OSU Physics Department
Comprehensive Examination #124

Monday, January 4, Tuesday, January 5, 2016

Winter 2016 Comprehensive Examination

PARTS 1, 2, 3 & 4

General Instructions

Comprehensive Examination consists of four separate parts of two problems each. Each problem caries equal weight (20 points each). The first part (Quantum Mechanics) is handed out at 9:00 am on Monday, January 4, and lasts three hours. The second part (Electricity and Magnetism) will be handed out at 1:00 pm on the same day and will also last three hours. The third (Statistical Mechanics) and fourth (Classical Mechanics) parts will be administered on Tuesday, January 5, at 9:00 am and 1:00 pm, respectively. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Use the last pages of your bluebooks for "scratch" work, separated by at least one empty page from your solutions. "Scratch" work will not be graded.

If you submit blue books for any given section, that section will be graded as part of your cumulative score. Unless you are taking the exam for practice, all sections not previously passed need to be attempted and submitted.

Time-dependent wavefunctions describing energy eigenstates of a simple harmonic oscillator can be written as

$$\psi_n(x,t) = \phi_n(x) \exp\left(-i\beta_n t\right).$$

For example, the first excited state of an oscillator with characteristic frequency ω is described by ϕ_1 and β_1 written as,

$$\phi_1(x) = \sqrt{\frac{2\alpha^3}{\sqrt{\pi}}}\, x \exp\left(-\frac{\alpha^2 x^2}{2}\right) \tag{1}$$

$$\beta_1 = \frac{3}{2}\omega \tag{2}$$

(a) Show that $\psi_o(x,t)$ is a solution to the Schrodinger equation for a simple harmonic oscillator when

$$\phi_0(x) = \sqrt{\frac{\alpha}{\pi}} \exp\left(-\frac{\alpha^2 x^2}{2}\right).$$

Find the values of α and β_0 in terms of the mass m of the oscillator, the characteristic frequency ω of the oscillator, and fundamental constants.

(b) At $t = 0$, the same simple harmonic oscillator is in the state

$$\psi(x, t=0) = \cos(\theta)\phi_0(x) + \sin(\theta)\phi_1(x)$$

where $\cos\theta$ and $\sin\theta$ are real-valued coefficients. Find the expectation value of position, $\langle x \rangle$, at a subsequent time t.

Useful result:

$$\int_{-\infty}^{\infty} u^2 e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}$$

QM question 1

a) S. Eqn $H\Psi_0 = i\hbar \frac{\partial \Psi_0}{\partial t}$ ——— ①

where $\Psi_0 = \phi_0(x) e^{-i\beta_0 t}$, $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$

$$\frac{\partial \Psi_0}{\partial t} = -i\beta_0 \phi_0 e^{-i\beta_0 t}$$

Plugging everything into ① gives

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\phi_0 + \frac{1}{2}m\omega^2 x^2 \phi_0 = i\hbar(-i\beta_0)\phi_0 \quad \text{——— ②}$$

We want to show that $\phi_0 = \sqrt{\frac{\alpha}{\pi}}\, e^{-\frac{\alpha^2 x^2}{2}}$ is a solution

$$\frac{d\phi_0}{dx} = \sqrt{\frac{\alpha}{\pi}}\, e^{-\frac{\alpha^2 x^2}{2}}(-\alpha^2 x)$$

$$\frac{d^2\phi_0}{dx^2} = \sqrt{\frac{\alpha}{\pi}}\left(-\alpha^2 e^{-\frac{\alpha^2 x^2}{2}} + e^{-\frac{\alpha^2 x^2}{2}}\alpha^4 x^2\right) = (-\alpha^2 + \alpha^4 x^2)\phi_0$$

Plug this into ②

$$-\frac{\hbar^2}{2m}(-\alpha^2 + \alpha^4 x^2) + \frac{1}{2}m\omega^2 x^2 = \hbar\beta_0$$

$$\frac{\alpha^2\hbar^2}{2m} - \underbrace{\frac{\hbar^2\alpha^4}{2m}x^2 + \frac{1}{2}m\omega^2 x^2}_{} = \hbar\beta_0 \quad \text{——— ③}$$

coefficients multiplying x^2 must be equal an opposite for equality to hold

$$\Rightarrow \frac{1}{2}m\omega^2 = \frac{\hbar^2\alpha^4}{2m}$$

$$\boxed{\alpha^2 = \frac{m\omega}{\hbar}}$$

Now (3) reduces to

$$\frac{\alpha^2\hbar^2}{2m} = \hbar\beta_0$$

$$\Rightarrow \hbar\beta_0 = \frac{\hbar^2}{2m}\frac{m\omega}{\hbar}$$

$$\boxed{\beta_0 = \frac{\omega}{2}}$$

b) $\langle x\rangle = \langle\psi|x|\psi\rangle$

$$= \left(\cos\theta\langle\psi_0| + \sin\theta\langle\psi_1|\right) x \left(\cos\theta|\psi_0\rangle + \sin\theta|\psi_1\rangle\right)$$

↑ ψ_0 & ψ_1 contain time dependence

$$= \cos^2\theta\langle\psi_0|x|\psi_0\rangle + \sin^2\theta\langle\psi_1|x|\psi_1\rangle + \cos\theta\sin\theta\langle\psi_0|x|\psi_1\rangle + \sin\theta\cos\theta\langle\psi_1|x|\psi_0\rangle$$

Note $\langle\psi_0|x|\psi_0\rangle = 0$

$\langle\psi_1|x|\psi_1\rangle = 0$

$\langle\psi_0|x|\psi_1\rangle = \langle\psi_1|x|\psi_0\rangle^*$

$$\Rightarrow \langle x\rangle = 2\cos\theta\sin\theta\,\mathrm{Re}\left\{\langle\psi_0|x|\psi_1\rangle\right\}$$

$$= 2\cos\theta\sin\theta\,\mathrm{Re}\left\{e^{i(\beta_0-\beta_1)t}\right\}\langle\phi_0|x|\phi_1\rangle$$

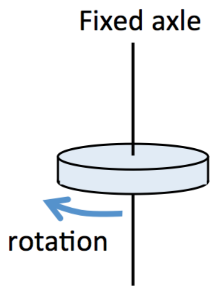
Note $\beta_0 = \frac{\omega}{2}$ and $\beta_1 = \frac{3\omega}{2}$ $\Rightarrow$ $\beta_0 - \beta_1 = \omega$

and $\langle\phi_0|x|\phi_1\rangle = \left(\frac{2\alpha^4}{\pi^{3/2}}\right)^{1/2}\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}dx$

$$\langle \phi_0 | x | \phi_1 \rangle = \frac{\sqrt{2}\,\alpha^{2}}{\pi^{3/4}} \frac{1}{\alpha^3} \underbrace{\int_{-\infty}^{\infty} u^2 e^{-u^2}\, du}_{= \frac{\sqrt{\pi}}{2}}$$

$$= \frac{1}{\sqrt{2}\,\alpha\,\pi^{1/4}}$$

Therefore $$\boxed{\langle x \rangle = \frac{\sqrt{2}}{\alpha\,\pi^{1/4}} \cos\theta \sin\theta \cos\omega t}$$



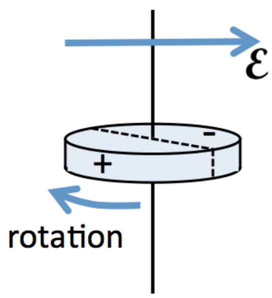
Consider a very small disk that is free to rotate on a fixed axle (see Figure above). The disk has moment of inertia I about this axis.

