OSU Physics Department
Comprehensive Examination #127
Solutions

Monday, January 9 and Tuesday, January 10, 2017

Winter 2017 Comprehensive Examination

Quantum Mechanics	9 AM-12 PM	Monday, January 9
Statistical Mechanics	1 PM-4 PM	Monday, January 9
Electricity and Magnetism	9 AM-12 PM	Tuesday, January 10
Clasical Mechanics	1 PM-4 PM	Tuesday, January 10

General Instructions

This Winter 2017 Comprehensive Examination consists of four separate parts of two problems each. Each problem caries equal weight (20 points each) and lasts three hours. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work on the provided pages, work each problem in its own labeled pages, and be certain that your chosen student letter (but not your name) is on the header of each page of your exam, including any unused pages. If you need additional paper for your work, use the blank pages provided. Each page of work should include the problem number, a page number, your chosen student letter, and the total number of pages actually used. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam and the collection of formulas distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please staple and return all pages of your exam—including unused pages—at the end of the exam.

Consider a 1D simple harmonic oscillator, which at $t = 0$ is in a state given by

$$|\psi(t=0)\rangle = e^{ikx}|0\rangle \tag{1.1}$$

where $|0\rangle$ is the ground state of the oscillator.

a) What is the expectation value $\langle x\rangle$ at $t = 0$?

b) What is $\frac{d\langle x\rangle}{dt}$ at $t = 0$?

Your solution will reflect a connection between quantum and classical mechanics. Provided k is sufficiently large, this wave function will follow a classical trajectory for quite a while before dispersion takes over. A more thorough exploration of this behavior is beyond the scope of this exam, but could consist of relating the time derivative of $\langle p\rangle$ to $\langle x\rangle$ at all times.

The normalized ground state of a simple harmonic oscillator is given by

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$

In addition, the following integrals may be useful:

$$\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} u^2 e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$$

a) The expectation value of x is given by

$$\begin{aligned}
\langle\psi| x |\psi\rangle &= \langle 0| e^{-ikx} x e^{ikx} |0\rangle && (1.2)\\
&= \langle 0| x |0\rangle && (1.3)\\
&= 0 && (1.4)
\end{aligned}$$

where the last step happened because the integral was odd. Or you could express x in terms of creation and annihilation operators, in which case it would be zero because $\langle 0|1\rangle = 0$.

b) To find the time derivative, we need to apply the product rule.

$$\begin{aligned}
\frac{d}{dt}\langle\psi| x |\psi\rangle &= \frac{d\langle\psi|}{dt} x |\psi\rangle + \langle\psi| x \frac{d|\psi\rangle}{dt} && (1.5)\\
&= \langle\psi| x \frac{d|\psi\rangle}{dt} + \text{c.c.} && (1.6)\\
&= 2\Re \langle\psi| x \frac{d|\psi\rangle}{dt} && (1.7)
\end{aligned}$$

where "c.c." means the complex conjugate of the previous bit and $\Re$ represents the real part. We now remind ourselves of Schrödinger's equation

$$\begin{aligned}
H|\psi\rangle &= i\hbar \frac{d}{dt}|\psi\rangle && (1.8)\\
\frac{d|\psi\rangle}{dt} &= \frac{H|\psi\rangle}{i\hbar} && (1.9)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\langle\psi| x |\psi\rangle &= 2\Re \langle\psi| x \frac{d|\psi\rangle}{dt} && (1.10)\\
&= 2\Re \frac{1}{i\hbar}\langle\psi| xH |\psi\rangle && (1.11)\\
&= 2\Re \frac{1}{i\hbar}\langle 0| e^{-ikx} x H e^{ikx} |0\rangle && (1.12)\\
&= 2\Re \frac{1}{i\hbar}\langle 0| e^{-ikx} x \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2\right) e^{ikx} |0\rangle && (1.13)\\
&= 2\Re - \frac{\hbar^2}{2m}\frac{1}{i\hbar}\langle 0| e^{-ikx} x \frac{d^2}{dx^2} e^{ikx} |0\rangle + 2\frac{1}{2}m\omega^2 \Re \frac{1}{i\hbar}\langle 0| e^{-ikx} x^3 e^{ikx} |0\rangle && (1.14)\\
&= 2\Re - \frac{\hbar^2}{2m}\frac{1}{i\hbar}\langle 0| e^{-ikx} x \frac{d^2}{dx^2} e^{ikx} |0\rangle + 2\frac{1}{2}m\omega^2 \Re \frac{1}{i\hbar}\cancelto{0}{\langle 0| x^3 |0\rangle} && (1.15)\\
&= 2\Re - \frac{\hbar^2}{2m}\frac{1}{i\hbar}\langle 0| e^{-ikx} x \frac{d^2}{dx^2} e^{ikx} |0\rangle && (1.16)
\end{aligned}$$

Now let's take a moment to just apply the second derivative above to our wave function.

$$
\begin{aligned}
\frac{d^2}{dx^2} e^{ikx}\phi_0(x) &= \frac{d}{dx}\left(ike^{ikx}\phi_0(x) + e^{ikx}\phi_0'(x)\right) && (1.17)\\
&= -k^2 e^{ikx}\phi_0(x) + 2ike^{ikx}\phi_0'(x) + e^{ikx}\phi_0''(x) && (1.18)\\
&= -k^2 e^{ikx}\phi_0(x) + 2ike^{ikx}\phi_0'(x) + e^{ikx}\phi_0''(x) && (1.19)
\end{aligned}
$$

Now we could evaluate all three terms here, but two of them will vanish when we stick them into the integral, so let's do that first.

$$
\begin{aligned}
\frac{d}{dt}\left\langle\psi\right| x \left|\psi\right\rangle &= 2\Re - \frac{\hbar^2}{2m}\frac{1}{i\hbar}\left\langle 0\right| e^{-ikx} x \frac{d^2}{dx^2} e^{ikx}\left|0\right\rangle && (1.20)\\
&= 2\Re - \frac{\hbar^2}{2m}\frac{1}{i\hbar}\int_{-\infty}^{\infty}\phi_0(x)e^{-ikx}x\left(-k^2 e^{ikx}\phi_0(x) + 2ike^{ikx}\phi_0'(x) + e^{ikx}\phi_0''(x)\right)dx && (1.21)\\
&= 2\Re - \frac{\hbar^2}{2m}\frac{1}{i\hbar}\int_{-\infty}^{\infty}\phi_0(x)x\left(\cancelto{\text{even}}{-k^2\phi_0(x)} + 2ik\phi_0'(x) + \cancelto{\text{even}}{\phi_0''(x)}\right)dx && (1.22)\\
&= 2\Re - \frac{\hbar^2}{2m}\frac{2k}{\hbar}\int_{-\infty}^{\infty}\phi_0(x)x\phi_0'(x)dx && (1.23)\\
&= 2\Re - \frac{\hbar k}{m}\int_{-\infty}^{\infty}\phi_0(x)x\left(-2\frac{m\omega}{2\hbar}x\phi_0(x)\right)dx && (1.24)\\
&= 2\Re - \frac{\hbar k}{m}\int_{-\infty}^{\infty}\phi_0(x)x\left(-2\frac{m\omega}{2\hbar}x\phi_0(x)\right)dx && (1.25)\\
&= 2\Re\frac{\hbar k}{m}\int_{-\infty}^{\infty}\frac{m\omega}{\hbar}x^2\sqrt{\frac{m\omega}{\pi\hbar}}e^{-\frac{m\omega}{\hbar}x^2}dx && (1.26)
\end{aligned}
$$

