OSU Physics Department
Comprehensive Examination #126
Solutions

Wednesday, September 21 and Thursday, September 22, 2016

Fall 2016 Comprehensive Examination

Statistical Mechanics	9 AM-12 PM	Wednesday, September 21
Electricity and Magnetism	1 PM-4 PM	Wednesday, September 21
Quantum Mechanics	9 AM-12 PM	Thursday, September 22
Clasical Mechanics	1 PM-4 PM	Thursday, September 22

General Instructions

This Fall 2016 Comprehensive Examination consists of four separate parts of two problems each. Each problem caries equal weight (20 points each) and lasts three hours. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work on the provided pages, work each problem in its own labeled pages, and be certain that your chosen student letter (but not your name) is on the header of each page of your exam, including any unused pages. If you need additional paper for your work, use the blank pages provided. Each page of work should include the problem number, a page number, your chosen student letter, and the total number of pages actually used. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam and the collection of formulas distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please staple and return all pages of your exam—including unused pages—at the end of the exam.

Consider an ideal molecular gas at high temperature (relative to τ defined below), whose internal energy is given by

$$U = \left(\frac{5}{2} - \frac{\tau^2}{T^2} \right) N k_B T$$

where τ is a constant with dimensions of temperature. This gas is in an insulated piston, and is slowly compressed to one half of its original volume. The initial temperature of the gas is T_0.

a) **(10 pts)** Is the final temperature T_f after the compression greater than, less than, or equal to the initial temperature T_0? Explain your reasoning. *Note: the bulk of the points will be given for your reasoning. Be sure to be explicit regarding* which physical law *underlies each step in your argument.*

b) **(10 pts)** Find the final temperature T_f in terms of T_0 and τ. An equation involving these three quantities that could be solved numerically is a sufficient answer.

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    Solution:
    The final temperature must be greater than T_0. By the First Law,

$$\Delta U = Q + W \tag{1.1}$$

    Since this process is adiabatic (or you could just say "thermally insulated") there is no heat flow and $Q = 0$. Thus the change in internal energy must be equal to the amount of work done on the gas. Since we are compressing the gas we are doing work on the gas, so its internal energy must increase. In order for the internal energy to increase, but the internal energy depends only on temperature for this system (or any ideal gas). Since the heat capacity cannot ever be negative (or you could alternatively argue that the heat capacity of this particular gas is positive based on the function for U given), this means that the temperature must increase.

b) **(10 pts)** Find the final temperature T_f in terms of T_0 and τ. An equation involving these three quantities that could be solved numerically is a sufficient answer.

    Solution:
    This is essentially asking us to quantify how much the temperature changes. We do so by invoking (as usual) the First Law in differential form (or the Thermodynamic Identity).

$$dU = T\,dS^{\,0} - p\,dV \tag{1.2}$$

    We do know that this is an ideal gas, so we can relate p to T and V (using $pV = N k_B T$), which is good. Similarly, because the process is adiabatic we know that the entropy isn't changing. And finally, we have an equation for U which allows us to relate changes to U to changes in temperature:

$$dU = N k_B \left(\frac{5}{2} + 2\frac{\tau^2}{T^2} \right) dT \tag{1.3}$$

Putting these equations together and then using the ideal gas law, we find that

$$-pdV = Nk_B \left(\frac{5}{2} + \frac{\tau^2}{T^2} \right) dT \tag{1.4}$$

$$-\frac{Nk_B T}{V} dV = Nk_B \left(\frac{5}{2} + \frac{\tau^2}{T^2} \right) dT \tag{1.5}$$

$$-\frac{1}{V} dV = \frac{\frac{5}{2} + \frac{\tau^2}{T^2}}{T} dT \tag{1.6}$$

$$-\frac{1}{V} dV = \left(\frac{5}{2} \frac{1}{T} + \frac{\tau^2}{T^3} \right) dT \tag{1.7}$$

At this point, we can relate changes in volume to changes in temperature, and we just need to integrate those changes to find the total change in temperature.

$$-\int_{V_0}^{\frac{V_0}{2}} \frac{1}{V} dV = \int_{T_0}^{T_f} \left(\frac{5}{2} \frac{1}{T} + \frac{\tau^2}{T^3} \right) dT \tag{1.8}$$

$$\int_{\frac{V_0}{2}}^{V_0} \frac{1}{V} dV = \int_{T_0}^{T_f} \left(\frac{5}{2} \frac{1}{T} + \frac{\tau^2}{T^3} \right) dT \tag{1.9}$$

We are almost there! You could indeed solve this equation numerically, but the problem asked for an equation involving only T_f, T_0, and τ, which means we still need to eliminate V_0 by solving the left-hand integral.

$$\ln(V_0) - \ln \left(\frac{V_0}{2} \right) = \int_{T_0}^{T_f} \left(\frac{5}{2} \frac{1}{T} + \frac{\tau^2}{T^3} \right) dT \tag{1.10}$$

$$\ln 2 = \int_{T_0}^{T_f} \left(\frac{5}{2} \frac{1}{T} + \frac{\tau^2}{T^3} \right) dT \tag{1.11}$$

At this point we have an acceptable answer. Although the problem did not explicitly request an equation with an integral, this is indeed an equation relating the requested quantities, which can be solved numerically. However, for good measure I'll do the integrals, which aren't hard.

$$\ln 2 = \frac{5}{2} \ln \frac{T_f}{T_0} - \frac{1}{2} \left(\frac{\tau^2}{T_f^2} - \frac{\tau^2}{T_0^2} \right) \tag{1.12}$$

Now this isn't too nasty an equation, but it's not one that I want to solve myself analytically. It would have been quite simple to find the fractional change in volume needed to double the temperature, but that is not what was asked.

Consider a system of N uncoupled harmonic oscillators with the same frequency ω, i.e. the Einstein model of a crystal.

a) **(10 pts)** Find the heat capacity of this system at low temperatures, being careful to keep at least the lowest order of temperature dependence (i.e. not just the constant term).

b) **(8 pts)** Find the heat capacity at high temperatures, again being careful to keep the lowest order of temperature dependence.

c) **(2 points)** Briefly explain the physical basis for such a large difference between the heat capacity at low and high temperatures.

Consider a system of N uncoupled harmonic oscillators with the same frequency ω, i.e. the Einstein model of a crystal.

a) **(10 pts)** Find the heat capacity of this system at low temperatures, being careful to keep at least the lowest order of temperature dependence (i.e. not just the constant term).