(a) Use quantum mechanics to find expressions for the:

   (i) Eigenenergies of the system

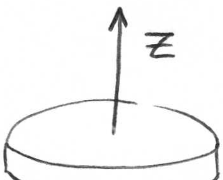
   (ii) Normalized wave function of the m^{th} energy eigenstate

(b) The disk has an electric dipole moment d (half the disk is positively charged, half the disk is negatively charged, see Figure below). Calculate the first non-zero correction to the eigenenergies when a *small* electric field is applied perpendicular to the axis of rotation.



QM question 2

a) Let angular momentum be along z-axis



Then energy is $\frac{L_z^2}{2I}$

Hamiltonian

$$H = \frac{\hat{L}_z^2}{2I} = -\frac{\hbar^2}{2I}\frac{d^2}{d\phi^2}$$

We look for a wavefunction, $\psi_m(\phi)$ that satisfies

$$H\psi_m = E_m \psi_m$$

and satisfies the boundary condition $\psi(\phi) = \psi(\phi + 2\pi)$

$$\psi_m(\phi) = Ae^{im\phi} \quad \text{and} \quad \boxed{E_m = \frac{\hbar^2 m^2}{2I}}$$

To normalize $\psi_m(\phi)$ we need

$$\int_{\phi=0}^{2\pi} |\psi(\phi)|^2 d\phi = 1$$

$$2\pi A^2 = 1$$

$$A = \frac{1}{\sqrt{2\pi}}$$

$$\boxed{\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}}$$

b) The perturbation Hamiltonian is $\underbrace{-\mathcal{E} d \cos\phi}_{\text{energy of dipole moment in the } \vec{\mathcal{E}}\text{-field.}} = H'$

First order Perturbation Theory

$$E_m^{(1)} = \langle \psi_m | H' | \psi_m \rangle = \frac{-1}{2\pi} \int_0^{2\pi} e^{-im\phi} \mathcal{E} d \cos\phi \, e^{+im\phi} \, d\phi = 0$$

~~Second~~ Second order perturbation theory

$$E_m^{(2)} = \sum_{n \neq m} \frac{|\langle n | H' | m \rangle|^2}{E_m^{(0)} - E_n^{(0)}}$$

$$\langle n | H' | m \rangle = -\frac{\mathcal{E} d}{2\pi} \int_0^{2\pi} e^{-i(m-n)\phi} \cos\phi \, d\phi$$

$$= -\frac{\mathcal{E} d}{4\pi} \int_0^{2\pi} \left(e^{-i(m-n-1)\phi} + e^{-i(m-n+1)\phi} \right) d\phi$$

The integral is only non-zero if $n = m-1$ or $n = m+1$

$$\langle m+1 | H' | m \rangle = -\frac{\mathcal{E} d}{4\pi} 2\pi = -\frac{\mathcal{E} d}{2}$$

$$\langle m-1 | H' | m \rangle = -\frac{\mathcal{E} d}{4\pi} 2\pi = -\frac{\mathcal{E} d}{2}$$

Therefore $E_m^{(2)} = \frac{\mathcal{E}^2 d^2}{4} \left(\frac{1}{E_m^{(0)} - E_{m+1}^{(0)}} \right) + \frac{\mathcal{E}^2 d^2}{4} \left(\frac{1}{E_m^{(0)} - E_{m-1}^{(0)}} \right)$

From part a) we know $E_m^{(0)} = \frac{\hbar^2 m^2}{2I}$

$$\Rightarrow E_m^{(2)} = \frac{\mathcal{E}^2 d^2}{4} \frac{2I}{\hbar^2} \left(\frac{1}{m^2 - (m-1)^2} + \frac{1}{m^2 - (m+1)^2} \right)$$

$$= \frac{\mathcal{E}^2 d^2 I}{2\hbar^2} \left(\frac{1}{2m-1} - \frac{1}{2m+1} \right)$$

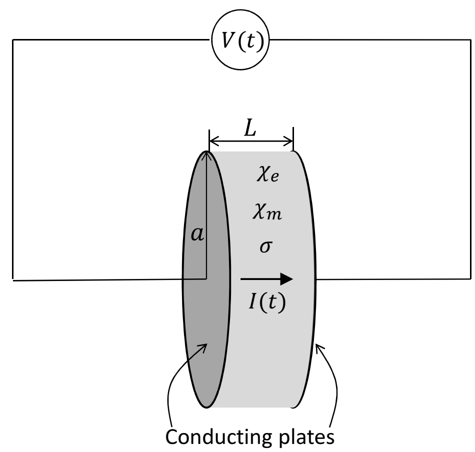
$$= \frac{\mathcal{E}^2 d^2 I}{2\hbar^2} \left(\frac{2}{4m^2-1} \right)$$

$$= \frac{\mathcal{E}^2 d^2 I}{\hbar^2} \frac{1}{4m^2-1}$$

~~Eigenstate~~

Energy is shifted up when the small $\mathcal{E}$-field is applied.

A slowly varying bias $V(t) = V_0 \cos \omega t$ ($\omega \ll \frac{2\pi c}{a}$, where c is the speed of light) is applied between two conducting plates between which a thin circular disk of radius a and length L ($a \gg L$) is placed. The disk has electrical susceptibility χ_e, magnetic susceptibility χ_m, and conductivity σ.



(a) Find the electric field **E** and the current density **J** inside the disk. You may assume that **E** and **J** are uniform inside the disk.

(b) Find the H-field **H** and the magnetic field **B** inside the disk.

