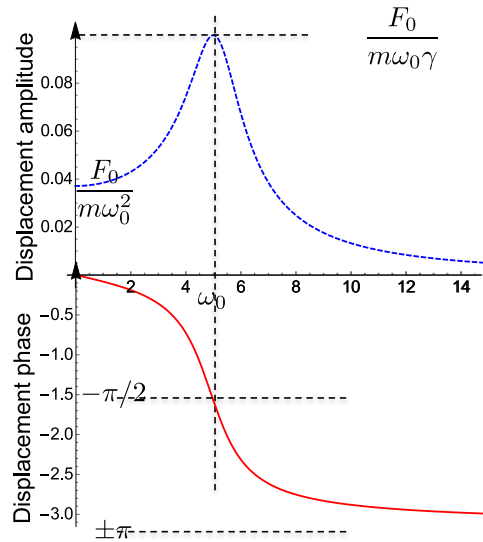
High frequency:

Displacement **amplitude** goes to zero as ω^{-2} at $\omega \gg \omega_0$

$$x_0(\omega \rightarrow \infty) = \frac{F_0 / m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} \rightarrow \frac{F_0}{m \omega^2} \rightarrow 0 \quad ;$$

Disp. **phase** is $-\pi$ or π (the same, physically). Either way, disp is 180° from the driving force.

$$\phi(\omega \rightarrow \infty) = \arctan\left(\frac{-\omega \gamma}{\omega_0^2 - \omega^2}\right) \Rightarrow \arctan\left(\frac{\gamma}{\omega}\right) \rightarrow \pi$$



- (c) At the highest frequencies, well above the resonant frequency $\omega \gg \omega_0$, the maximum acceleration becomes large in proportion to ω^2 (from Eq. 5).

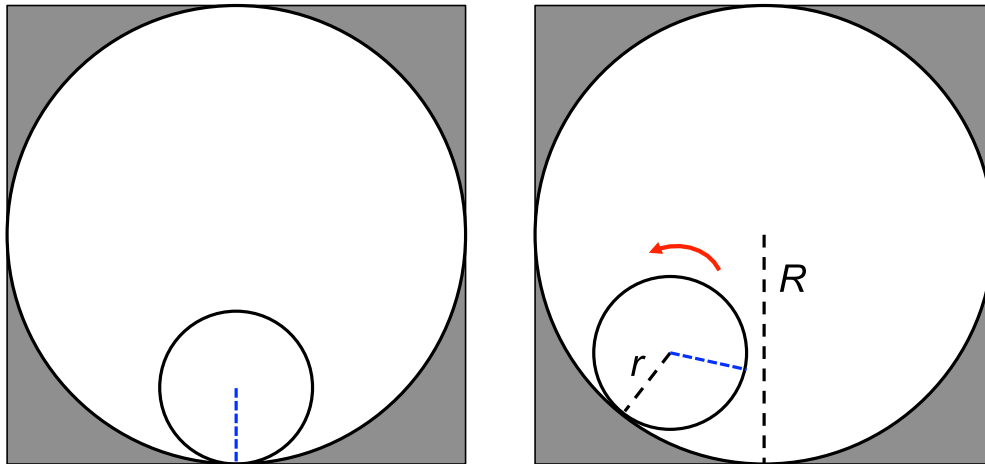
$$\ddot{x} = -\omega^2 x; \quad \ddot{x}_{\max} = -\omega^2 x_0;$$

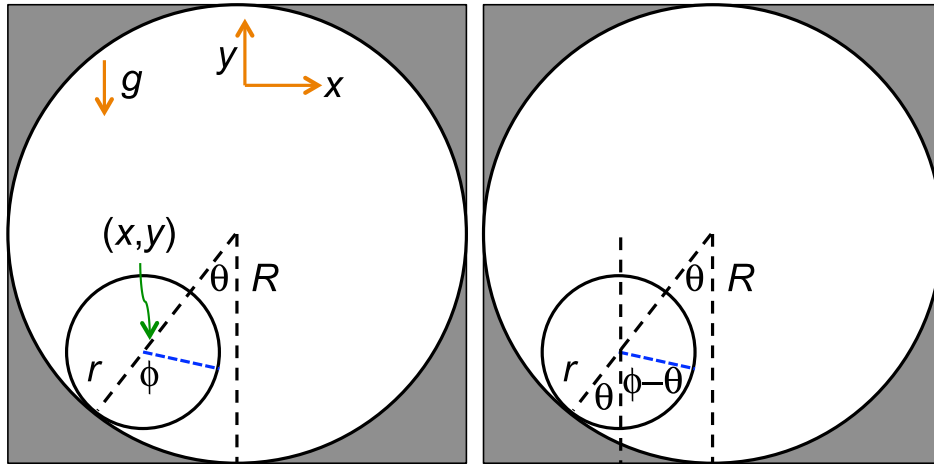
Of course the displacement x_0 is small, but we will see which term dominates. Some force has to provide that acceleration.