Solution:
There are multiple approaches we could use here. I will show what I consider a pretty simple method: find the internal energy as a weighted average of energy, and then find the temperature derivative of that. The internal energy is given by

$$U \equiv \sum_i^{\text{all states}} P_i E_i \tag{2.1}$$

Now the "all states" sum is over all possible states of the N-oscillator system. Fortunately, since the oscillators don't interact, we can treat each one separately, and just find the mean energy for a single oscillator and then multiply by N.

$$U_1 = \sum_{n=0}^{\infty} P_n \hbar\omega \left(n + \frac{1}{2} \right) \tag{2.2}$$

$$= \frac{1}{2}\hbar\omega + \hbar\omega \sum_{n=0}^{\infty} P_n n \tag{2.3}$$

Here we pull out the Boltzmann ratio for probabilities to get:

$$P_n = \frac{e^{-\beta E_n}}{\sum_m e^{-\beta E_m}} \tag{2.4}$$

$$= \frac{e^{-\beta\hbar\omega(n+1/2)}}{\sum_m e^{-\beta\hbar\omega(m+1/2)}} \tag{2.5}$$

$$= \frac{e^{-\beta\hbar\omega n}}{\sum_m e^{-\beta\hbar\omega m}} \tag{2.6}$$

Plugging this in we see that

$$U_1 = \frac{1}{2}\hbar\omega + \hbar\omega \frac{\sum_{n=0}^{\infty} n e^{-\beta\hbar\omega n}}{\sum_{n=0}^{\infty} e^{-\beta\hbar\omega n}} \tag{2.7}$$

$$U = N\frac{1}{2}\hbar\omega + N\hbar\omega \frac{\sum_{n=0}^{\infty} n e^{-\beta\hbar\omega n}}{\sum_{n=0}^{\infty} e^{-\beta\hbar\omega n}} \tag{2.8}$$

Now there are several approaches we can use here. If we remember how to add a geometric series, we just find the answer analytically (and then find

the low-temperature limit). However, this portion of the problem only asks for the low-temperature limit, so I will show instead how to find the low-temperature limit directly (which is easier).

At low temperatures, $\beta\hbar\omega \gg 1$, which means we are taking the exponential of a large and negative value, making each exponential very small (except the first, which is 1). Since each exponential has an n in it, each exponential is way smaller than the last, so we can keep just the first few terms.

$$U = \frac{N}{2}\hbar\omega + N\hbar\omega\frac{e^{-\beta\hbar\omega} + 2e^{-2\beta\hbar\omega} + \cdots}{1 + e^{-\beta\hbar\omega} + e^{-2\beta\hbar\omega} + \cdots} \tag{2.9}$$

$$= \frac{N}{2}\hbar\omega + N\hbar\omega\left(e^{-\beta\hbar\omega} + 2e^{-2\beta\hbar\omega} + \cdots\right)\left(1 - (e^{-\beta\hbar\omega} + e^{-2\beta\hbar\omega} + \cdots) + (e^{-\beta\hbar\omega} + \cdots)^2 + \cdots\right) \tag{2.10}$$

$$= \frac{N}{2}\hbar\omega + N\hbar\omega\left(e^{-\beta\hbar\omega} + 2e^{-2\beta\hbar\omega} + \cdots\right)\left(1 - e^{-\beta\hbar\omega} + \cdots\right) \tag{2.11}$$

$$= \frac{N}{2}\hbar\omega + N\hbar\omega\left(e^{-\beta\hbar\omega} + e^{-2\beta\hbar\omega} + \cdots\right) \tag{2.12}$$

To find C_V we just need to take a temperature derivative...

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V \tag{2.13}$$

$$= N\frac{\hbar^2\omega^2}{k_B T^2}\left(e^{-\beta\hbar\omega} + 2e^{-2\beta\hbar\omega} + \cdots\right) \tag{2.14}$$

Note that the problem only asked for the lowest order of temperature dependence, so I have an extra term.

b) **(8 pts)** Find the heat capacity at high temperatures, again being careful to keep the lowest order of temperature dependence.

Solution:
At high temperatures we have a more interesting problem to solve. You might try turning the sum in the partition function into an integral, which is very natural (and the first way I attempted this problem). The trouble is that this predicts that the heat capacity is a constant, independent of temperature, but the problem asked for the lowest order of temperature dependence. ⌣̈ Fortunately, it is not hard to solve the problem analytically, and then just perform a Taylor expansion.

The key to solving a geometric series analytically is to multiply by what is in this case a Boltzmann factor. Then you can recognize that the difference between this product and the sum itself is just a constant.

Note that I am using below for convenience a definition of Z that does not incorporate the zero-point energy. This would be a problem if I were

to assume that $F = -kT \ln Z$ or similar, but I won't do that here... when I find U I will connect this Z with the formula (Eq. 2.8, see also Eq. **??**) above that already factors out the zero-point energy.

$$Z \equiv \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} \tag{2.15}$$

$$Z e^{-\beta\hbar\omega} = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1)} \tag{2.16}$$

$$= \sum_{n=1}^{\infty} e^{-\beta\hbar\omega n} \tag{2.17}$$

$$= Z - 1 \tag{2.18}$$

$$Z\left(e^{-\beta\hbar\omega} - 1\right) = -1 \tag{2.19}$$

$$Z = \frac{1}{1 - e^{-\beta\hbar\omega}} \tag{2.20}$$

Once we have the partition function, everything else is pretty easy. We can recognize that the numerator of the internal energy is actually a simple derivative of the partition function.

$$U = \frac{N}{2}\hbar\omega + N\frac{\sum_{n=0}^{\infty} \hbar\omega n e^{-\beta\hbar\omega n}}{Z} \tag{2.21}$$

$$= \frac{N}{2}\hbar\omega + N\frac{-\frac{\partial Z}{\partial \beta}}{Z} \tag{2.22}$$

$$= \frac{N}{2}\hbar\omega + N\frac{Z^2 \hbar\omega e^{-\beta\hbar\omega}}{Z} \tag{2.23}$$

$$= \frac{N}{2}\hbar\omega + N\hbar\omega\frac{e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} \tag{2.24}$$

$$= \frac{N}{2}\hbar\omega + N\hbar\omega\frac{1}{e^{\beta\hbar\omega} - 1} \tag{2.25}$$

Now that we have our analytic solution for U, we can either take a derivative of U to find C_V analytically and then find the power series of C_V, or we could do things in the reverse order, finding the power series of U and then taking a temperature derivative of that to find C_V. Either is a comparable amount of effort, but I will do the derivative first, just so I won't have to keep copying and pasting the zero-point energy.

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V \tag{2.26}$$

$$= N\hbar\omega\frac{e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega} - 1\right)^2}\frac{\hbar\omega}{k_B T^2} \tag{2.27}$$

$$= Nk_B(\beta\hbar\omega)^2\frac{e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega} - 1\right)^2} \tag{2.28}$$

Now we have the joy of doing a power series approximation, given that $\beta\hbar\omega \ll 1$. Normally I would just recursively expand power series that I know (the exponential, the second power, the product, etc), but that seems rather tedious to write out in LaTeX. So I will just take a few derivatives and do the Taylor series the old-fashioned way.

$$f(x) \equiv x^2 \frac{e^x}{(e^x - 1)^2} \tag{2.29}$$

$$f(0) = 1 \tag{2.30}$$

$$f'(x) = f(x) + 2\frac{f(x)}{x} - 2\frac{x^2 e^{2x}}{(e^x - 1)^3} \nearrow^{0} \tag{2.31}$$

$$f'(0) = 1 \tag{2.32}$$

$$f(x) = 1 + x + \cdots \tag{2.33}$$

Okay, that is a surprisingly simple power series, so long as we only look at two terms. The harder bits in the above were the evaluating $f(0)$ and $f'(0)$, which required either using l'Hopital's rule or doing a quick (but relatively simple) power series expansion in one's head to determine what the $e^x - 1$ is doing as $x \to 0$. Thus we find that:

$$C_V = Nk_B \left(1 + \beta\hbar\omega + \cdots\right) \tag{2.34}$$

$$= Nk_B \left(1 + \frac{\hbar\omega}{k_B T} + \cdots\right) \tag{2.35}$$

So the heat capacity is approaching Nk_B from above with T^{-1} behavior as the temperature rises.