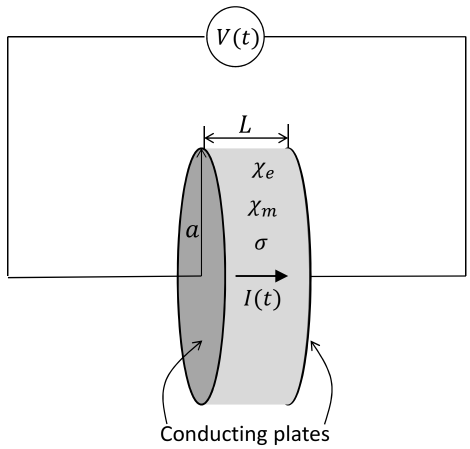
(c) We consider the energy dissipation of the circuit.

(i) Calculate the time-averaged energy dissipation inside the disk.

(ii) Find the time-averaged Poynting vector inside the disk and explain the physical meaning of the direction of the Poynting vector.

(iii) Show that the energy dissipation is equal to the surface integration of the Poynting vector over the surface of the disk, where the surface normal is pointing inward. Explain the physical meaning of the result.

A slowly varying bias $V(t) = V_0 \cos \omega t$ ($\omega \ll \frac{2\pi c}{a}$, where c is the speed of light) is applied between two conducting plates between which a thin circular disk of radius a and length L ($a \gg L$) is placed. The disk has electrical susceptibility χ_e, magnetic susceptibility χ_m, and conductivity σ.



(a) Find the electric field $\mathbf{E}$ and the current density $\mathbf{J}$ inside the disk. You may assume that $\mathbf{E}$ and $\mathbf{J}$ are uniform inside the disk.

Solution:
In the low frequency limit, we may apply the static field approximation, i.e., the Maxwell's equation reduces to

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \rightarrow \nabla \times \mathbf{E} \cong 0$$

and the field can be obtained from the scalar potential,

$$\mathbf{E} = -\nabla V$$

Since the field and current density are uniform inside the disk

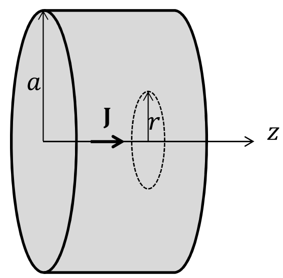
$$\mathbf{E}(t) = \frac{V(t)}{L}\hat{z} = \frac{V_0}{L}\cos \omega t \hat{z}$$

$$\mathbf{J}(t) = \sigma \mathbf{E}(t) = \frac{\sigma V_0}{L}\cos \omega t \hat{z}$$

(b) Find the H-field $\mathbf{H}$ and the magnetic field $\mathbf{B}$ inside the disk.

Solution:
In the low frequency limit,

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \rightarrow \nabla \times \mathbf{H} \cong \mathbf{J}$$



Using the Ampere's law

$$\int_C \mathbf{H} \cdot d\mathbf{l} = I$$

and $\mathbf{H} = H\hat{\phi}$ due to the cylindrical symmetry,

$$\begin{aligned} 2\pi r H &= J \cdot \pi r^2 \\ \rightarrow H &= \frac{r}{2} J = \frac{\sigma V_0}{2L} r \cos\omega t \\ \rightarrow \mathbf{H} &= \hat{\phi}\frac{\sigma V_0}{2L} r \cos\omega t \end{aligned}$$

and

$$\begin{aligned} \mathbf{B} &= \mu\mathbf{H} = \mu_0(1+\chi_m)\mathbf{H} \\ &= \hat{\phi}\mu_0(1+\chi_m)\frac{\sigma V_0}{2L} r \cos\omega t \end{aligned}$$

(c) We consider the energy dissipation of the circuit.

(i) Calculate the time-averaged energy dissipation inside the disk.

(ii) Find the time-averaged Poynting vector inside the disk and explain the physical meaning of the direction of the Poynting vector.

(iii) Show that the energy dissipation is equal to the surface integration of the Poynting vector over the surface of the disk, where the surface normal is pointing inward. Explain the physical meaning of the result.

Solution:
(i) The time-averaged energy dissipation is

$$\begin{aligned} P &= \left\langle \int \mathbf{J} \cdot \mathbf{E} dv \right\rangle \\ &= \frac{\sigma V_0}{L}\frac{V_0}{L} \cdot \pi a^2 \cdot L \langle \cos^2 \omega t \rangle \\ &= \frac{1}{2}\pi a^2 \sigma \frac{V_0^2}{L} \end{aligned}$$

(ii) The time-averaged Poynting vector is

$$\begin{aligned}\langle \mathbf{S} \rangle &= \langle \mathbf{E} \times \mathbf{H} \rangle = \frac{V_0}{L}\frac{\sigma V_0}{2L} r \langle \cos^2 \omega t \rangle (\hat{z} \times \hat{\phi}) \\ &= -\frac{\sigma V_0^2}{4L^2} r \hat{r}\end{aligned}$$

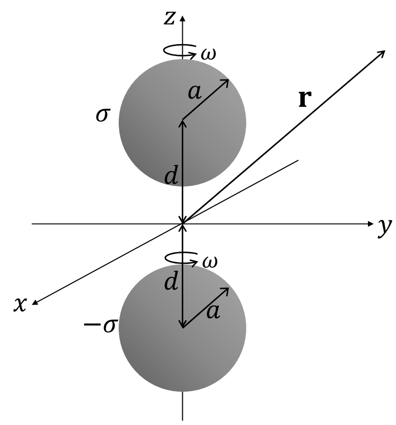
The Poynting vector is pointing inwards, meaning that energy is flowing into the disk.

(iii) The surface integration of the Poynting vector can be decomposed into three parts for the two end surfaces and the side surface. The integrations over the end surfaces vanish because the Poynting vector ($\langle \mathbf{S} \rangle \parallel -\hat{r}$) is perpendicular to the end surfaces ($\hat{n} \parallel \pm\hat{z}$). On the other hand, the inward-pointing normal of the side surface is $-\hat{r}$, and hence

$$\begin{aligned}\int \langle \mathbf{S} \rangle \cdot d\mathbf{a} &= \int (-\frac{a\sigma V_0^2}{4L^2}\hat{r}) \cdot da(-\hat{r}) \\ &= \frac{a\sigma V_0^2}{4L^2} \cdot 2\pi a L = \frac{1}{2}\pi a^2 \sigma \frac{V_0^2}{L} = P\end{aligned}$$

This result confirms energy conservation: the loss of electrical energy (energy dissipation) is equal to the electromagnetic energy flowing into the disk.

Two hollow spheres of radius a are centered at $(0,0,d)$ and $(0,0,-d)$, where $d > a$. They carry uniform surface charge density of σ and $-\sigma$, respectively, and rotate around z-axis at angular frequency ω.



(a) Find the potential, $V(\mathbf{r})$, and the electric field, $\mathbf{E}(\mathbf{r})$, inside and outside the spheres.