We do a simple u substitution:

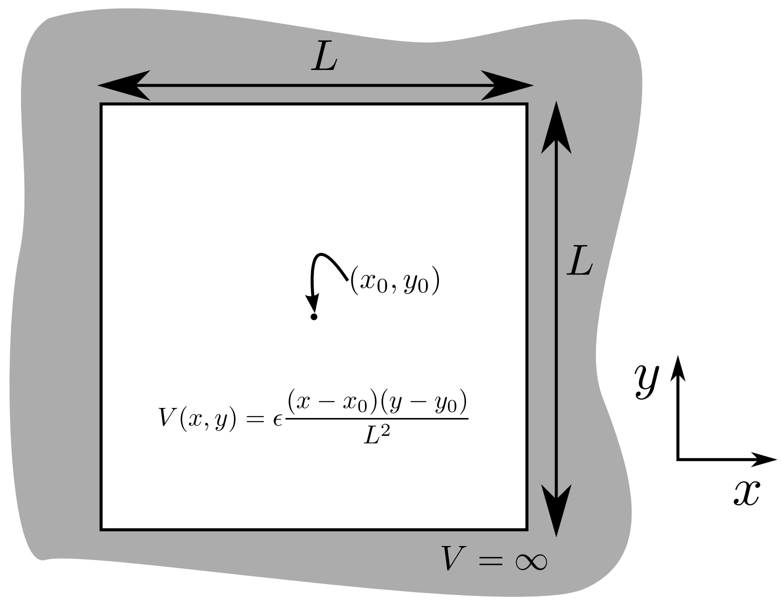
$$
u = \sqrt{\frac{m\omega}{\hbar}}x \qquad\qquad du = \sqrt{\frac{m\omega}{\hbar}}dx \tag{1.27}
$$

$$
\begin{aligned}
\frac{d}{dt}\left\langle\psi\right| x \left|\psi\right\rangle &= 2\frac{\hbar k}{m}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}u^2 e^{-u^2}du && (1.28)\\
&= 2\frac{\hbar k}{m}\frac{1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} && (1.29)\\
&= \frac{\hbar k}{m} && (1.30)
\end{aligned}
$$

In other words, $\hbar k$ is the momentum, and $p = mv$.

Problem 2

Consider a particle in a 2D infinite well with side length L. This well has a potential given in the following figure



where the point (x_0, y_0) is defined to be the center of the box (you may choose your own coordinate system). You may further assume that

$$\epsilon \ll \frac{\hbar^2}{mL^2}$$

What are the energies of the lowest 3 eigenstates?

Your final answer may contain a dimensionless definite integral that is independent of any physical parameters, and is non-zero.

To solve for the lowest 3 eigenstates, we will use perturbation theory, since we know that $\epsilon l \frac{\hbar^2}{mL^2}$. The first three unperturbed eigenstates are given by

$$\psi_{11} = A_{11} \cos\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi y}{L}\right) \tag{2.1}$$

$$\psi_{12} = A_{21} \cos\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi y}{L}\right) \tag{2.2}$$

$$\psi_{21} = A_{21} \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{\pi y}{L}\right) \tag{2.3}$$

where we choose to put the origin in the center of the box. These are the longest-wavelength modes that satisfy our boundary conditions that the wave-function must go to zero at the edges of the box. A_{11} and A_{12} are normalization constants. The unperturbed energies of these states are

$$E_{11}^{(0)} = \frac{\hbar^2}{2m}\left(\frac{\pi^2}{L^2} + \frac{\pi^2}{L^2}\right) \tag{2.4}$$

$$= \frac{\hbar^2\pi^2}{mL^2} \tag{2.5}$$

$$E_{21}^{(0)} = E_{12}^{(0)} \tag{2.6}$$

$$= \frac{\hbar^2}{2m}\left(\frac{\pi^2}{L^2} + \frac{2^2\pi^2}{L^2}\right) \tag{2.7}$$

$$= \frac{5}{2}\frac{\hbar^2\pi^2}{mL^2} \tag{2.8}$$

$$= \frac{5}{2}E_{11} \tag{2.9}$$

Now we need to find the shift in energy due to the saddle-point perturbation. The first one is easy because it is non-degenerate, but the second two states are degenerate so we'll have to apply degenerate perturbation theory.

$$E_{11} = E_{11}^{(0)} + \cancelto{0}{\left\langle \psi_{11} \middle| \epsilon\frac{xy}{L^2} \middle| \psi_{11} \right\rangle} \tag{2.10}$$

$$= E_{11}^{(0)} \tag{2.11}$$

The integral is odd... twice-odd actually, and thus comes out to zero. So our ground state energy is unshifted to first order.

The same logic applies to the diagonal matrix elements:

$$\left\langle \psi_{21} \middle| \epsilon\frac{xy}{L^2} \middle| \psi_{21} \right\rangle = 0 \tag{2.12}$$

$$\left\langle \psi_{12} \middle| \epsilon\frac{xy}{L^2} \middle| \psi_{12} \right\rangle = 0 \tag{2.13}$$

Therefore, if we were not using degenerate perturbation theory, we would conclude that our perturbation has no effect to first order. However, the first two excited states *are* degenerate, so we need to ask about the nondiagonal matrix element.

$$\left\langle \psi_{21} \left| \epsilon \frac{xy}{L^2} \right| \psi_{12} \right\rangle = \frac{\epsilon A_{21}^2}{L^2} \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy \cos\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi y}{L}\right) xy \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{\pi y}{L}\right) \tag{2.14}$$

$$= \frac{\epsilon A_{21}^2}{L^2} \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy\, x \cos\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) y \cos\left(\frac{\pi y}{L}\right) \sin\left(\frac{2\pi y}{L}\right) \tag{2.15}$$

$$= \frac{\epsilon}{L^2} \left(A_{21} \int_{-L/2}^{L/2} dx\, x \cos\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) \right)^2 \tag{2.16}$$

At this point we have a solution that is in terms of a single definite integral, but that integral still contains physical terms. We can remove them with a substitution.

$$u = \frac{\pi x}{L} \qquad du = \pi \frac{dx}{L} \tag{2.17}$$

$$\left\langle \psi_{21} \left| \epsilon \frac{xy}{L^2} \right| \psi_{12} \right\rangle = \epsilon L^2 A_{21}^2 \left(\frac{1}{\pi^2} \int_{-\pi/2}^{\pi/2} du\, u \cos u \sin(2u) \right)^2 \tag{2.18}$$

Now the integral is a pure dimensionless number that is independent of the parameters of our problem (notably length). Unfortunately, we still have the normalization factor annoying us, and making it potentially unclear as to how our energy shift will depend on the size of the box. To address this (which we could do later), let us normalize the wave functions.

$$1 = A_{21}^2 \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy \cos^2\left(\frac{\pi x}{L}\right) \sin^2\left(\frac{2\pi y}{L}\right) \tag{2.19}$$

$$= A_{21}^2 \frac{L}{2} \frac{L}{2} \tag{2.20}$$

$$A_{21} = \frac{2}{L} \tag{2.21}$$

In the first step, I used the fact that the average value of a square of a sinusoid is $\frac{1}{2}$. You can solve for this by recognizing that the sum of the squares of sine and cosine must be 1, which means that their averages must be $\frac{1}{2}$, provided you average over a half-integer number of wavelengths.