c) **(2 points)** Briefly explain the physical basis for such a large difference between the heat capacity at low and high temperatures.
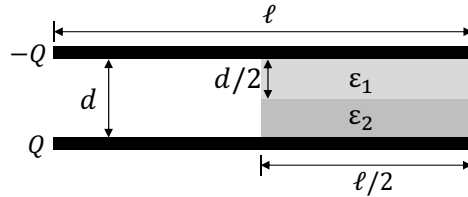
Solution:
We see that at low temperatures the heat capacity drops to zero, while at high temperatures the heat capacity approaches k_B per mode. This is because at low temperatures there is not enough thermal energy to excite even a single phonon (since $k_B T \ll \hbar\omega$), so we have a surpressed heat capacity, since changing the temperature still doesn't give much chance of exciting a single phonon. At high temperatures we expect to approach the equipartition theorem limit, which is corresponds to a heat capacity of $\frac{1}{2}k_B$ for each independent quadratic term in the energy, which in this case is just the kinetic and potential energy for each mode, thus k_B per mode.

Problem 3

Two parallel conducting plates (each plate is a square of side ℓ) are separated by a small distance d (i.e., $d \ll \ell$), and carry charges, Q and $-Q$, respectively. Two dielectric slabs of electric permittivities ε_1 and ε_2 and of uniform thickness $d/2$ are inserted between the plates, but the slabs fill only a half of the volume between the plates as shown in the figure below.
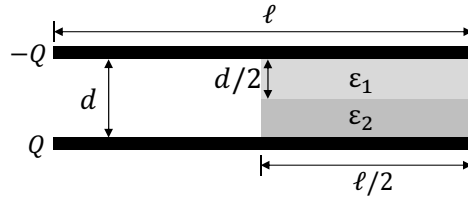
a) Find the surface charge density on the interior surfaces of the conducting plates.

b) Find the electric fields in the dielectric media and the bound surface charge density at the interface between the two dielectric slabs.

c) What is the capacitance of the system?

Two parallel conducting plates (each plate is a square of side ℓ) are separated by a small distance d (i.e., $d \ll \ell$), and carry charges, Q and $-Q$, respectively. Two dielectric slabs of electric permittivities ε_1 and ε_2 and of uniform thickness $d/2$ are inserted between the plates, but the slabs fill only a half of the volume between the plates as shown in the figure below.
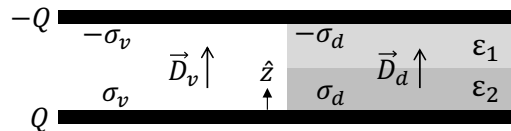

a) Find the surface charge density on the interior surfaces of the conducting plates.

Solution:

Displacement field and surface charge density

First of all, we show that the surface charge density in contact with the medium 1 $(-\sigma_d)$ has the same magnitude but opposite sign with that in contact with the medium 2 (σ_d). The displacement field $\overrightarrow{D}$ is normal to the plates in the parallel plate configuration and must be continuous at the dielectric interface because $\nabla \cdot \overrightarrow{D} = 0$ near the interface. In other words, the displacement field in the dielectric media $\overrightarrow{D}_d$ is uniform and normal to the plates.

Using $\nabla \cdot \overrightarrow{D} = \rho$ near the conducting plates, we obtain the relations, $\overrightarrow{D}_d \cdot \hat{z} = D_d = \sigma_d$ on the lower plate and $\overrightarrow{D}_d \cdot (-\hat{z}) = -D_d = -\sigma_d$ on the upper plate. Here we get a relation,

$$\sigma_d = D_d. \tag{3.1}$$

Likewise, the surface charge density on the upper plate in contact with the vacuum $(-\sigma_v)$ has the same magnitude but opposite sign with that on the lower plate (σ_v), and

$$\sigma_v = D_v. \tag{3.2}$$

Charge conservation

Now the charge in the conducting plate can be expressed as

$$Q = \sigma_v \times \frac{\ell^2}{2} + \sigma_d \times \frac{\ell^2}{2} = (\sigma_v + \sigma_d)\frac{\ell^2}{2} \tag{3.3}$$

Potential difference

The potential difference between the plates V is constant, and

$$V = E_v d = E_1 \frac{d}{2} + E_2 \frac{d}{2} \Rightarrow E_v = \frac{1}{2}(E_1 + E_2), \tag{3.4}$$

where E_v, E_1, E_2 are electric fields in vacuum, medium 1, and medium 2, respectively. Since $D = \varepsilon E$,

$$\frac{D_v}{\varepsilon_0} = \frac{D_d}{2}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right), \tag{3.5}$$

where ε_0 is the vacuum permittivity. Using Eqs.(1) and (2), we obtain

$$\sigma_v = \frac{\varepsilon_0}{2}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\sigma_d \tag{3.6}$$

Inserting Eq.(6) into Eq.(3), we get

$$Q = \left[\frac{\varepsilon_0}{2}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right) + 1\right]\sigma_d \frac{\ell^2}{2}. \tag{3.7}$$

Thus,

$$\sigma_d = \frac{2Q}{\left[\frac{\varepsilon_0}{2}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right) + 1\right]\ell^2} \tag{3.8}$$

and

$$\sigma_v = \frac{\varepsilon_0}{2}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\sigma_d = \frac{\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)Q}{\left[\varepsilon_0\frac{\varepsilon_0}{2}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right) + 1\right]\ell^2} \tag{3.9}$$

b) Find the electric fields in the dielectric media and the bound surface charge density at the interface between the two dielectric slabs.

Solution:

The electric fields in the dielectric media are

$$E_{1,2} = \frac{D_d}{\varepsilon_{1,2}} = \frac{\sigma_d}{\varepsilon_{1,2}} = \frac{2Q}{\left[\frac{\varepsilon_0}{2}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right) + 1\right]\varepsilon_{1,2}\ell^2} \tag{3.10}$$

Using the Gauss law, $\nabla \cdot \vec{E} = \rho/\varepsilon_0$, we get

$$\sigma_b = \varepsilon_0(E_1 - E_2) = \frac{2\varepsilon_0\left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}\right)Q}{\left[\frac{\varepsilon_0}{2}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right) + 1\right]\ell^2} \tag{3.11}$$

c) What is the capacitance of the system?

Solution:
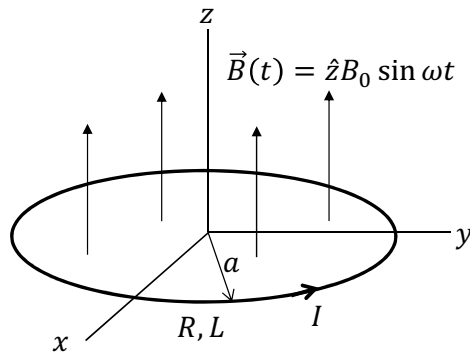
The capacitance is

$$C = \frac{Q}{V} = \frac{Q}{E_v d} = \frac{\varepsilon_0 Q}{D_v d} = \frac{\varepsilon_0 Q}{\sigma_v d}. \tag{3.12}$$

Using Eq.(9), we get

$$C = \frac{\left[\frac{\varepsilon_0}{2}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right) + 1\right]\ell^2}{\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)d}. \tag{3.13}$$

A circular metal wire of radius a, resistance R, and self-inductance L lies on the xy plane as shown in the figure below. An oscillating magnetic field $\vec{B}(t) = \hat{z}B_0 \sin \omega t$ is applied along the z-axis.

a) Find the time-dependent current $I(t)$ through the wire.

b) What is the time-averaged power dissipation in the circular wire?