(b) Show that

$$V(r,\theta,\phi) \cong \frac{2\sigma a^2 d}{\varepsilon_0 r^2}\cos\theta$$

and

$$\mathbf{E}(r,\theta,\phi) \cong \frac{2\sigma a^2 d}{\varepsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

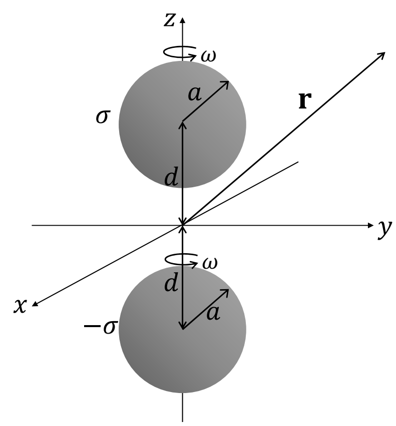
in the far zone for $r \gg d$, where $\mathbf{r} = (r,\theta,\phi)$ in the spherical coordinate.

(c) Find the magnetic field $\mathbf{B}$ (i) at $\mathbf{r} = 0$ and (ii) on z-axis for $|z| >> d$.

Useful formula: The magnetic field $\mathbf{B}(\mathbf{r})$ at position $\mathbf{r}$ generated by a steady current I is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_C \frac{I\mathbf{dl}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \text{ - Biot-Savart Law}$$

Two hollow spheres of radius a are centered at $(0,0,d)$ and $(0,0,-d)$, where $d > a$. They carry uniform surface charge density of σ and $-\sigma$, respectively, and rotate around z-axis at angular frequency ω.



(a) Find the potential, $V(\mathbf{r})$, and the electric field, $\mathbf{E}(\mathbf{r})$, inside and outside the spheres.

Solution:
The total charges on sphere 1 and 2 are Q and $-Q$, respectively, where $Q = 4\pi\sigma a^2$. The charge distribution on each sphere is spherically symmetric, therefore, applying the Gauss law, we obtain the potential and the electric field induced by the sphere 1,

$$V_1(\mathbf{r}) = \begin{cases} \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\mathbf{r}-\mathbf{d}|} & \text{outside the sphere 1} \\ \frac{1}{4\pi\varepsilon_0}\frac{Q}{a} & \text{inside the sphere 1} \end{cases}$$

$$\mathbf{E}_1(\mathbf{r}) = \begin{cases} \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{r}-\mathbf{d}}{|\mathbf{r}-\mathbf{d}|^3} & \text{outside the sphere 1} \\ 0 & \text{inside the sphere 1} \end{cases}$$

and the potential and the electric field induced by the sphere 2,

$$V_2(\mathbf{r}) = \begin{cases} -\frac{1}{4\pi\varepsilon_0}\frac{Q}{|\mathbf{r}+\mathbf{d}|} & \text{outside the sphere 1} \\ -\frac{1}{4\pi\varepsilon_0}\frac{Q}{a} & \text{inside the sphere 1} \end{cases}$$

$$\mathbf{E}_2(\mathbf{r}) = \begin{cases} -\frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{r}+\mathbf{d}}{|\mathbf{r}+\mathbf{d}|^3} & \text{outside the sphere 1} \\ 0 & \text{inside the sphere 1} \end{cases}$$

where $\mathbf{d} = d\hat{z}$.

Applying the superposition principle, we get

$$V(\mathbf{r}) = V_1(\mathbf{r}) + V_2(\mathbf{r}) = \begin{cases} \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{|\mathbf{r}-\mathbf{d}|} - \frac{1}{|\mathbf{r}+\mathbf{d}|}\right\} & \text{outside the spheres 1 and 2} \\ \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{a} - \frac{1}{|\mathbf{r}+\mathbf{d}|}\right\} & \text{inside the sphere 1} \\ \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{|\mathbf{r}-\mathbf{d}|} - \frac{1}{a}\right\} & \text{inside the sphere 2} \end{cases}$$

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r}) = \begin{cases} \frac{Q}{4\pi\varepsilon_0}\left\{\frac{\mathbf{r}-\mathbf{d}}{|\mathbf{r}-\mathbf{d}|^3} - \frac{\mathbf{r}+\mathbf{d}}{|\mathbf{r}+\mathbf{d}|^3}\right\} & \text{outside the spheres 1 and 2} \\ -\frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{r}+\mathbf{d}}{|\mathbf{r}+\mathbf{d}|^3} & \text{inside the sphere 1} \\ \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{r}-\mathbf{d}}{|\mathbf{r}-\mathbf{d}|^3} & \text{inside the sphere 2} \end{cases}$$

(b) Show that

$$V(r,\theta,\phi) \cong \frac{2\sigma a^2 d}{\varepsilon_0 r^2}\cos\theta$$

and

$$\mathbf{E}(r,\theta,\phi) \cong \frac{2\sigma a^2 d}{\varepsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

in the far zone for $r \gg d$, where $\mathbf{r} = (r,\theta,\phi)$ in the spherical coordinate.

Solution:
In the spherical coordinate,

$$\begin{aligned} |\mathbf{r}-\mathbf{d}| &= (r^2 + d^2 - 2rd\cos\theta)^{1/2} \\ |\mathbf{r}+\mathbf{d}| &= (r^2 + d^2 + 2rd\cos\theta)^{1/2} \\ \mathbf{r}-\mathbf{d} &= r\hat{r} - d\hat{z} = (r - d\cos\theta)\hat{r} + d\sin\theta\hat{\theta} \\ \mathbf{r}+\mathbf{d} &= r\hat{r} + d\hat{z} = (r + d\cos\theta)\hat{r} - d\sin\theta\hat{\theta} \end{aligned}$$

Therefore,

$$V(\mathbf{r}) = V_1(\mathbf{r}) + V_2(\mathbf{r}) = \begin{cases} \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{(r^2+d^2-2rd\cos\theta)^{1/2}} - \frac{1}{(r^2+d^2+2rd\cos\theta)^{1/2}}\right\} & \text{outside the spheres 1 and 2} \\ \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{a} - \frac{1}{(r^2+d^2+2rd\cos\theta)^{1/2}}\right\} & \text{inside the sphere 1} \\ \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{(r^2+d^2-2rd\cos\theta)^{1/2}} - \frac{1}{a}\right\} & \text{inside the sphere 2} \end{cases}$$