With this normalization in hand, we find that

$$\left\langle \psi_{21} \left| \epsilon \frac{xy}{L^2} \right| \psi_{12} \right\rangle = \frac{\epsilon}{4} \left(\frac{1}{\pi^2} \int_{-\pi/2}^{\pi/2} du\, u \cos u \sin(2u) \right)^2 \tag{2.22}$$

$$= \frac{\epsilon}{4} I \tag{2.23}$$

$$I \equiv \left(\frac{1}{\pi^2} \int_{-\pi/2}^{\pi/2} du\, u \cos u \sin(2u) \right)^2 \tag{2.24}$$

Okay, now for some degenerate perturbation theory. We have a perturbation Hamiltonian that in our degenerate subspace looks like

$$H = \begin{bmatrix} 0 & \frac{\epsilon}{4} I \\ \frac{\epsilon}{4} I & 0 \end{bmatrix} \tag{2.25}$$

The eigenstates of this Hamiltonian are clear by inspection:

$$\psi_\pm = \frac{\psi_{21} \pm \psi_{12}}{\sqrt{2}} \tag{2.26}$$

and their eigenvalues are

$$E_\pm = \pm \frac{\epsilon}{4} I \tag{2.27}$$

You can check these by multiplying the above matrix by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Our three lowest energies thus come out to:

$$E_{11} = \frac{\hbar^2 \pi^2}{m L^2} \tag{2.28}$$

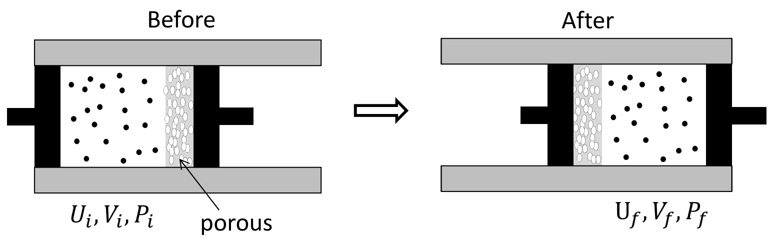
$$E_- = \frac{5}{2} \frac{\hbar^2 \pi^2}{m L^2} - \frac{\epsilon}{4} I \tag{2.29}$$

$$E_+ = \frac{5}{2} \frac{\hbar^2 \pi^2}{m L^2} + \frac{\epsilon}{4} I \tag{2.30}$$

where the dimensionless integral I is given above.

Problem 3

A gas initially at a pressure P_i, temperature T_i, and volume V_i is forced through a porous plug into another chamber, maintained at pressure $P_f < P_i$, as shown in the figure below. P_i and P_f are kept constant, and all chambers and the plug are insulated, $\Delta Q = 0$, for the process. Assume that the process is slow enough to maintain uniform temperature and pressure in each chamber.



a) Show that the initial state enthalpy $H_i = U_i + P_i V_i$ is same with the final state enthalpy $H_f = U_f + P_f V_f$, where U_i and U_f are the initial and final state internal energies. This means that the throttling process takes place at constant enthalpy.

b) A differential change in H with T and P as independent variables can be expressed as

$$dH = C_p dT + \left[V - T \left(\frac{\partial V}{\partial T} \right)_P \right] dP.$$

Using this relation, show that

$$\left(\frac{\partial T}{\partial P} \right)_H = 0$$

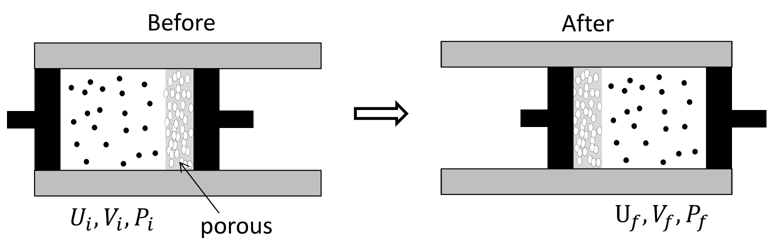
for an ideal gas with the equation of state $PV = RT$ for one mole where $R = Nk_B$. Based on this result, find the relation between the initial temperature T_i and the final temperature T_f. What is the work done by the ideal gas during the process?

c) In general, the coefficient $\left(\frac{\partial T}{\partial P}\right)_H$ does not vanish for an interacting gas. We consider the van der Waals equation of state written for one mole of substance,

$$\left(P + \frac{a}{V^2} \right) (V - b) = RT,$$

where $a \ll RTV$ and $b \ll V$ are characteristic constants for a given substance. Find the coefficient $\left(\frac{\partial T}{\partial P}\right)_H$. Using the result and assuming $C_p = \frac{5}{2}R$, show that the gas cools down when the initial temperature T_i is lower than $\frac{2a}{bR}$.

A gas initially at a pressure P_i, temperature T_i, and volume V_i is forced through a porous plug into another chamber, maintained at pressure $P_f < P_i$, as shown in the figure below. P_i and P_f are kept constant, and all chambers and the plug are insulated, $\Delta Q = 0$, for the process. Assume that the process is slow enough to maintain uniform temperature and pressure in each chamber.



a) Show that the initial state enthalpy $H_i = U_i + P_iV_i$ is same with the final state enthalpy $H_f = U_f + P_fV_f$, where U_i and U_f are the initial and final state internal energies. This means that the throttling process takes place at constant enthalpy.

Solution:
The net work driven by the gas is

$$\Delta W = \int_0^{V_f} P_f dV_f + \int_{V_i}^0 P_i dV_i = P_fV_f - P_iV_i$$

Since $\Delta Q = 0$,

$$\Delta U = -\Delta W \Rightarrow U_f - U_i = P_iV_i - P_fV_f \Rightarrow U_i + P_iV_i = U_f + P_fV_f.$$

b) A differential change in H with T and P as independent variables can be expressed as

$$dH = C_p dT + \left[V - T\left(\frac{\partial V}{\partial T}\right)_P\right] dP.$$

Using this relation, show that

$$\left(\frac{\partial T}{\partial P}\right)_H = 0$$

for an ideal gas with the equation of state $PV = RT$ for one mole where $R = Nk_B$. Based on this result, find the relation between the initial temperature T_i and the final temperature T_f. What is the work done by the ideal gas during the process?