A circular metal wire of radius a, resistance R, and self-inductance L lies on the xy plane as shown in the figure below. An oscillating magnetic field $\vec{B}(t) = \hat{z}B_0 \sin \omega t$ is applied along the z-axis.
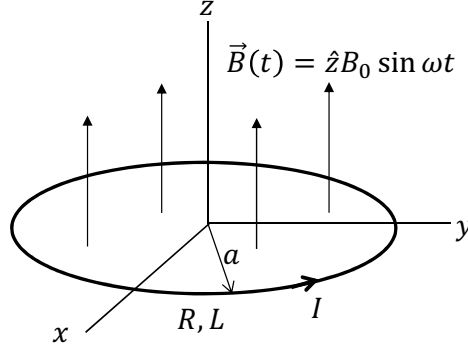
a) Find the time-dependent current $I(t)$ through the wire.

Solution:
The *emf* around the circuit is

$$V(t) = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{s} = -\frac{d}{dt}(\pi a^2 B_0 \sin \omega t) = -\pi a^2 B_0 \omega \cos \omega t \tag{4.1}$$

We use the complex number formalism for convenience:

$$V(t) = \Re\{V_0 e^{-i\omega t}\}, \quad \text{where } V_0 = -\pi a^2 \omega B_0. \tag{4.2}$$

Since the circuit has the resistance R and the inductance L,

$$V(t) = I(t)R + L\frac{dI(t)}{dt} \tag{4.3}$$

with $I(t) = \Re\left\{I_0 e^{-i\omega t}\right\}$, the differential equation becomes

$$V_0 e^{-i\omega t} = I_0 R e^{-i\omega t} + LI_0(-i\omega)e^{-i\omega t} \tag{4.4}$$

$$V_0 = I_0(R - i\omega L) \tag{4.5}$$

$$I_0 = \frac{V_0}{R - i\omega L} \tag{4.6}$$

Therefore,

$$
\begin{aligned}
I(t) &= \Re\left\{\frac{V_0}{R - i\omega L}e^{-i\omega t}\right\} \\
&= \frac{V_0}{R^2 + \omega^2 L^2}(R\cos\omega t + \omega L\sin\omega t) \\
&= \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}\cos(\omega t - \phi) \\
&= -\frac{\pi a^2 \omega B_0}{\sqrt{R^2 + \omega^2 L^2}}\cos(\omega t - \phi) \tag{4.7}
\end{aligned}
$$

where $\tan\phi = \frac{\omega L}{R}$.

b) What is the time-averaged power dissipation in the circular wire?

Solution:
The time-averaged power dissipation of harmonic fields and current is

$$
\begin{aligned}
P &= \Re\left\{\frac{1}{2}\int \vec{J}\cdot\vec{E}\,dv\right\} = \Re\left\{\frac{1}{2}I_0^* V_0\right\} \\
&= \Re\left\{\frac{1}{2}\frac{V_0}{R - i\omega L}V_0\right\} = \Re\left\{\frac{1}{2}V_0^2\frac{R + i\omega L}{R^2 + \omega^2 L^2}\right\} = \frac{1}{2}V_0^2\frac{R}{R^2 + \omega^2 L^2} \\
\Rightarrow P &= \frac{\pi^2 a^4 \omega^2 B_0^2 R}{2\left(R^2 + \omega^2 L^2\right)} \tag{4.8}
\end{aligned}
$$

A particle of mass m is "free", except that it is confined to a rectangular region of space in the $z = 0$ plane. The region has size L_x in the x direction and L_y in the y direction. The algebra is easiest if you choose one corner of the rectangular region at $(x, y) = (0,0)$. Spin is not relevant in this problem.

(a) (9 points) Solve the energy eigenvalue equation to find the energy eigenvalues and the corresponding eigenfunctions. Explain the steps and concepts applied for full credit.

(b) (4 points) Discuss the degeneracy of the energy eigenstates. Is it possible for the ground state to be degenerate?

(c) (7 points) You want to model a non-ideal case, so you add a perturbation potential with the following properties: (i) there is no perturbation on the edges of the confining region, (ii) the potential is largest at the center of the region, (iii) the perturbation is smooth (*i.e.* differentiable everywhere, except perhaps at the boundary) and (iv) the perturbation is "small".

Write a simple potential energy function that has these properties, and define what is meant by "small" in this case. Evaluate the first order correction to the <u>ground state</u> energy.

Integrals that may be useful:

$$\int \sin^2 ax\, dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} + C$$

$$\int \sin^3 ax\, dx = \frac{\cos(3ax)}{12a} - \frac{\cos(ax)}{4a} + C$$

$$\int \sin b_1 x \sin b_2 x\, dx = \frac{\sin((b_2 - b_1)x)}{2(b_2 - b_1)} - \frac{\sin((b_2 + b_1)x)}{2(b_2 + b_1)} + C; \quad b_1 \neq b_2$$

$$\int \sin b_1 x \cos b_2 x\, dx = -\frac{\cos((b_1 - b_2)x)}{2(b_1 - b_2)} - \frac{\cos((b_1 + b_2)x)}{2(b_1 + b_2)} + C; \quad |b_1| \neq |b_2|$$

Comprehensive Exam, Fall 2016 QM Undergraduate (Solution)

(a) Eigenstates $\varphi(\vec{r})$ and energy eigenvalues E are determined by the eigenvalue equation

$$\hat{H}\varphi(\vec{r}) = E\varphi(\vec{r})$$

where the Hamiltonian or energy operator is

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})$$

For a free particle, confined to the regions described, the potential energy is

$$V(\vec{r}) = \begin{cases} 0 & 0 < x < L_x \text{ and } 0 < y < L_y \\ \infty & \text{otherwise} \end{cases}$$

For the two-dimensional, rectangular geometry, Cartesian coordinates x and y are appropriate. The eigenvalue equation becomes:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi(x,y) = E\varphi(x,y)$$

The Hamiltonian is separable, so assume an eigenfunction that is a product of a function of x and a function of y

$$\varphi(x,y) = X(x)Y(y).$$

Substituting,

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)X(x)Y(y) = E\,X(x)Y(y)$$

$$-\frac{\hbar^2}{2m}\left(Y(y)\frac{\partial^2}{\partial x^2}X(x) + X(x)\frac{\partial^2}{\partial y^2}Y(y)\right) = E\,X(x)Y(y)$$

$$-\frac{\hbar^2}{2m}\left(\frac{1}{X(x)}\frac{\partial^2}{\partial x^2}X(x) + \frac{1}{Y(y)}\frac{\partial^2}{\partial y^2}Y(y)\right) = E$$

Because E is constant, each term in the parentheses must be constant because x and y can vary independently. Choose the constant associated with the x term to be E_x.