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r}) = \begin{cases} \frac{Q}{4\pi\varepsilon_0}\left\{\frac{(r-d\cos\theta)\hat{r}+d\sin\theta\hat{\theta}}{(r^2+d^2-2rd\cos\theta)^{3/2}} - \frac{(r+d\cos\theta)\hat{r}-d\sin\theta\hat{\theta}}{(r^2+d^2+2rd\cos\theta)^{3/2}}\right\} & \text{outside the spheres 1 and 2} \\ -\frac{Q}{4\pi\varepsilon_0}\frac{(r+d\cos\theta)\hat{r}-d\sin\theta\hat{\theta}}{(r^2+d^2+2rd\cos\theta)^{3/2}} & \text{inside the sphere 1} \\ \frac{Q}{4\pi\varepsilon_0}\frac{(r-d\cos\theta)\hat{r}+d\sin\theta\hat{\theta}}{(r^2+d^2-2rd\cos\theta)^{3/2}} & \text{inside the sphere 2} \end{cases}$$

When $r >> d$, $\mathbf{r}$ is outside the spheres, thus

$$\begin{aligned} V(r,\theta) &= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r}\left(1+\frac{d^2}{r^2}-2\frac{d}{r}\cos\theta\right)^{-1/2}-\frac{1}{r}\left(1+\frac{d^2}{r^2}+2\frac{d}{r}\cos\theta\right)^{-1/2}\right] \\ &\cong \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r}\left(1-2\frac{d}{r}\cos\theta\right)^{-1/2}-\frac{1}{r}\left(1+2\frac{d}{r}\cos\theta\right)^{-1/2}\right] \\ &\cong \frac{Q}{4\pi\varepsilon_0 r}\left[\left(1+\frac{d}{r}\cos\theta\right)-\left(1-\frac{d}{r}\cos\theta\right)\right] \\ &= \frac{Q}{4\pi\varepsilon_0 r}\left(\frac{2d}{r}\cos\theta\right)=\frac{2dQ}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2} \\ &= \frac{2d\sigma a^2}{\varepsilon_0}\frac{\cos\theta}{r^2} \end{aligned}$$

and

$$\begin{aligned} \mathbf{E}(r,\theta) &= -\nabla V(r,\theta) \cong -\nabla\left(\frac{2d\sigma a^2}{\varepsilon_0}\frac{\cos\theta}{r^2}\right) \\ &= -\frac{2d\sigma a^2}{\varepsilon_0}\left[\hat{r}\frac{\partial}{\partial r}\left(\frac{\cos\theta}{r^2}\right)+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{r^2}\right)\right] \\ &= \frac{2d\sigma a^2}{\varepsilon_0}\left[\hat{r}\left(\frac{2\cos\theta}{r^3}\right)+\hat{\theta}\left(\frac{\sin\theta}{r^3}\right)\right] \\ &= \frac{2d\sigma a^2}{\varepsilon_0 r^3}\left(2\cos\theta\hat{r}+\sin\theta\hat{\theta}\right) \end{aligned}$$

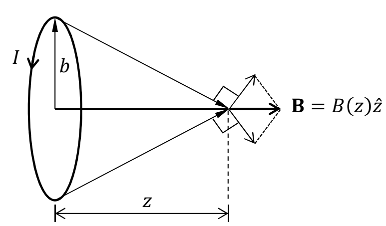
(c) Find the magnetic field $\mathbf{B}$ (i) at $\mathbf{r}=0$ and (ii) on z-axis for $|z| >> d$.

Solution:
(i) Since the currents on the spheres 1 and 2 flow in the opposite directions, the magnetic fields induced by the two spheres are antiparallel. Therefore, they are canceled out at the mid point and the total magnetic field vanishes at $\mathbf{r}=0$.

(ii) First, we calculate the magnetic field on z-axis generated by a circular ring current I, using the Biot-Savart law. Because of the circular symmetry, on z axis the magnetic field has only the z component.

$$B(z)=\frac{\mu_0 b^2 I}{2}\frac{1}{(z^2+b^2)^{3/2}}$$

where b is the ring radius.

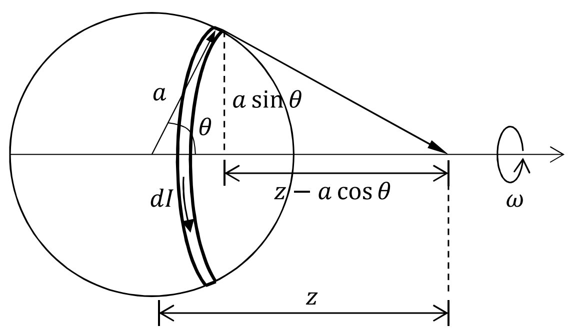


Second, we calculate the magnetic field on z-axis generated by a sphere. You can consider the rotating spherical shell as a collection of continuously attached rotating rings. As shown in the figure below, the current flowing through a thin circular strip on the sphere surface is

$$dI = \frac{dQ}{T} = \sigma\omega a^2 \sin\theta d\theta,$$

where $dQ = 2\pi\sigma a^2 d\theta$ and $T = 2\pi/\omega$. Thus, the magnetic field is

$$\begin{aligned} B(z) &= \frac{\mu_0}{2}\int \frac{a^2\sin^2\theta}{[(z-a\cos\theta)^2 + a^2\sin^2\theta]^{3/2}} dI \\ &= \frac{1}{2}\mu_0\sigma\omega a^2 \int_0^\pi \frac{sin^3\theta}{(z^2+a^2-2az\cos\theta)^{3/2}} d\theta \\ &\cong \frac{\mu_0\sigma\omega a^2}{2|z|^3}\int_0^\pi \sin^3\theta d\theta \\ &= \frac{2\mu_0\sigma\omega a^2}{3|z|^3} \end{aligned}$$



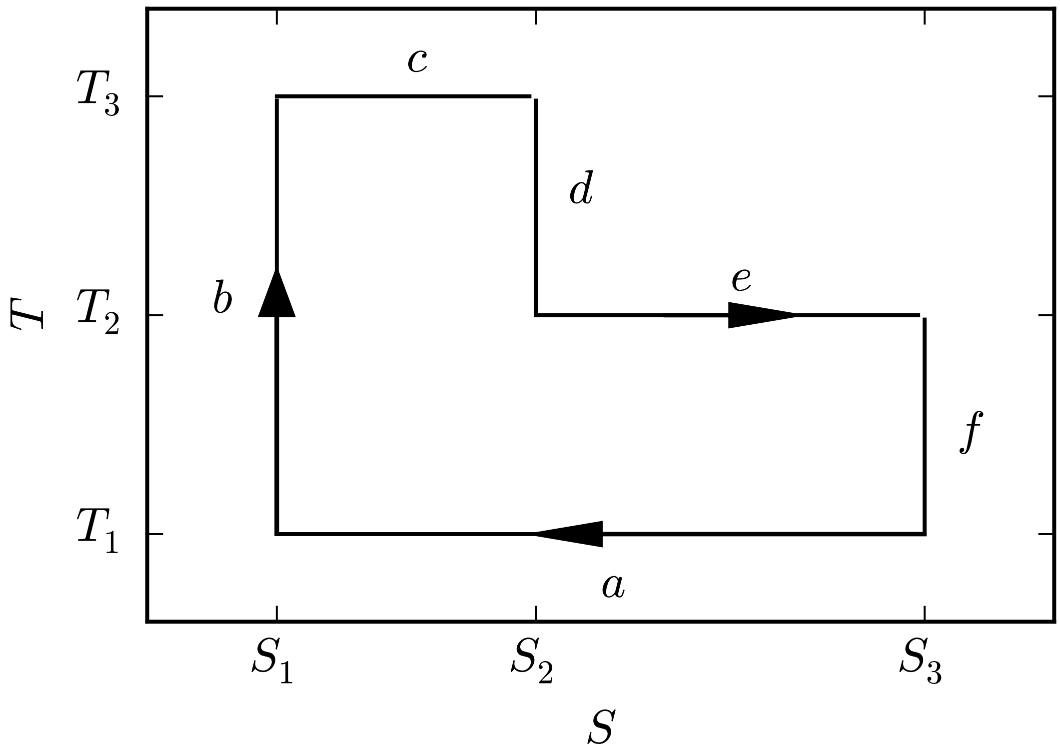
Third, the magnetic field generated by the two spheres is