Solution:
When $dH = 0$,

$$C_p dT = \left[T\left(\frac{\partial V}{\partial T}\right)_P - V\right] dP \Rightarrow \left(\frac{\partial T}{\partial P}\right)_H = \frac{1}{C_P}\left[T\left(\frac{\partial V}{\partial T}\right)_P - V\right]$$

For an ideal gas,

$$PV = RT \Rightarrow P\left(\frac{\partial V}{\partial T}\right)_P = R \Rightarrow T\left(\frac{\partial V}{\partial T}\right)_P = \frac{RT}{P} = V$$

Therefore,

$$\left(\frac{\partial T}{\partial P}\right)_H = \frac{1}{C_P}\left[T\left(\frac{\partial V}{\partial T}\right)_P - V\right] = \frac{1}{C_P}[V - V] = 0$$

This means that there is no temperature change during the process, i.e., $T_f = T_i$.

Since $T_i = T_f$, $P_iV_i = P_fV_f$, thus $\Delta W = P_iV_i - P_fV_f = 0$. There is no work done by the ideal gas during the process.

c) In general, the coefficient $\left(\frac{\partial T}{\partial P}\right)_H$ does not vanish for an interacting gas. We consider the van der Waals equation of state written for one mole of substance,

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT,$$

where $a \ll RTV$ and $b \ll V$ are characteristic constants for a given substance. Find the coefficient $\left(\frac{\partial T}{\partial P}\right)_H$. Using the result and assuming $C_p = \frac{5}{2}R$, show that the gas cools down when the initial temperature T_i is lower than $\frac{2a}{bR}$.

Solution:
Since $a \ll RTV$ and $b \ll V$, the equation of state reduces to

$$\begin{aligned}
& PV\left(1 + \frac{a}{RTV}\right)\left(1 - \frac{b}{V}\right) \cong RT \Rightarrow PV\left(1 + \frac{a}{RTV} - \frac{b}{V}\right) \cong RT \\
\Rightarrow \quad & PV + \frac{P}{RT}a - Pb = RT \Rightarrow PV = -P\left(\frac{a}{RT} - b\right) + RT \\
\Rightarrow \quad & V = b - \frac{a}{RT} + \frac{RT}{P},
\end{aligned}$$

and, then,

$$\left(\frac{\partial V}{\partial T}\right)_P = \frac{R}{P} + \frac{a}{RT^2}.$$

Therefore,using the equation in (b), we obtain

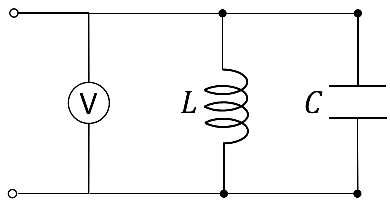
$$\begin{aligned}
\left(\frac{\partial T}{\partial P}\right)_H &= \frac{1}{5R/2}\left[T\left(\frac{R}{P} + \frac{a}{RT^2}\right) - \left(b - \frac{a}{RT} + \frac{RT}{P}\right)\right] \\
&= \frac{2}{5R}\left(\frac{2a}{RT} - b\right)
\end{aligned}$$

This equation tells us that

$$\left(\frac{\partial T}{\partial P}\right)_H > 0$$

for $T < \frac{2a}{bR}$. This means that the pressure drop ($\Delta P = P_f - P_i < 0$) induces cooling ($\Delta T = T_f - T_i < 0$) during the throttling process at low temperature $T < \frac{2a}{bR}$.

A quantum thermometer modeled as a microscopic LC circuit measures the thermal noise of the voltage across an inductor and capacitor in parallel.



In terms of the charge Q on the capacitor, the Hamiltonian governing oscillations is

$$H = \frac{1}{2}LI^2 + \frac{1}{2C}Q^2 = \frac{1}{2}L\left(\frac{dQ}{dt}\right)^2 + \frac{1}{2C}Q^2$$

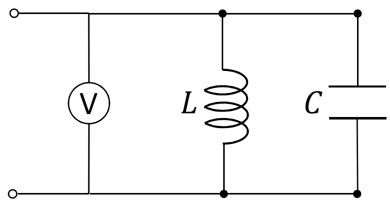
and the oscillation frequency is $\omega = 1/\sqrt{LC}$. The circuit is attached to a thermal reservoir of temperature T.

a) The LC circuit has discrete energy levels. Write down its energy eigenvalues. What is the average energy of the circuit in thermal equilibrium at T?

b) What are the average energies stored in the capacitor and in the inductor?

c) Find the rms noise voltage $\langle V^2\rangle^{1/2}$ and the rms noise current $\langle I^2\rangle^{1/2}$. Obtain the expressions for $\langle V^2\rangle^{1/2}$ in the classical limit ($k_BT \gg \hbar\omega$) and in the quantum limit ($k_BT \ll \hbar\omega$).

A quantum thermometer modeled as a microscopic LC circuit measures the thermal noise of the voltage across an inductor and capacitor in parallel.



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and the oscillation frequency is $\omega = 1/\sqrt{LC}$. The circuit is attached to a thermal reservoir of temperature T.

a) The LC circuit has discrete energy levels. Write down its energy eigenvalues. What is the average energy of the circuit in thermal equilibrium at T?

Solution:
The LC circuit is a simple harmonic oscillator with the resonance frequency ω. Thus the energy eigenvalues are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \qquad n = 0, 1, 2, \cdots .$$

The average energy in the circuit is

$$U = \langle E \rangle = \frac{\sum_{n=0}^{\infty} E_n e^{-E_n/k_B T}}{\sum_{n=0}^{\infty} e^{-E_n/k_B T}} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta},$$

where $Z = \sum_{n=0}^{\infty} e^{-E_n/k_B T}$ and $\beta = 1/k_B T$. Since

$$Z = e^{-\beta\hbar\omega/2}\sum_{n=0}^{\infty} e^{-n\beta\hbar\omega} = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}} = \frac{1}{e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}} = \frac{1}{2\sinh\left(\frac{\hbar\omega}{2k_B T}\right)},$$

$$U = \frac{1}{2}\hbar\omega\frac{\cosh\left(\frac{\hbar\omega}{2k_B T}\right)}{\sinh\left(\frac{\hbar\omega}{2k_B T}\right)} = \frac{1}{2}\hbar\omega\coth\left(\frac{\hbar\omega}{2k_B T}\right)$$

b) What are the average energies stored in the capacitor and in the inductor?

Solution:
In a simple harmonic oscillator, the two components have same average energy:

$$\left\langle \frac{1}{2}LI^2 \right\rangle = \left\langle \frac{1}{2}CV^2 \right\rangle = \frac{U}{2} = \frac{1}{4}\hbar\omega \coth\left(\frac{\hbar\omega}{2k_BT}\right)$$

c) Find the rms noise voltage $\left\langle V^2 \right\rangle^{1/2}$ and the rms noise current $\left\langle I^2 \right\rangle^{1/2}$. Obtain the expressions for $\left\langle V^2 \right\rangle^{1/2}$ in the classical limit ($k_BT \gg \hbar\omega$) and in the quantum limit ($k_BT \ll \hbar\omega$).