$$-\frac{\hbar^2}{2m}\left(\frac{1}{X(x)}\frac{\partial^2}{\partial x^2}X(x)\right) = E_x$$

which requires that

$$-\frac{\hbar^2}{2m}\left(\frac{1}{Y(y)}\frac{\partial^2}{\partial y^2}Y(y)\right) = E - E_x \equiv E_y$$

with

$$E = E_x + E_y$$

The solutions to these equations are

$$\frac{\partial^2}{\partial x^2}X(x) = -\frac{2mE_x}{\hbar^2}X(x)$$

$$\Rightarrow X(x) = A\sin\left(\sqrt{\frac{2mE_x}{\hbar^2}}x\right) + B\cos\left(\sqrt{\frac{2mE_x}{\hbar^2}}x\right)$$

and

$$\frac{\partial^2}{\partial y^2}Y(y) = -\frac{2mE_y}{\hbar^2}Y(y)$$

$$\Rightarrow Y(y) = C\sin\left(\sqrt{\frac{2mE_y}{\hbar^2}}y\right) + D\cos\left(\sqrt{\frac{2mE_y}{\hbar^2}}y\right)$$

Apply boundary conditions. Set the wave functions to zero at the boundaries to ensure continuity with the zero of the wave function in the forbidden region. With the suggested choice of boundaries,

$$\varphi(0,y) = \varphi(x,0) = \varphi(L_x,y) = \varphi(x,L_y) = 0.$$

The cosine term coefficients vanish to ensure a zero on the left and bottom edges: $B = D = 0$.
To ensure a zero on the top and right edges, the sine terms must have arguments that are (non-zero) integer multiples of π. Call these integers $\mathbf{n_x}$ **and** $\mathbf{n_y}$. Then

$$\sqrt{\frac{2mE_x}{\hbar^2}}L = n_x\pi \Rightarrow E_x = \frac{n_x^2\,\hbar^2\pi^2}{2mL_x^2}$$

$$\sqrt{\frac{2mE_y}{\hbar^2}}L = n_y\,\pi \Rightarrow E_y = \frac{n_y^2\,\hbar^2\pi^2}{2mL_y^2}$$

and relabel the energies to reflect the new quantum numbers

$$E_{n_x} = \frac{n_x^2\,\hbar^2\pi^2}{2mL_x^2}; E_{n_y} = \frac{n_y^2\,\hbar^2\pi^2}{2mL_y^2}$$

$$\boxed{\text{The total energy is } E_{n_x,n_y} = \frac{n_x^2\,\hbar^2\pi^2}{2mL_x^2} + \frac{n_y^2\,\hbar^2\pi^2}{2mL_y^2}}$$

and also label the wave functions

$$X_{n_x}(x) = A\sin\left(\sqrt{\frac{2mE_{n_x}}{\hbar^2}}x\right); Y_{n_y}(y) = C\sin\left(\sqrt{\frac{2mE_{n_y}}{\hbar^2}}y\right)$$

The total wave function (including normalization) is

$$\boxed{\varphi_{n_x,n_y}(x,y) = \frac{2}{\sqrt{L_xL_y}}\sin\left(\sqrt{\frac{2mE_{n_x}}{\hbar^2}}x\right)\sin\left(\sqrt{\frac{2mE_{n_y}}{\hbar^2}}y\right)}$$

(b) Degeneracy occurs if there are <u>distinct</u> wave functions specified by sets of quantum numbers, say $\{n_x, n_y\}$ and $\{m_x, m_y\}$ that give the same energy. That is $\dfrac{n_x^2}{L_x^2} + \dfrac{n_y^2}{L_y^2} = \dfrac{m_x^2}{L_x^2} + \dfrac{m_y^2}{L_y^2}$. In general, this equality is hard to achieve. If the sides of the rectangle are the same, then any state φ_{n_x, n_y} that has different quantum numbers such as (1,2) is degenerate with the state with the numbers interchanged (2,1). The wave functions are different but the energy is the same. But a state with the same quantum numbers like (1,1) – the ground state – is non-degenerate, because there is only one wave function.

If the lengths are different, then they must have a special integer ratio to achieve the same effect (we call this case accidental degeneracy), and for most length ratios, it is impossible to achieve degeneracy. In any case, the ground state is non-degenerate.

(c) The simplest (and easiest to integrate) perturbation function that fits the criteria is

$$H'(x,y) = \begin{cases} V_0 \sin\dfrac{\pi x}{L_x} \sin\dfrac{\pi y}{L_y} & 0 < x < L_x \text{ and } 0 < y < L_y \\ \\ \infty & \text{otherwise} \end{cases}$$

where "small" is defined by the perturbation amplitude V_0 being much smaller than the ground state energy: $V_0 \ll \dfrac{\hbar^2 \pi^2}{2mL_x^2} + \dfrac{\hbar^2 \pi^2}{2mL_y^2}$

The perturbation to the energy of any non-degenerate state is (equation sheet)
$E_{n_x n_y}^{(1)} = \left\langle X_{n_x} Y_{n_y} \middle| H' \middle| X_{n_x} Y_{n_y} \right\rangle$ and to the ground state is $E_{11}^{(1)} = \left\langle X_1 Y_1 \middle| H' \middle| X_1 Y_1 \right\rangle$

Evaluate, changing variables $x' = \dfrac{x}{L_x}; \, y' = \dfrac{y}{L_y}$ and using the integrals provided

$$E_{11}^{(1)} = \frac{4V_0}{L_x L_y} \int_0^{L_x} \sin^3 \frac{\pi x}{L_x} \, dx \int_0^{L_y} \sin^3 \frac{\pi y}{L_y} \, dy = 4V_0 \left(\int_0^1 \sin^3 \pi x' \, dx' \right)^2$$

$$= 4V_0 \left(\left. \frac{\cos(3\pi x')}{12\pi} \right|_0^1 - \left. \frac{\cos(\pi x')}{4\pi} \right|_0^1 \right)^2 = 4V_0 \left(\frac{-2}{12\pi} - \frac{-2}{4\pi} \right)^2 = 4V_0 \left(\frac{1}{3\pi} \right)^2 = \frac{4V_0}{9\pi^2}$$

$$\boxed{E_{11}^{(1)} = \frac{4V_0}{9\pi^2}}$$

The correction is about 20 times smaller than V_0 which in turn is much smaller than the ground state energy. So the correction is also small compared to the difference between the ground and first excited states, which is good if perturbation theory is to be used.

An electron and a positron interact via a potential energy whose spin-dependent part is $\alpha \mathbf{S}_1 \cdot \mathbf{S}_2$, where $\mathbf{S}_1$ is the spin angular momentum of the electron, $\mathbf{S}_2$ is the spin angular momentum of the positron and α is a constant with appropriate dimensions that measures the strength of the interaction. It is also useful to define the total angular momentum of the system, $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$.

To ensure that you focus ONLY on the spin-dependent part of the Hamiltonian, assume that the particles are fixed in space, so that there is no kinetic energy to worry about and the Coulomb interaction gives a constant contribution and can be ignored.

(a) Find the energy eigenvalues and eigenstates of the system.
(b) Add a magnetic field $\mathbf{B}$ in the $\hat{z}$ direction and find the new energy eigenvalues of the system. Give a physical explanation of why two of the states are insensitive to the magnetic field.