$$\begin{aligned} B(z) &= \frac{2\mu_0\sigma\omega a^2}{3|z-d|^3} - \frac{2\mu_0\sigma\omega a^2}{3|z+d|^3} \\ &= \frac{2\mu_0\sigma\omega a^2}{3|z|^3}\left[\left(1-\frac{d}{|z|}\right)^3 - \left(1+\frac{d}{|z|}\right)^3\right] \\ &\cong \frac{2\mu_0\sigma\omega a^2}{3|z|^3}\left[\left(1+3\frac{d}{|z|}\right) - \left(1-3\frac{d}{|z|}\right)\right] \\ &= \frac{4\mu_0\sigma\omega d a^2}{z^4} \end{aligned}$$

Useful formula: The magnetic field $\mathbf{B}(\mathbf{r})$ at position $\mathbf{r}$ generated by a steady current I is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_C \frac{I\mathbf{dl}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \text{ - Biot-Savart Law}$$

Consider the following cycle with a non-ideal gas such as freon.



(a) **(6 pts)** What is the energy transferred by heating in each step?

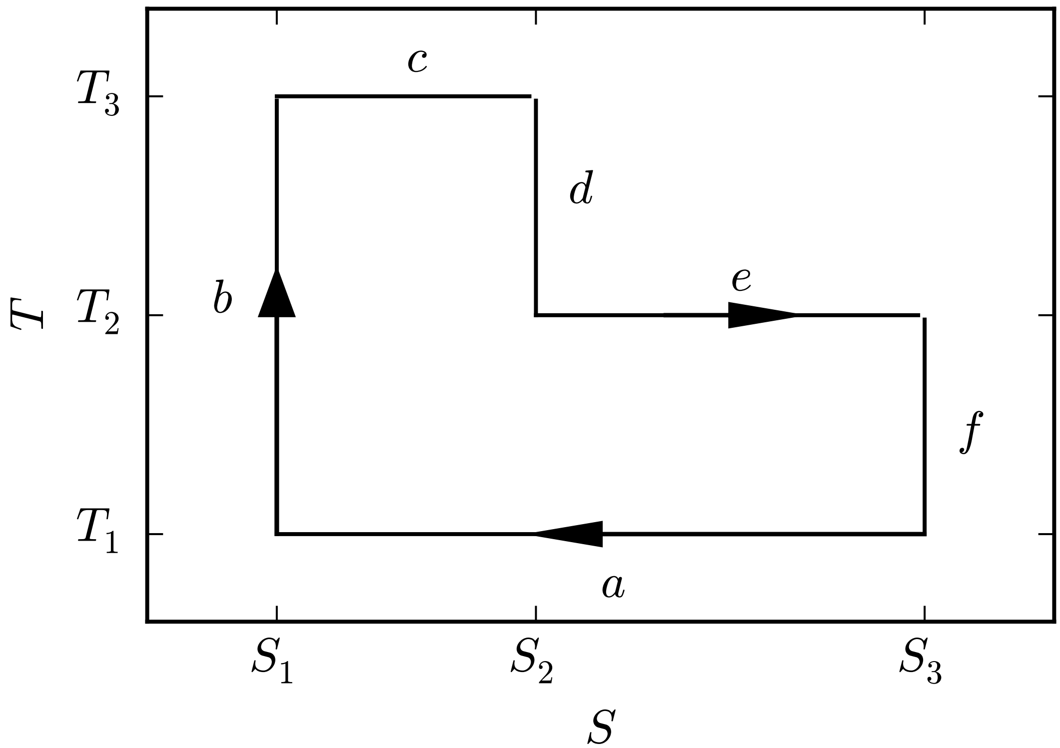
(b) **(3 pts)** What is the net transfer of energy by heating over one cycle? Clarify whether this energy is gained or lost by the system.

(c) **(3 pts)** What is the net transfer of energy by working over one cycle? Clarify whether this energy is gained or lost by the system.

(d) **(5 pts)** If this cycle is used as a heat engine, what is its efficiency?

(e) **(3 pts)** If you can change only S_2, how could you maximize the efficiency of the heat engine? (keeping $S_1 < S_2 < S_3$)

Consider the following cycle with a non-ideal gas such as freon.



(a) **(6 pts)** What is the energy transferred by heating in each step?

Solution:
Remember that for a quasistatic process, $Q = \int TdS$, which is easy to compute for each step of this curve on the TS diagram, since the integrals are each just rectangles.

$$Q_a = -T_1(S_3 - S_1)$$
$$Q_b = 0$$
$$Q_c = T_3(S_2 - S_1)$$
$$Q_d = 0$$
$$Q_e = T_2(S_3 - S_2)$$
$$Q_f = 0$$

A positive value of Q here means energy transfered *to* the system by heating, so it is isothermally heated in steps c and e, and isothermally cooled in step a.

(b) **(3 pts)** What is the net transfer of energy by heating over one cycle? Clarify whether this energy is gained or lost by the system.

Solution:
Obviously we just have to add up the above heat transfers. The total comes out to

$$\begin{aligned} Q &= T_3(S_2 - S_1) + T_2(S_3 - S_2) - T_1(S_3 - S_1) \\ &= (T_3 - T_1)(S_2 - S_1) + (T_2 - T_1)(S_3 - S_2) \end{aligned}$$

This is a positive value, so the system is heated by a complete cycle.

(c) **(3 pts)** What is the net transfer of energy by working over one cycle? Clarify whether this energy is gained or lost by the system.