The spring provides a maximum restoring force of $F = kx_0 = m\omega_0^2 x_0$ and the restoring force (magnitude) required to produce the acceleration is $F = m\omega^2 x_0 \gg m\omega_0^2 x_0$.

So for any amplitude, the spring alone cannot provide the required restoring force, so the driving force has to provide it. A restoring force is opposite to the displacement (pulls the mass back to equilibrium), so the driving force is therefore in antiphase (180 degrees out of phase; opposite in phase) with the displacement.

A smaller cylinder of radius r rolls without slipping on the inside cylindrical surface of radius R of a hollowed-out block. Assume that the cylinders always remain in contact. The large block is fixed in space; it is massive enough that it does not slide against its supporting surface. The small cylinder has mass m and moment of inertia I . Let g be the acceleration due to gravity, directed downwards in the drawing. Find the equation of motion of the center of mass of the small cylinder, and the frequency of small oscillations.





General strategy: Define coordinates and constraints. Find the Lagrangian and use Lagrange's equations to get the equation of motion. Make a small angle approximation to get a harmonic oscillator equation and identify a frequency.

Set up x - y coordinate system as shown. θ is the angle between the point of contact and the center of the large cylinder. The position and velocity of the center of mass of the small cylinder are:

$$x = (R-r)\sin\theta \quad y = R - (R-r)\cos\theta \quad (\text{check } x_{\theta=0} = 0; \quad y_{\theta=0} = r)$$

$$\dot{x} = (R-r)\dot{\theta}\cos\theta \quad \dot{y} = (R-r)\dot{\theta}\sin\theta$$

ϕ is the angle between the point of contact and the fiducial mark on the small cylinder.

Cylinders remain in contact so the arc length on the small cylinder $s = r\phi$ equals the arc length on the large cylinder $s = R\theta$.

Constraint:

$$r\phi = R\theta \Rightarrow \phi = \frac{R}{r}\theta$$

Angular velocity must be measured in the fixed frame, so it is the rate of change of the angle $\phi - \theta$. (Sign doesn't matter because it will be squared in the energy term.)

$$\text{Thus } \omega = \dot{\phi} - \dot{\theta} = \dot{\theta} \left(\frac{R-r}{r} \right)$$

Kinetic energy (KE **of** the center of mass plus KE **about** center of mass):

$$\begin{aligned}
T &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\omega^2 \\
&= \frac{1}{2}m((R-r)\dot{\theta}\cos\theta)^2 + \frac{1}{2}m((R-r)\dot{\theta}\sin\theta)^2 + \frac{1}{2}I\left(\dot{\theta}\left(\frac{R-r}{r}\right)\right)^2 \\
&= \frac{1}{2}m(R-r)^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta) + \frac{1}{2}I\frac{(R-r)^2}{r^2}\dot{\theta}^2 \\
&= \frac{1}{2}\left(m + \frac{I}{r^2}\right)(R-r)^2\dot{\theta}^2
\end{aligned}$$

Potential energy:

$$\begin{aligned}
U &= mg(y - y_{\theta=0}) \\
&= mg(R - (R-r)\cos\theta - r) \\
&= mg(R-r)(1 - \cos\theta)
\end{aligned}$$

Lagrangian:

$$\begin{aligned}
L &= T - U \\
&= \frac{1}{2}\left(m + \frac{I}{r^2}\right)(R-r)^2\dot{\theta}^2 - mg(R-r)(1 - \cos\theta)
\end{aligned}$$

Lagrange's equation:

$$\begin{aligned}
\frac{\partial L}{\partial \theta} &= \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} \\
-mg(R-r)\sin\theta &= \frac{1}{2}\left(m + \frac{I}{r^2}\right)(R-r)^2 2 \frac{d}{dt} \dot{\theta}
\end{aligned}$$

Equation of motion:

$$\ddot{\theta} = -\frac{mg}{\left(m + \frac{I}{r^2}\right)(R-r)} \sin\theta$$

Small angle approximation $\sin\theta \approx \theta$ and with particular choice for moment of inertia of solid uniform cylinder about long axis $I = \frac{1}{2}mr^2$:

$$\begin{aligned}
\ddot{\theta} &\approx -\frac{mg}{\left(m + \frac{m}{2}\right)(R-r)} \theta = -\frac{2g}{3(R-r)} \theta \equiv -\omega^2 \theta \\
\omega &= \sqrt{\frac{2g}{3(R-r)}}
\end{aligned}$$

A point charge q is placed inside of a conducting spherical shell of radius a . For simplicity, we may assume the charge to be on the z axis and at a distance of d from the center of the sphere. The shell does not have any net charge, and we would like to characterize the electrostatics of the system.