Notation and additional information to help solve the problem:
• The interaction Hamiltonian of particle 1 (particle 2) with a magnetic field is

$H_{B1(2)} = -g_{1(2)} \dfrac{q_{1(2)}}{2m_{1(2)}} \mathbf{S}_{1(2)} \cdot \mathbf{B}$ where $g_{1(2)}$ is the gyromagnetic ratio, $q_{1(2)}$ the charge, $m_{1(2)}$ the mass

and $\mathbf{S}_{1(2)}$ the spin angular momentum of particle 1 (particle 2). A positron has the same gyromagnetic ratio, mass and spin as the electron, and opposite charge.

• One basis in which to describe the system is $|++\rangle$, $|+-\rangle$, $|-+\rangle$, $|--\rangle$. The first symbol refers to the z-component of the spin of the electron, the second to the z-component of the spin of the positron. These four states are all eigenstates of the individual angular momenta (squared) S_1^2, S_2^2 and of the individual z-components of angular momenta S_{1z}, S_{2z}. So, for example, $S_{1z}|+-\rangle = \frac{1}{2}\hbar|+-\rangle$ and $S_{2z}|+-\rangle = -\frac{1}{2}\hbar|+-\rangle$.

• Another basis is the set of states that are eigenstates of the total angular momentum (squared) S^2, the z-component of total spin angular momentum S_z and of the individual angular momenta (squared) S_1^2, S_2^2. These basis states are designated $|S,M_s\rangle$, and the specific states are $|11\rangle$, $|10\rangle$, $|1,-1\rangle$, $|00\rangle$.

• Either basis could be useful (or both bases), and you will have to decide which to use. Notice that in either basis, the labels associated with S_1^2, and S_2^2 are suppressed because S_1 and S_2 are always equal to ½.

• The relationship between the bases is:
$$|11\rangle = |++\rangle$$
$$|10\rangle = \tfrac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right)$$
$$|1-1\rangle = |--\rangle$$
$$|00\rangle = \tfrac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

Comprehensive Exam, Fall 2016 QM Graduate (Solution)

(a) The Hamiltonian is $H = \alpha \mathbf{S}_1 \cdot \mathbf{S}_2$.
It follows from

$$\mathbf{S}^2 = \left(\mathbf{S}_1 + \mathbf{S}_2\right)^2 = \left(\mathbf{S}_1^2 + \mathbf{S}_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2\right)$$
$$\Rightarrow \mathbf{S}_1 \cdot \mathbf{S}_2 = \tfrac{1}{2}\left(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2\right)$$

that

$$H = \frac{\alpha}{2}\left(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2\right)$$

Energy Eigenstates:
H contains the total spin (squared) and the spin (squared) of the individual angular momenta. Therefore the appropriate eigenstates are $\left|S, M_s\right\rangle$. The spin of the electron is $S_1 = \tfrac{1}{2}$, and spin of the positron is $S_2 = \tfrac{1}{2}$, so the allowed values of the total angular momentum $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ are $S = 1, 0$. For $S = 1$, $M_S = 1, 0, -1$ and for $S = 0$, $M_S = 0$. As expected, there are 4 states.

$$\left|S, M_S\right\rangle : \left|1,1\right\rangle, \left|1,0\right\rangle, \left|1,-1\right\rangle, \left|0,0\right\rangle$$

Energy Eigenvalues:
This Hamiltonian matrix elements are:

$$\left\langle S' M_S' \middle| H \middle| S M_S \right\rangle = \frac{\alpha}{2} \left\langle S' M_S' \middle| \mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2 \middle| S M_S \right\rangle$$
$$= \frac{\alpha}{2}\left[S(S+1) - S_1(S_1+1) - S_2(S_2+1)\right]\hbar^2 \left\langle S' M_S' \middle| S M_S \right\rangle$$
$$= \frac{\alpha\hbar^2}{2}\left[S(S+1) - S_1(S_1+1) - S_2(S_2+1)\right]\delta_{S'S}\delta_{M_S'M_S}$$

The matrix is diagonal, so the **eigenvalues are the diagonal elements**. There are 3 states with *S*=1 (triplet) one with *S*=0, (singlet). The energies are:

$$\left\langle S M_S \middle| H \middle| S M_S \right\rangle = \begin{cases} \dfrac{1}{4}\alpha\hbar^2; & S = 1 \\[2mm] -\dfrac{3}{4}\alpha\hbar^2; & S = 0 \end{cases}$$

and are separated by $\Delta E = \alpha\hbar^2$. The full matrix is

$$H \doteq \frac{\alpha\hbar^2}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{matrix} 1,1 \\ 1,0 \\ 1,-1 \\ 0,0 \end{matrix}$$

(b) In the presence of the magnetic field, the Hamiltonian is

$$H = \alpha \mathbf{S}_1 \cdot \mathbf{S}_2 - g_1 \frac{q_1}{2m_1} \mathbf{S}_1 \cdot \mathbf{B} - g_2 \frac{q_2}{2m_2} \mathbf{S}_2 \cdot \mathbf{B}$$

$$= \alpha \mathbf{S}_1 \cdot \mathbf{S}_2 - \frac{g}{2m}(q_1 S_{1z} + q_2 S_{2z}) B$$

because the field is in the z direction and the gyromagnetic ratio and mass of the electron and the positron are the same. Their charges are opposite, so with e a positive number representing the magnitude of the electron charge,

$$H = \alpha \mathbf{S}_1 \cdot \mathbf{S}_2 - \frac{ge}{2m}(-S_{1z} + S_{2z}) B$$

This Hamiltonian is not diagonal in either basis, so you have to choose one, find the matrix and its eigenvalues. Given the relationship between the bases, the coupled (total angular momentum) basis is a useful choice.

The matrix for the first term was found in part (a). For the second term (call it H_B), you will find that of the 16 possible matrix elements, only 2 are non-zero:

$$\langle S' M_S' | H_B | S M_S \rangle = -\frac{ge}{2m} B \langle S' M_S' | (-S_{1z} + S_{2z}) | S M_S \rangle$$

$$\langle S' M_S' | H_B | 11 \rangle = -\frac{ge}{2m} B \langle S M_S | (-S_{1z} + S_{2z}) | 11 \rangle$$

$$= -\frac{ge}{2m} B \langle S' M_S' | (-S_{1z} + S_{2z}) | + + \rangle \qquad\qquad \text{No states couple to state } |11\rangle$$

$$= -\frac{ge}{2m} B \langle S' M_S' | \underbrace{(-1+1)}_{0} \frac{\hbar}{2} | + + \rangle = 0$$

$$\langle S' M_S' | H_B | 10 \rangle = -\frac{ge}{2m} B \langle S M_S | (-S_{1z} + S_{2z}) | 10 \rangle$$

$$= -\frac{ge}{2m} B \langle S' M_S' | (-S_{1z} + S_{2z}) \frac{1}{\sqrt{2}} (|+-\rangle + |-+\rangle)$$

$$= -\frac{ge}{2m} \frac{\hbar}{2} B \langle S' M_S' | \frac{1}{\sqrt{2}} [(-1-1)|+-\rangle + (+1+1)|-+\rangle]$$

$$= +\frac{ge}{2m} \frac{2\hbar}{2} B \langle S' M_S' | \frac{1}{\sqrt{2}} [|+-\rangle - |-+\rangle] \qquad\qquad \text{The only non-zero possibility}$$
is $S'=0, M_S'=0$, so