Solution:
Because this is a cycle, the change in internal energy must be zero, which means that the energy transfer by working must be opposite to the energy transfer by heating. Thus

$$\begin{aligned} W &= -Q \\ &= -(T_3 - T_1)(S_2 - S_1) - (T_2 - T_1)(S_3 - S_2) \end{aligned}$$

i.e. the system does work on its environment, which makes this a kind of a heat engine.

(d) **(5 pts)** If this cycle is used as a heat engine, what is its efficiency?

Solution:
The efficiency is the ratio of what you get out to what you put in. In a heat engine, you get out work, and put in heat energy. The key is that the energy *added to the system* by heating rather than the net heat, since you can't recycle the energy lost due to heating the cool bath (without doing extra work).

$$\begin{aligned} \epsilon &= \frac{|W|}{|Q_{added}|} \\ &= \frac{(T_3 - T_1)(S_2 - S_1) + (T_2 - T_1)(S_3 - S_2)}{Q_c + Q_e} \\ &= \frac{(T_3 - T_1)(S_2 - S_1) + (T_2 - T_1)(S_3 - S_2)}{T_3(S_2 - S_1) + T_2(S_3 - S_2)} \end{aligned}$$

(e) **(3 pts)** If you can change only S_2, how could you maximize the efficiency of the heat engine? (keeping $S_1 < S_2 < S_3$)

Solution:
To maximize the efficiency by only changing S_2, we can use one of two approaches.

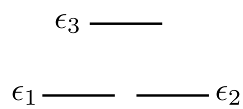
Firstly, we can recognize that a Carnot cycle with a maximum ratio between high and low temperature is optimal, which in this case means that we want all the heating (and cooling) done at T_3 and T_1. We can achieve this by making $S_2 = S_3$.

Alternatively, we could examine the form of our answer to the previous question. Because $T_3 > T_2$,

$$\frac{T_3 - T_1}{T_3} > \frac{T_2 - T_1}{T_2}$$

we can maximize the efficiency by maximizing $(S_2 - S_1)$ at the expense of making $(S_3 - S_2) = 0$.

Consider a system consisting of N distinguishable noninteracting particles. Each particle has just 3 energy eigenstates, with energies $\epsilon_1 = \epsilon_2 < \epsilon_3$, i.e. two of the three states have the same energy, and the other state has a higher energy.



(a) **(10 pts)** What is the entropy of this system as a function of temperature?

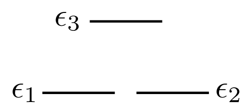
(b) **(4 pts)** What is the entropy in the limit of low temperatures?
Clarify what you mean by "low temperature."

(c) **(4 pts)** What is the entropy in the limit of high temperatures?
Clarify what you mean by "high temperature."

(d) **(2 pts)** Is there an upper bound on the entropy of this system?
Why or why not?

Note: it is possible to solve (b), (c), and (d) correctly (and receive full credit for them) without solving (a).

Consider a system consisting of N distinguishable noninteracting particles. Each particle has just 3 energy eigenstates, with energies $\epsilon_1 = \epsilon_2 < \epsilon_3$, i.e. two of the three states have the same energy, and the other state has a higher energy.



(a) **(10 pts)** What is the entropy of this system as a function of temperature?

Solution:
Because these are distinguishable noninteracting particles, the partition function separates such that

$$Z_{tot} = Z_1^N$$

where Z_1 is the partition function for a single particle. We can find this from the definition of the total partition function:

$$\begin{aligned}
Z_{tot} &= \sum_{1}^{\text{all states}} e^{-\beta E_i} \\
&= \sum_{i_1=1}^{3} \cdots \sum_{i_N=1}^{3} e^{-\beta(\epsilon_{i1} + \cdots + \epsilon_{iN})} \\
&= \sum_{i_1=1}^{3} \cdots \sum_{i_N=1}^{3} e^{-\beta\epsilon_{i1}} \cdots e^{-\beta\epsilon_{iN}} \\
&= \left(\sum_{i_1=1}^{3} e^{-\beta\epsilon_{i1}}\right) \cdots \left(\sum_{i_N=1}^{3} e^{-\beta\epsilon_{iN}}\right) \\
&= \left(\sum_{i=1}^{3} e^{-\beta\epsilon_i}\right)^N
\end{aligned}$$

I always like to start with the Helmholtz free energy, which is given by

$$\begin{aligned}
F &= -kT \ln Z_{tot} \\
&= -NkT \ln\left(\sum_{i=1}^{3} e^{-\beta\epsilon_i}\right) \\
&= -NkT \ln\left(2e^{-\beta\epsilon_1} + e^{-\beta\epsilon_3}\right) \\
&= -NkT \ln\left(e^{-\beta\epsilon_1}\left(2 + e^{-\beta(\epsilon_3 - \epsilon_1)}\right)\right) \\
&= -NkT\left(-\beta\epsilon_1 + \ln\left(2 + e^{-\beta(\epsilon_3 - \epsilon_1)}\right)\right) \\
&= \epsilon_1 - NkT \ln\left(2 + e^{-\beta(\epsilon_3 - \epsilon_1)}\right)
\end{aligned}$$

Towards the end, I simplified the equation by writing it in terms of the energy difference, which is all that really matters for this problem. Now we can remember that the entropy is related to the free energy by

$$\begin{aligned} S &= -\left(\frac{\partial F}{\partial T}\right)_{V,N} \\ &= Nk\ln\left(2 + e^{-\beta(\epsilon_3-\epsilon_1)}\right) + N\frac{\epsilon_3-\epsilon_1}{T}\frac{e^{-\beta(\epsilon_3-\epsilon_1)}}{2+e^{-\beta(\epsilon_3-\epsilon_1)}} \end{aligned}$$

The next two questions do the answer-checking that I would normally do at this stage to check whether this answer seems sensible. I'll just do one additional check here, which is the case that $\epsilon_1 = \epsilon_3$. This is functionally equivalent to high temperatures, but we can also think of it as a different system that is a limiting case of this one. In this case we have a system with three degenerate levels. Since there is no meaningful energy scale in the system, its entropy cannot be temperature-dependent, so it is a relief that the second term exactly drops out in this limit. The result becomes $S = Nk\ln 3$, which is exactly what we would expect for N particles, each of which can be in each of 3 states with equal probability. We also note that the entropy when the energies differ is always lower than this, which is good, because those energy differences can only reduce the number of accessible states.

(b) **(4 pts)** What is the entropy in the limit of low temperatures?
Clarify what you mean by "low temperature."

Solution:
By low temperature, I mean that $\beta(\epsilon_3 - \epsilon_1) \gg 1$. In this limit, the exponential terms vanish, leaving us with

$$S \approx Nk\ln 2$$

This value makes sense, as we know that at very low temperatures only the two degenerate ground states of each particle are accessible, so the entropy should be (using Boltzmann's relation $S = k\ln\Omega$) $k\ln 2$ for each particle.