Solution:
Using the average energies stored in the capacitor and the inductor, we get

$$\left\langle \frac{1}{2}CV^2 \right\rangle = \frac{U}{2} \Rightarrow \left\langle V^2 \right\rangle = \frac{U}{C} \Rightarrow \left\langle V^2 \right\rangle^{1/2} = \sqrt{\frac{U}{C}}$$

and

$$\left\langle \frac{1}{2}LI^2 \right\rangle = \frac{U}{2} \Rightarrow \left\langle I^2 \right\rangle = \frac{U}{I} \Rightarrow \left\langle I^2 \right\rangle^{1/2} = \sqrt{\frac{U}{I}}$$

In the classical limit ($k_BT \gg \hbar\omega$),

$$U = \frac{1}{2}\hbar\omega \coth\left(\frac{\hbar\omega}{2k_BT}\right) \cong \frac{1}{2}\hbar\omega \frac{1}{\frac{\hbar\omega}{2k_BT}} = k_BT$$

Thus

$$\left\langle V^2 \right\rangle^{1/2} \cong \sqrt{\frac{k_BT}{C}}.$$

In the quantum limit ($k_BT \ll \hbar\omega$),

$$U = \frac{1}{2}\hbar\omega \coth\left(\frac{\hbar\omega}{2k_BT}\right) \cong \frac{1}{2}\hbar\omega$$

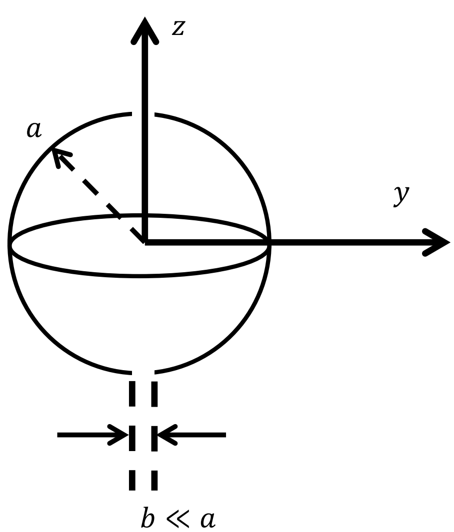
Thus

$$\left\langle V^2 \right\rangle^{1/2} \cong \sqrt{\frac{\hbar\omega}{2C}}.$$

Problem 5

A spherical shell of radius *a* is uniformly charged with a total charge *Q*. However, when taking a close look, one finds that there are two holes in the north and south poles with the same size as shown in the figure below. Using the coordinate system indicated in the figure,

(1) What is the electric field on the z axis and at $z \gg a$? (keep terms up to the leading terms in $\frac{b}{a}$)
(2) What is the electric field on the z axis just outside of the sphere, ($a < z$ and $z - a \ll b$, keep the leading terms in $\frac{b}{a}$)
(3) What is the electric field on the z axis just inside of the sphere, ($a > z$ and $a - z \ll b$, keep the leading terms in in $\frac{b}{a}$)
(4) Sketch the potential along the z axis from -∞ to +∞.



Keeping up to the linear terms, the charge density is

$$\rho = \frac{Q}{4\pi a^2 - \frac{\pi b^2}{2}} \tag{5.1}$$

We should consider the holey sphere as a full sphere with charge density ρ plus two patches that are charged with $-\rho$. The E field and potential of the full sphere is a familiar result, the two patches can be approximated as either point charges or infinite planes in problem (1) to (3). For simplicity, we will assume $Q > 0$.

(1) Amplitude E of the electric field is

$$E \approx \frac{1}{4\pi\epsilon_0}\frac{4\pi a^2\rho}{z^2} - \frac{1}{4\pi\epsilon_0}\frac{\pi b^2\rho}{4(z+a)^2} - \frac{1}{4\pi\epsilon_0}\frac{\pi b^2\rho}{4(z-a)^2} \tag{5.2}$$

This is sufficient but if we continue take leading terms this can be further simplified

$$E \approx \frac{Q}{4\pi\epsilon_0}[\frac{1}{z^2}(1+\frac{b^2}{8a^2}) - \frac{b^2}{16a^2}(\frac{1}{(z+a)^2} + \frac{1}{(z-a)^2})] \tag{5.3}$$

(2) Similar as in (1), except that now one of the patch should be considered as an infinitely large plane, therefore we have

$$E \approx \frac{1}{4\pi\epsilon_0}\frac{4\pi a^2\rho}{z^2} - \frac{1}{4\pi\epsilon_0}\frac{\pi b^2\rho}{4(z+a)^2} - \frac{\rho}{2\epsilon_0} \tag{5.4}$$

(3) Similar as in (2) except that the direction of the E field from the nearby patch changes direction, therefore we have

$$E \approx \frac{1}{4\pi\epsilon_0}\frac{4\pi a^2\rho}{z^2} - \frac{1}{4\pi\epsilon_0}\frac{\pi b^2\rho}{4(z+a)^2} + \frac{\rho}{2\epsilon_0} \tag{5.5}$$

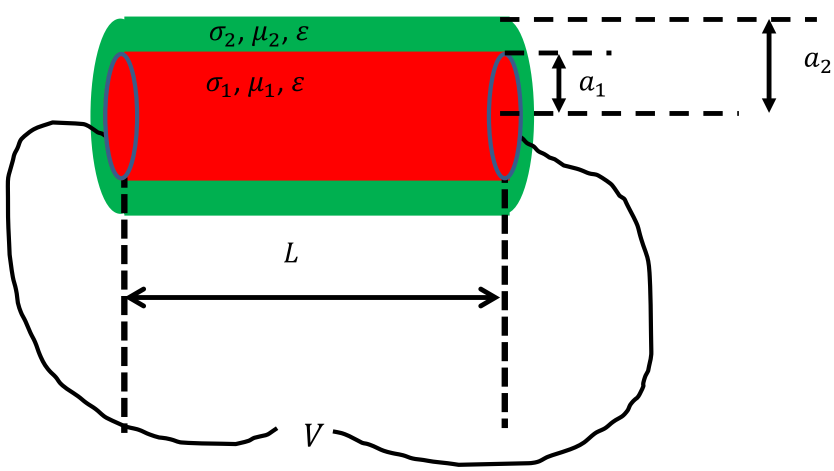
(4) Key points: the potential profile is symmetric; the potential slope is discontinuous at $z = a$ and $z = -a$, namely kinks; the potential decays as $1/z$ far from the origin; the potential has a local maximum (between $z = -a$ and $z = a$) at $z = 0$.

A cylindrical conducting wire of length L and conductance σ_1 is uniformly coated with another layer with conductance σ_2. The radius of the inner layer is a_1 and the radius of the outer layer is a_2.

Assuming both layers have the same dielectric constant ε and their magnetic permeabilities are $\mu_1$and μ_2.