$$= +\frac{ge}{2m} \frac{2\hbar}{2} B \langle S' M_S' | 00 \rangle \qquad\qquad \langle 00 | H_B | 10 \rangle = +\frac{ge\hbar}{2m} B$$

$$= +\frac{ge\hbar}{2m} B \langle S' M_S' | 00 \rangle$$

$$\langle S'M_S|H_B|1,-1\rangle = -\frac{ge}{2m}B\langle SM_S|(-S_{1z}+S_{2z})|1,-1\rangle$$

$$= -\frac{ge}{2m}B\langle S'M_S|(-S_{1z}+S_{2z})|--\rangle$$

$$= -\frac{ge}{2m}B\langle S'M_S|(+1-1)\frac{\hbar}{2}|--\rangle$$

$$= 0$$

No states couple to state $|1,-1\rangle$

$$\langle S'M_S|H_B|00\rangle = -\frac{ge}{2m}B\langle S'M_S|(-S_{1z}+S_{2z})|00\rangle$$

$$= -\frac{ge}{2m}B\langle S'M_S|(-S_{1z}+S_{2z})\frac{1}{\sqrt{2}}(|+-\rangle-|-+\rangle)$$

$$= -\frac{ge}{2m}\frac{\hbar}{2}B\langle S'M_S|\frac{1}{\sqrt{2}}[(-1-1)|+-\rangle-(+1+1)|-+\rangle]$$

$$= +\frac{ge}{2m}\frac{2\hbar}{2}B\langle S'M_S|\frac{1}{\sqrt{2}}[|+-\rangle+|-+\rangle]$$

$$= +\frac{ge}{2m}\frac{2\hbar}{2}B\langle S'M_S|10\rangle$$

$$= +\frac{ge\hbar}{2m}B\langle S'M_S|10\rangle$$

The only non-zero possibility is $S'=1$ $M_S'=0$, so $\langle 10|H_B|00\rangle = +\frac{ge\hbar}{2m}B$, which you can also

deduce from the fact that the Hamiltonian is Hermitian, but nice to confirm.

$$H_B \doteq \frac{ge\hbar}{2m}B\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{matrix} 1,1 \\ 1,0 \\ 1,-1 \\ 0,0 \end{matrix}$$

Collect terms to get the full Hamiltonian:

$$H_{full} \doteq \underbrace{\frac{\alpha\hbar^2}{4}}_{\gamma}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} + \underbrace{\frac{ge\hbar}{2m}B}_{\delta}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{matrix} 1,1 \\ 1,0 \\ 1,-1 \\ 0,0 \end{matrix}$$

$$= \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \gamma & 0 & \delta \\ 0 & 0 & \gamma & 0 \\ 0 & \delta & 0 & -3\gamma \end{pmatrix}\begin{matrix} 1,1 \\ 1,0 \\ 1,-1 \\ 0,0 \end{matrix}$$

Rearrange to see that the diagonalization is easy:

$$H_{full} \doteq \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & \delta \\ 0 & 0 & \delta & -3\gamma \end{pmatrix} \begin{matrix} 1,1 \\ 1,-1 \\ 1,0 \\ 0,0 \end{matrix}$$

Two eigenvalues are γ, just as before in part (a). **These are the states in which the electron and positron have parallel spins and therefore opposite magnetic moments (because g and m are the same and q is opposite), so those states are not influenced by the magnetic field.**

The other two eigenvalues, for states with opposite spins and non-zero moment, are found by diagonalizing the 2x2 submatrix to obtain

$$(\gamma - E)(-3\gamma - E) - \delta^2 = 0$$

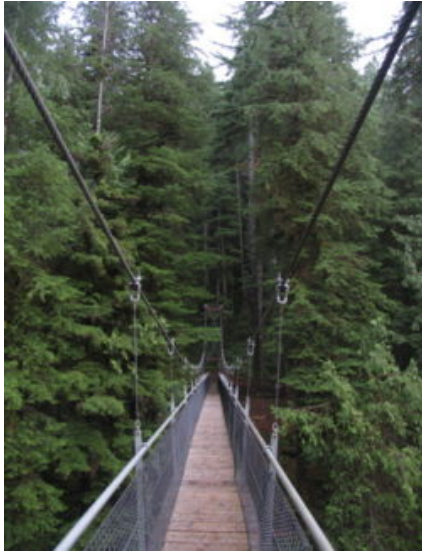$$E^2 + 2E\gamma - 3\gamma^2 - \delta^2 = 0$$

$$E = -\gamma \pm \sqrt{\gamma^2 + (3\gamma^2 + \delta^2)} = -\gamma \pm \sqrt{4\gamma^2 + \delta^2}$$

The expression reduces to the correct values for zero field, $\delta = 0$, and as the field becomes large, one state goes up in energy and the other down, as we expect for a dipole in a field. The plot below is schematic.
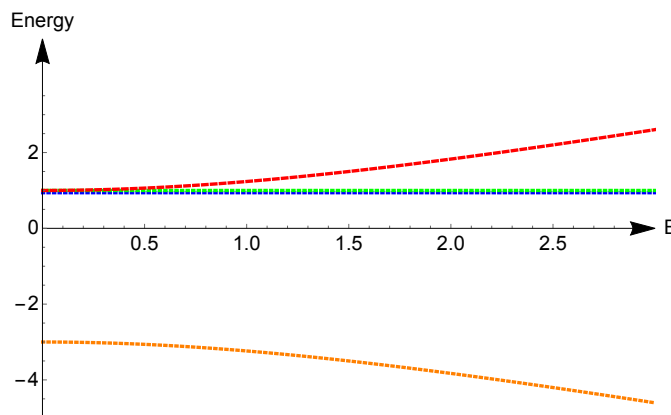
Classical Mechanics

Problem 1: The Drift Creek Falls suspension bridge is a popular spot for hikers in Oregon. The shape of the bridge follows the so-called catenary curve $y = a \cosh\left(\frac{x}{a}\right) = a\frac{e^{x/a}+e^{-x/a}}{2}$ which can be derived based on the following simple assumptions: (a) the bridge can be considered as a string of uniform mass density ρ, (b) the bridge of total length L is anchored at equal height between two posts, (c) the two posts are separated by a distance of D.

(1) With the above assumptions, show that the catenary curve shape satisfies the requirement of static force balance.

(2) Consider the case where $D \ll a$. A bicycle wheel is released from one end of the bridge at zero velocity, how long does it take to reach the middle point of the bridge? Assuming the total mass m of the bicycle wheel is uniformly distributed along a ring of radius r, and the wheel rolls without slipping.
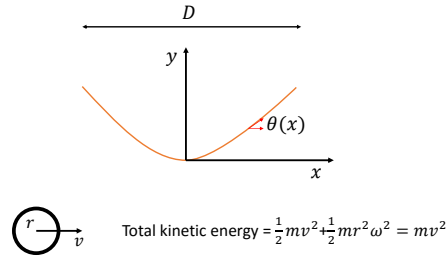
Problem 8
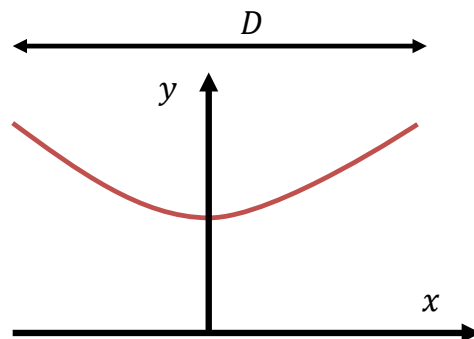
Figure 1:

Using the notation of Fig. 1. First of all, for any segment of the bridge, the horizontal component of the tension at its two ends must balance, therefore the horizontal component of the tension is constant everywhere. We call this constant T_o, which is also the tension at the middle point of the bridge.