- a) Approximate the electric field at the center of the sphere in two cases (i) when the point charge is very close to the center of the sphere (ii) when the point charge is very close to the boundary of the sphere. Keep the leading order terms, and offer physical explanation of the results.
- b) Calculate the potential both inside and outside of the sphere, you can leave the results as an infinite series but the coefficients should be explicitly expressed in terms of a , d , q and other physical constants.

Note: you may find the Legendre polynomial useful:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_n \frac{r'^n}{r^{n+1}} P_n(\cos \theta) \quad (7.1)$$

- a) When q is close to the center, Coulomb field dominates. When q is close to the shell, the shell feels flat and we can expect an image charge of $-q$ outside the sphere. Together, these two charges generate a dipole field at the origin.

In general, the field inside the sphere can be considered as a superposition of three contributions: (i) charge q , (ii) an image charge outside the sphere to make the potential on the sphere equals to zero, (iii) a uniform distribution of charge on the shell to account for the fact that the shell is not grounded.

- b) Because of the axial symmetry, we can use the general expansion in terms of Legendre polynomials. First assuming the potential on the shell is v_0 , for outside of the shell, we have

$$\phi(\mathbf{r}) = \sum_n \frac{B_n}{r^{n+1}} P_n(\cos \theta) \quad (7.2)$$

Applying the boundary condition, it is easy to see that

$$\phi(\mathbf{r}) = \frac{v_0 a}{r} \quad (7.3)$$

Now for the potential inside the sphere,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{r} - \mathbf{d}|} + \sum_n A_n r^n P_n(\cos \theta) \quad (7.4)$$

Here $\mathbf{d} = d\hat{z}$.

Applying the boundary condition we have

$$v_0 = \frac{q}{4\pi\epsilon_0} \sum_n \frac{d^n}{a^{n+1}} P_n(\cos \theta) + \sum_n A_n a^n P_n(\cos \theta) \quad (7.5)$$

This tells us

$$A_{n,n>0} = -\frac{q}{4\pi\epsilon_0} \frac{d^n}{a^{2n+1}}, A_0 = v_0 - \frac{q}{4\pi\epsilon_0 a} \quad (7.6)$$

We still have to solve for v_0 . Using Gaussian theorem on a spherical surface outside of the shell, we know $v_0 = \frac{q}{4\pi\epsilon_0 a}$. That means $A_0 = 0$.

A non-conducting ring with total mass M and total charge Q is hung with a string so that the string is normal to the ring surface, and the ring can freely rotate around the string. The ring has a radius of a and is inside of a long solenoid. The solenoid has radius r , total of N tightly-wound rounds over its length of L .

- a) If the current through the solenoid changes at a rate of \dot{I} , what is the torque exerted on the ring.
- b) When the current through the solenoid slowly increase from 0 to I , what is the final angular velocity of the ring?

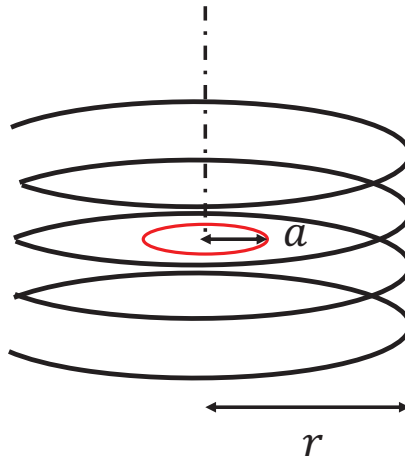


Figure 1: Schematics of ring in solenoid.

- a) Using Ampere loop, it is easy to derive that the magnetic field inside of a solenoid is constant, and equals to

$$B = \mu_0 IN/L \quad (8.1)$$

If current changes, the EMF generated along the ring equals to (we will only consider the magnitude from now on)

$$2\pi aE = \pi a^2 \mu_0 \dot{I}N/L \quad (8.2)$$

Therefore the torque equals to

$$T = QaE = \frac{1}{2}a^2 \mu_0 Q \dot{I}N/L \quad (8.3)$$

- b) From conservation of angular momentum, we can also relate the torque to change of angular velocity:

$$Ma^2 \dot{\omega} = \frac{1}{2}a^2 \mu_0 Q \dot{I}N/L \quad (8.4)$$

Therefore the final angular velocity equals to $\frac{1}{2M} \mu_0 QIN/L$.