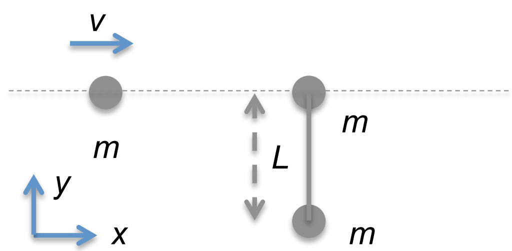
(1) If the wire is connected to a DC voltage source of V_0, what are the electric, magnetic and Poynting vector everywhere in the wire?
(2) If the wire is connected to a AC voltage source of $V_0\cos(\omega t)$, what are the electric, magnetic, and Poynting vector everywhere in the wire? Assuming that the frequency ω is very small such that conductivity, electric and magnetic permeability stay the same as the DC case.

You can assume that the current is uniform in each of the two layers, neglect boundary effects.



Problem 7

An elastic collision occurs in free space, with no gravitational forces to be considered. A point particle of mass m is incident upon a dumbbell (a system of two point masses, each of mass m, connected by a rigid, massless rod of length L). The dumbbell is initially at rest. The incident particle has velocity $\vec{v} = v\hat{x}$ and it makes a head-on collision with one of the masses as depicted below:



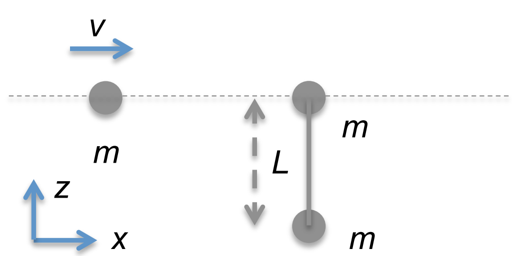
(a) What conservation laws are applicable (and why)?
(b) Find in terms of the known quantities v, L and m
- the magnitude and direction of the velocity of the incident particle after the collision,
- the magnitude and direction of the velocity of the dumbbell's center of mass, and
- the magnitude and direction of the angular velocity of the dumbbell.

(c) Qualitatively describe the motion of the three particles after the collision.

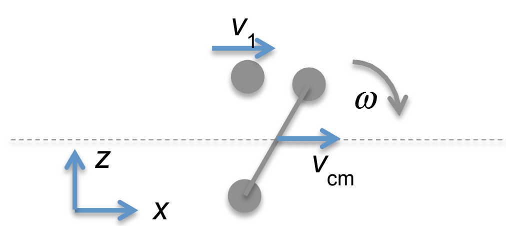
SOLUTION

An <u>elastic</u> collision occurs in free space, with <u>no</u> gravitational forces to be considered. A point particle of mass m is incident upon a dumbbell (a system of two point masses, each of mass m, connected by a rigid, massless rod of length L). The dumbbell is initially at rest. The incident particle has velocity $\vec{v} = v\hat{x}$ and it makes a head-on collision with one of the masses.

Initial (y into page for right-handed system):



Final:



(a) What conservation laws are applicable (and why)?
(i) Conservation of linear momentum because there are no external forces.
(ii) Conservation of angular momentum because there are no external torques.
(iii) Conservation of kinetic energy because the collision is elastic.

(d) Use the conservation laws to find the magnitude and direction of the velocity of the incident particle after the collision, the magnitude and direction of the velocity of the dumbbell's center of mass, and the magnitude and direction of the angular velocity of the dumbbell in terms of the known quantities v, L and m.

Conservation of linear momentum:

$$m\vec{v} = m\vec{v}_1 + 2m\vec{v}_{cm} \qquad (1)$$

where $\vec{v}_1$ is the velocity of the incident mass after the collision and $\vec{v}_{cm}$ is the velocity of the center of mass of the dumbbell after the collision. There is a head-on collision between the two masses, and the internal forces in such a collision are exclusively in the x direction, so the change in momentum of each mass can be only in the x direction at the instant of collision. Thus mass #1 (the single mass) moves in the x direction after the collision as does the center of mass of the dumbbell.

$$v = v_1 + 2mv_{cm} \qquad \mathbf{(2)}$$

(v is known, v_1 and v_{cm} are to be determined)

(ii) Conservation of angular momentum (about an axis perpendicular to page and through center of mass of dumbbell):

$$mv\hat{x} \times \frac{L}{2}\hat{z} = \underbrace{mv_1\hat{x} \times \frac{L}{2}\hat{z}}_{L \text{ of particle \#1}} + \underbrace{2m\left(v_{cm}\hat{x}\right) \times 0}_{L \text{ of center of mass}} + \underbrace{I\omega\hat{y}}_{L \text{ about center of mass}} \qquad (3)$$

with I the moment of inertia and ω the angular velocity of the dumbbell about the same axis:

$$I = m\left(\frac{L}{2}\right)^2 + m\left(\frac{L}{2}\right)^2 = \frac{mL^2}{2} \qquad (4)$$

Combining (3) and (4)

$$\frac{mvL}{2} = \frac{mv_1L}{2} + \frac{mL^2\omega}{2} \Rightarrow v = v_1 + L\omega \qquad \mathbf{(5)}$$

(v and L are known, v_1 and ω are to be determined)

(iii) Conservation of kinetic energy:

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}2mv_{cm}^2 + \frac{1}{2}I\omega^2 \qquad (6)$$

and using (4) and some clean-up

$$v^2 = v_1^2 + 2v_{cm}^2 + \frac{L^2}{2}\omega^2 \qquad \mathbf{(7)}$$

(v and L are known, v_1 , v_{cm} and ω are to be determined)

Now solve (2), (5) and (7) for the 3 unknowns v_1, v_{cm} and ω.

(2) gives $\quad v = v_1 + 2v_{cm} \Rightarrow v - v_1 = 2v_{cm}$ (8)

(5) gives $\quad v = v_1 + L\omega \Rightarrow v - v_1 = L\omega$ (9)

(8) and (9) give $\quad \omega = \dfrac{2v_{cm}}{L}$ (10)

Kinetic energy (7) along with (10) for ω gives

$$v^2 - v_1^2 = 2v_{cm}^2 + \frac{L^2}{2}\left(\frac{2v_{cm}}{L}\right)^2 = 2v_{cm}^2 + 2v_{cm}^2 = 4v_{cm}^2 \qquad (11)$$

Now take (11) $(v - v_1)(v + v_1) = 4v_{cm}^2$ and (8) $v - v_1 = 2v_{cm}$

to get $(v + v_1) = 2v_{cm}$ (12)

(8) says $v - v_1 = 2v_{cm}$

Adding and subtracting (12) and (8)

$$\boxed{v_{cm} = \frac{v}{2}; \quad v_1 = 0}$$

$$\boxed{\text{and putting into (10) gives } \omega = \frac{2v_{cm}}{L} = \frac{v}{L}}$$

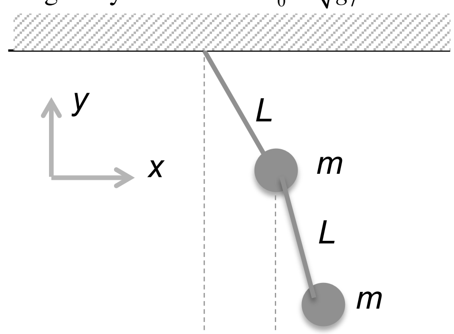
(c) So mass #1 m hits the dumbbell and **stops**.