We now consider an infinitesimal segment of the bridge between x and $x+\delta x$. The force balance in the vertical direction is then:

$$T_o \tan\theta + \rho g \frac{\delta x}{\cos\theta} = T_o \tan\theta' \tag{7.1}$$

$$\tan\theta = \frac{dy}{dx}|_x \tag{7.2}$$

$$\tan\theta' = \frac{dy}{dx}|_{x+\delta x} \tag{7.3}$$

$$\tag{7.4}$$

Which is equivalent of:

$$\frac{d^2 y}{dx^2} = \frac{\rho g}{T_o}\sqrt{1 + (\frac{dy}{dx})^2} \tag{7.5}$$

It is easy to verify that the Catenary equation is a solution of this ODE with $a = \frac{T_o}{\rho g}$.

If $D << a$, we can indeed approximate a catenary as a parabolic curve
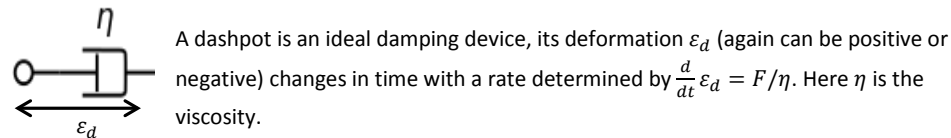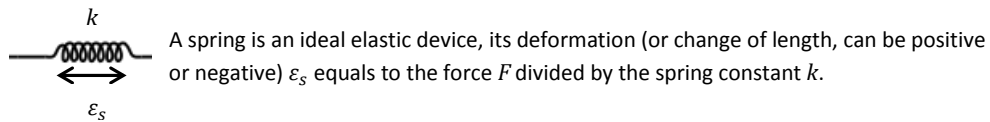
$$y \approx \frac{a}{2}(2 + (x/a)^2) \tag{7.6}$$

From conservation of energy, we find the velocity of the wheel v satisfies

$$mv^2 = mg\frac{a}{2}[(D/2a)^2 - (x/a)^2]$$

$$v_x = v\cos\theta = \frac{v}{\sqrt{1 + (x/a)^2}} \approx v \tag{7.7}$$

Therefore the motion of the wheel is approximately a harmonic oscillator with frequency $\omega = \sqrt{g/2a}$, and the time it takes to reach the middle point of the bridge is $\frac{\pi}{2\omega}$.

Problem 2: In baseball, the batter likes to hit the ball as far as possible, knowing that the ball is rather inelastic. On the other hand, when hit with a faster speed, the ball behaves more elastically. This phenomenon, termed viscoelasticity, suggests that material can exhibit inelastic property (like water), and elastic property (like rubber) at the same time depending on how fast it is deformed. To understand viscoelasticity, we will develop simple models based on the following two simple elements:

A spring is an ideal elastic device, its deformation (or change of length, can be positive or negative) ε_s equals to the force F divided by the spring constant k.

A dashpot is an ideal damping device, its deformation ε_d (again can be positive or negative) changes in time with a rate determined by $\frac{d}{dt}\varepsilon_d = F/\eta$. Here η is the viscosity.

Using these elements we can build two models: (1) the spring and dashpot connect in series (Maxwell model) or (2) the spring and dashpot connect in parallel (Kelvin–Voigt model, the spring and dashpot always have the same length)
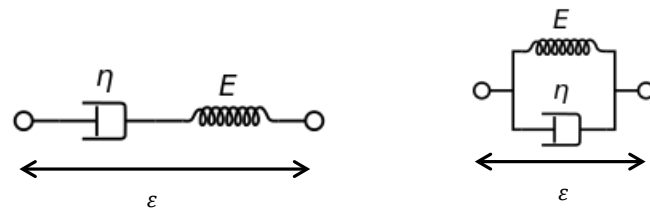
(1) Material response to a pulse signal is often very useful, so let's examine how these two models behave in this case. When a sudden stretching force $F = f\delta(t)$ is applied, derive the deformation ε as a function of time for both models.

(2) When an oscillatory force $F = f sin(\omega t)$ is applied, derive the deformation ε as a function of time for both models.

(3) The elasticity of a material can be characterized by the dynamic modulus: when an oscillatory force $F = f sin(\omega t)$ is applied, the deformation can be described as $\varepsilon = asin(\omega t) + bcos(\omega t)$. The dynamic elastic modulus equals to f/a. Calculate the dynamic elastic modulus as a function of frequency ω for both models. Which one is a better model for a baseball?

When a δ function of force is applied, for the Maxwell model, we have

$$\epsilon = F/k + L_d(t)$$
$$\frac{d}{dt}L_d(t) = F/\eta$$

(8.1)

Solving the equation, we have

$$\epsilon = f\delta(t)/k + fH(t)/\eta, t \geq 0 \tag{8.2}$$

Here $H(t)$ is a step function $H(t < 0) = 0$, $H(t \geq 0) = 1$.

For the Kelvin-Voigt model, we have

$$F = k\epsilon + \eta\frac{d\epsilon}{dt} \tag{8.3}$$

In case you do not remember, the standard procedure to solve this type of first order ODE is to assume

$$\epsilon = \epsilon_a + \epsilon_b \tag{8.4}$$
$$\eta\frac{d\epsilon_a}{dt} = f\delta(t) \tag{8.5}$$
$$k\epsilon_b + \eta\frac{d\epsilon_b}{dt} = -k\epsilon_a \tag{8.6}$$

It is clear that $\epsilon_a = fH(t)/\eta$, therefore we have for $t > 0$: $k\epsilon_b + \eta\frac{d\epsilon_b}{dt} = -kf/\eta$. Solving the equation, we have

$$\epsilon_b = -f/\eta(1 - e^{-\frac{kt}{\eta}}) \tag{8.7}$$

Therefore

$$\epsilon = fH(t)/\eta - f/\eta(1 - e^{-\frac{kt}{\eta}}) \tag{8.8}$$

When an oscillatory force is applied, for the Maxwell model, we have

$$\epsilon = f\sin\omega t/k + L_d(t)$$
$$\frac{d}{dt}L_d(t) = f\sin\omega t/\eta$$

(8.9)

Which can be solved to give us

$$\epsilon = f\sin\omega t/k - \frac{f}{\omega\eta}\cos\omega t \tag{8.10}$$

Therefore the dynamic elastic modulus is k, independent of frequency.

For the KelvinVVoigt model, we have

$$f \sin \omega t = k\epsilon + \eta \frac{d\epsilon}{dt} \tag{8.11}$$

Which gives us

$$\epsilon = \frac{kf}{k^2 + \eta^2\omega^2} \sin \omega t - \frac{\eta\omega}{k^2 + \eta^2\omega^2} \cos \omega t \tag{8.12}$$

The dynamic elastic modulus increase with frequency

$$E(\omega) = \frac{k^2 + \eta^2\omega^2}{k} \tag{8.13}$$

and therefore is a better model for the baseball.