The dumbbell picks up all the linear momentum in the x direction, and since it has mass $2m$, the **center of mass moves with velocity *v*/2 in the *x* direction**. The two **dumbbell masses rotate about the axis perpendicular to the *xz* plane with angular velocity** $\vec{\omega} = \frac{v}{L}\hat{y}$

(clockwise as viewed from our vantage point).

Problem 8

A double pendulum is made from a string and two masses. The string is attached to the ceiling and has length $2L$. A mass $m_1 = m$ is attached in the middle of the string and a mass $m_2 = m$ is attached at the end. The usual assumptions are applicable to make the problem tractable: the string is massless and cannot be stretched. The masses are point-like. There is no dissipation. <u>Only small-displacement motion from equilibrium</u> is to be considered.
Let g be the acceleration due to gravity and define $\omega_0 = \sqrt{g/L}$.



(a) Show that the small-displacement motion of the masses is given by:

$$x_1(t) = A\cos(\omega_a t + \varphi_a) + B\cos(\omega_b t + \varphi_b)$$

$$x_2(t) = \left(1+\sqrt{2}\right)A\cos(\omega_a t + \varphi_a) + \left(1-\sqrt{2}\right)B\cos(\omega_b t + \varphi_a)$$

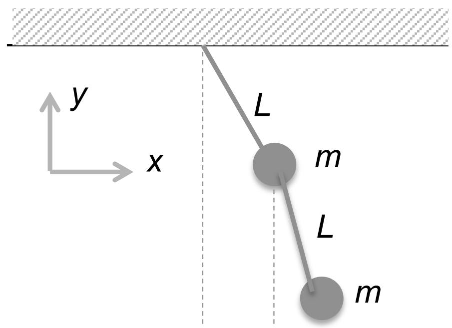
where A, B, φ_a, φ_b are arbitrary constants. Find the frequencies ω_a and ω_b.

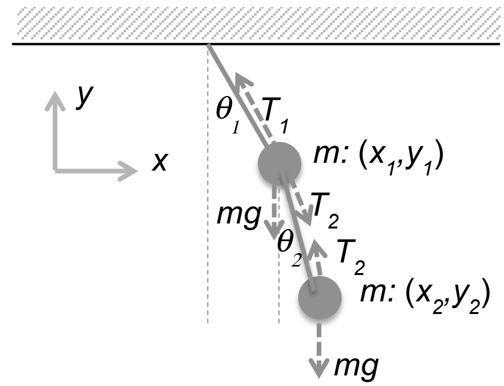
(b) Pick <u>one</u> of the following initial conditions and comment on the motion:

(i) $\quad x_2(0) = \left(1+\sqrt{2}\right)x_1(0); \quad v_1(0) = v_2(0) = 0$

(ii) $\quad x_2(0) = \left(1-\sqrt{2}\right)x_1(0); \quad v_1(0) = v_2(0) = 0$

SOULTION





(a) Show that the small-displacement motion of the masses is given by:

$$x_1(t) = A\cos(\omega_a t + \varphi_a) + B\cos(\omega_b t + \varphi_b)$$

$$x_2(t) = \left(1+\sqrt{2}\right)A\cos(\omega_a t + \varphi_a) + \left(1-\sqrt{2}\right)B\cos(\omega_b t + \varphi_a)$$

where A, B, φ_a, φ_b are arbitrary constants. Find the frequencies ω_a and ω_b.

Coordinates are defined in the figure above.

- Relate x_i and angles for small displacements:

$$\frac{x_1}{L} = \sin\theta_1 \approx \theta_1 \Rightarrow x_1 \approx L\theta_1 \text{ and } \frac{x_2 - x_1}{L} = \sin\theta_2 \approx \theta_2 \Rightarrow x_2 \approx x_1 + L\theta_2$$

- Small angle approximations: $\sin\theta \approx \theta$; $\cos\theta \approx 1 - \frac{\theta^2}{2}$;

motion approximately in x direction only so $y \approx$ constant $\dot{y} \approx small$; $\ddot{y} \approx 0$

Lagrangian method

Kinetic energy ($\dot{y}^2 \approx 0$ is second order so neglect)

$$K \approx \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 \tag{1'}$$

Potential energy

$$V = mgL(1-\cos\theta_1) + mgL(1-\cos\theta_1 + 1 - \cos\theta_2)$$

$$V \approx \frac{mgL}{2}\theta_1^2 + \frac{mgL}{2}\left(\theta_1^2 + \theta_2^2\right) = \frac{2mg}{2L}x_1^2 + \frac{mgL}{2}\left(\frac{x_2 - x_1}{L}\right)^2 \tag{2'}$$

$$V \approx \frac{2mg}{2L}x_1^2 + \frac{mg}{2L}\left(x_2^2 - 2x_1x_2 + x_1^2\right) = \frac{mg}{2L}\left(3x_1^2 - 2x_1x_2 + x_2^2\right)$$

Lagrangian

$$L = K - V = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{mg}{2L}\left(3x_1^2 - 2x_1x_2 + x_2^2\right) \tag{3'}$$

Equations of motion

$$\dot{p}_1 = \frac{\partial L}{\partial x_1} = -\frac{mg}{L}\left(3x_1 - x_2\right) = m\ddot{x}_1$$
$$\dot{p}_2 = \frac{\partial L}{\partial x_2} = -\frac{mg}{L}\left(-x_1 + x_2\right) = m\ddot{x}_2 \qquad (4')$$

Rearrange (4) and define $\omega_0^2 = g/L$ to get

$$\boxed{\begin{aligned}\ddot{x}_1 + \omega_0^2\left(3x_1 - x_2\right) = 0\\ \ddot{x}_2 + \omega_0^2\left(x_2 - x_1\right) = 0\end{aligned} \qquad (5')}$$

Newton's law method

• Forces:

$$m\ddot{x}_1 = -T_1 \sin\theta_1 + T_2 \sin\theta_2 \approx -T_1\theta_1 + T_2\theta_2 = -T_1\frac{x_1}{L} + T_2\frac{x_2 - x_1}{L} = -\left(T_1 + T_2\right)\frac{x_1}{L} + T_2\frac{x_2}{L} \qquad (1)$$

$$m\ddot{y}_1 = mg - T_1\cos\theta_1 + T_2\cos\theta_2 \approx mg - T_1 + T_2 \qquad (2)$$

$$m\ddot{x}_2 = -T_2\sin\theta_2 \approx -T_2\theta_2 = -T_2\frac{x_2 - x_1}{L} \qquad (3)$$

$$m\ddot{y}_2 = mg - T_2\cos\theta_2 \approx mg - T_2 \qquad (4)$$

(4) with $\ddot{y}_2 = 0$ gives $T_2 = mg$ (5)

(2) with $\ddot{y}_1 = 0$ and (5) gives $T_1 = T_2 + mg \Rightarrow T_1 = 2mg$ (6)

(1), (5), (6) give $\ddot{x}_1 = -\frac{3g}{L}x_1 + \frac{g}{L}x_2$ (7)

(3), (5), (6) give $\ddot{x}_2 = -\frac{g}{L}\left(x_2 - x_1\right)$ (8)

Rearrange (7), (8) and define $\omega_0^2 = g/L$

$$\boxed{\begin{aligned}\ddot{x}_1 + \omega_0^2\left(3x_1 - x_2\right) = 0\\ \ddot{x}_2 + \omega_0^2\left(x_2 - x_1\right) = 0\end{aligned} \qquad (10)}$$

which is the same as (5')

<u>Normal mode solutions:</u> We seek solutions where both masses oscillate with the same frequency. Using a complex coefficient A_i,

$$x_i\left(t\right) = A_i e^{i\omega t}$$
$$\ddot{x}_i = -\omega^2 x_i \qquad (11)$$

Recast (10) as

$$\begin{pmatrix} 3\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \qquad (12)$$

Set the determinant of the matrix equal to zero to find the eigenvalues.

$$\begin{vmatrix} 3\omega_0^2-\omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2-\omega^2 \end{vmatrix} = 0$$

$$\left(3\omega_0^2-\omega^2\right)\left(\omega_0^2-\omega^2\right)-\omega_0^4=0$$

$$2\omega_0^4-4\omega_0^2\omega^2+\omega^4=0$$

$$\frac{\omega^2}{\omega_0^2}=\frac{4\pm\sqrt{16-8}}{2}=2\pm\sqrt{2}$$

There are two normal mode frequencies

$$\boxed{\begin{aligned}\omega_a^2&=\left(2-\sqrt{2}\right)\omega_0^2\\ \omega_b^2&=\left(2+\sqrt{2}\right)\omega_0^2\end{aligned}} \qquad (13)$$

The first eigenvector is, with eigenvalue

$$\omega^2=\omega_b^2=\left(2+\sqrt{2}\right)\omega_0^2$$

$$\begin{pmatrix} 3\omega_0^2-\left(2+\sqrt{2}\right)\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2-\left(2+\sqrt{2}\right)\omega_0^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0$$

$$\begin{pmatrix} 1-\sqrt{2} & -1 \\ -1 & -1-\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0$$

Either of the 2 equations gives $x_{1,b}=\dfrac{x_{2,b}}{1-\sqrt{2}}$, (14)

so in normal mode *b* the oscillations are

$$\boxed{\begin{aligned}x_{1,b}(t)&=B\cos\left(\omega_b t+\varphi_b\right)\\ x_{2,b}(t)&=\left(1-\sqrt{2}\right)B\cos\left(\omega_b t+\varphi_b\right)\end{aligned}\quad \text{where } B \text{ and } \varphi_b \text{ are arbitrary constants.}} \qquad (15)$$

The second eigenvector is, with eigenvalue

$$\omega^2=\omega_a^2=\left(2-\sqrt{2}\right)\omega_0^2$$

$$\begin{pmatrix} 3\omega_0^2-\left(2-\sqrt{2}\right)\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2-\left(2-\sqrt{2}\right)\omega_0^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0$$

$$\begin{pmatrix} 1+\sqrt{2} & -1 \\ -1 & -1+\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0$$

Either of the 2 equations gives $x_{1,a} = \dfrac{x_{2,a}}{1+\sqrt{2}}$, (16)

so in normal mode a the oscillations are

$$\boxed{\begin{aligned} x_{1,a}(t) &= A\cos(\omega_a t+\varphi_a) \\ x_{2,a}(t) &= \left(1+\sqrt{2}\right)A\cos(\omega_a t+\varphi_a) \end{aligned} \quad \text{where } A \text{ and } \varphi_a \text{ are arbitrary constants}} \tag{17}$$

The general motion of either mass is the sum of its normal mode motions with appropriate coefficients. That is, the sum of (15) and (17), which gives the desired result:

$$\boxed{\begin{aligned} x_1(t) &= A\cos(\omega_a t+\varphi_a) + B\cos(\omega_b t+\varphi_b) \\ x_2(t) &= \left(1+\sqrt{2}\right)A\cos(\omega_a t+\varphi_a) + \left(1-\sqrt{2}\right)B\cos(\omega_b t+\varphi_a) \end{aligned}} \tag{18}$$

(b) Explore the following initial conditions and comment on the motion:
From the equations of motion:

$$x_1(0) = A\cos(\varphi_a) + B\cos(\varphi_b)$$

$$x_2(0) = \left(\sqrt{2}+1\right)A\cos(\varphi_a) - \left(\sqrt{2}-1\right)B\cos(\varphi_b)$$

$$v_1(0) = -\omega_a A\sin(\varphi_a) - \omega_b B\sin(\varphi_b)$$

$$v_2(0) = -\omega_a\left(\sqrt{2}+1\right)A\sin(\varphi_a) + \omega_b\left(\sqrt{2}-1\right)B\sin(\varphi_b)$$

(i) From the initial conditions:

$$x_2(0) = \left(\sqrt{2}+1\right)x_1(0) \Rightarrow$$

$$\left(\sqrt{2}+1\right)A\cos(\varphi_a) - \left(\sqrt{2}-1\right)B\cos(\varphi_b) = \left(\sqrt{2}+1\right)A\cos(\varphi_a) + \left(\sqrt{2}+1\right)B\cos(\varphi_b)$$

$$\Rightarrow B = 0 \ \left(\text{cannot be true that} \quad -\left(\sqrt{2}-1\right) = \sqrt{2}+1\right)$$

$$v_1(0) = 0 \ (\text{with } B = 0)$$

$$v_1(0) = -\omega_a A\sin(\varphi_a) = 0 \Rightarrow \varphi_a = 0 \text{ for } A \neq 0$$

$$\boxed{\begin{aligned} x_1(t) &= A\cos(\omega_a t) \\ x_2(t) &= \left(\sqrt{2}+1\right)A\cos(\omega_a t) \\ &\text{symmetric mode; masses oscillate in phase with frequency } \omega_a. \end{aligned}}$$

(ii) From the initial conditions:

$x_2(0) = -\left(\sqrt{2}-1\right)x_1(0) \Rightarrow$

$\left(\sqrt{2}+1\right)A\cos(\varphi_a) - \left(\sqrt{2}-1\right)B\cos(\varphi_b) = -\left(\sqrt{2}-1\right)A\cos(\varphi_a) - \left(\sqrt{2}-1\right)B\cos(\varphi_b)$

$\Rightarrow A = 0$ $\Big($cannot be true that $\sqrt{2}+1 = -\left(\sqrt{2}-1\right)\Big)$

$v_2(0) = 0$ (with $A = 0$)

$v_2(0) = -\omega_b B\sin(\varphi_b) = 0 \Rightarrow \varphi_b = 0$ for $B \neq 0$

$$\boxed{\begin{array}{l} x_1(t) = B\cos(\omega_b t) \\ x_2(t) = -\left(\sqrt{2}-1\right)B\cos(\omega_b t) \\ \text{antisymmetric mode; masses oscillate in antiphase with frequency } \omega_b. \end{array}}$$