It is interesting to note that this answer may seem to contradict the Third Law of Thermodynamics, since the entropy doesn't approach zero as the temperature approaches zero. This is why careful statements of the Third Law specify that the entropy of a *perfect crystal* approaches zero as the temperature approaches zero.

(c) **(4 pts)** What is the entropy in the limit of high temperatures?
Clarify what you mean by "high temperature."

Solution:
High temperature is defined by $\beta(\epsilon_3 - \epsilon_1) \ll 1$. In this limit, the exponentials approach 1, and the entropy becomes

$$S \approx Nk\,(\ln 3)$$

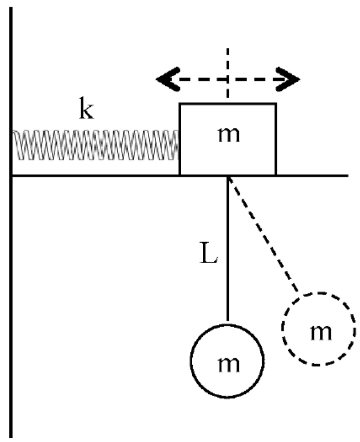
So, S naturally approaches $Nk\ln 3$, as we should expect from Boltzmann's expression for entropy, since at high temperature all three states are fully accessible.

(d) **(2 pts)** Is there an upper bound on the entropy of this system?
Why or why not?

Solution:
There is an upper bound of the entropy which is $Nk \ln 3$. This makes sense because there are a finite number of states (3 per particle). If there were an infinite number of energy states possible, there would be no upper bound for the entropy. Because kinetic energy is always a possibility with no upper bound, no physical material can have a finite upper bound on its entropy.

Note: it is possible to solve (b), (c), and (d) correctly (and receive full credit for them) without solving (a).

Pendulum-Spring Oscillator. Consider a bob-pendulum of length L and mass m that is attached to a block also of mass m. This block is free to move horizontally on a frictionless surface. This block is further connected to a wall with a spring of spring constant k.
For simplicity, you may imagine the system is constructed such that the characteristic frequencies (ω_o) of the uncoupled block or pendulum are equal, i.e. $\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{L}}$.



(a) Find the frequencies of the normal modes of this system. You may assume small oscillations around the equilibrium position of the pendulum.

(b) Find mathematically and describe in words, and the relative motions of the normal modes of the two masses.

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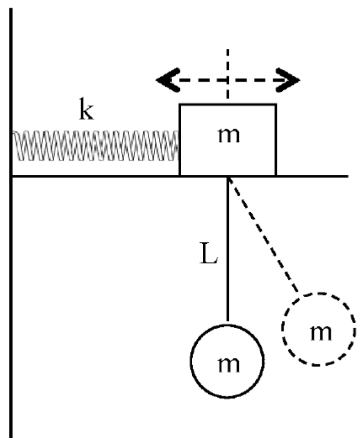

(a) Find the frequencies of the normal modes of this system. You may assume small oscillations around the equilibrium position of the pendulum.

Solution:
Question credit: M. Shaevitz, Columbia Qual Exam Jan. 2015

Write the Lagrangian, $\mathcal{L} = T - U$.
First we define a coordinate system in terms of the pendulum-mass angular displacement from vertical, θ and the block's linear displacement, x.

The kinetic energy (T) is that of the block plus the pendulum's kinetic energy terms (remember we need to add the linear velocity terms for the pendulum to get the total velocity and calculate the kinetic energy). Specifically in the small oscillation approximation for a pendulum, all kinetic motion is in the horizontal (x) plane, giving

$$T = \frac{1}{2}m(L\dot{\theta} + \dot{x})^2 + \frac{1}{2}m\dot{x}^2 \tag{5}$$

The potential energy is the harmonic spring potential and the angular gravitational potential of the pendulum height displacement given as,

$$U = \frac{1}{2}kx^2 + mgL(1 - \cos\theta) \tag{6}$$

$$\cong \frac{1}{2}kx^2 + mgL\frac{\theta^2}{2} \tag{7}$$

where in the last step we used the small angle approximation to Taylor expand the cosine term, i.e. $1 - \cos\theta \cong \theta^2/2$.

The system Lagrangian is then:

$$\mathcal{L} = T - U \tag{8}$$
$$= \frac{1}{2}m(L\dot{\theta} + \dot{x})^2 + \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - mgL\frac{\theta^2}{2} \tag{9}$$

We then use the Euler-Lagrange equations to get the equation for motion. With respect to θ we get,

$$0 = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} \tag{10}$$
$$0 = mL^2\ddot{\theta} + mL\ddot{x} + mgL\theta \tag{11}$$
$$0 = L^2\ddot{\theta} + L\ddot{x} + L^2\omega_o^2\theta \tag{12}$$

Likewise with respect to x we get,

$$0 = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) - \frac{\partial\mathcal{L}}{\partial x} \tag{13}$$
$$0 = mL\ddot{\theta} + 2m\ddot{x} + kx \tag{14}$$
$$0 = L\ddot{\theta} + 2\ddot{x} + \omega_o^2 x \tag{15}$$

The resulting coupled harmonic oscillator equations of motion above may be solved by assuming an exponential form solution, $\theta(t) = A\exp(i\omega t)$ and $x(t) = B\exp(i\omega t)$, which gives,

$$0 = -AL^2\omega^2 - BL\omega^2 + AL^2\omega_o^2 \tag{16}$$
$$0 = -AL\omega^2 - 2B\omega^2 + B\omega_o^2 \tag{17}$$

solving this system of equations by method of determinants we get a matrix system,
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -L^2\omega^2 + L^2\omega_o^2 & -L\omega^2 \\ -L\omega^2 & -2\omega^2 + \omega_o^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$
We can now solve for the eigenvalues (or characteristic frequencies) of matrix by the determinant,

$$2L^2\omega^4 - L^2\omega^2\omega_o^2 - 2L^2\omega_o^2\omega^2 + L^2\omega_o^4 - L^2\omega^4 = 0 \tag{18}$$
$$\Rightarrow \omega^4 - 3\omega_o^2\omega^2 + \omega_o^4 = 0 \tag{19}$$

Hence, the characteristic frequencies of this coupled system are $\omega^2 = \frac{1}{2}\left(3 \pm \sqrt{5}\right)\omega_o^2$ or $\omega = \sqrt{\frac{1}{2}\left(3 \pm \sqrt{5}\right)\frac{g}{L}}$.

(b) Find mathematically and describe in words, and the relative motions of the normal modes of the two masses.

Solution:
Here we can find the normal modes by the eigenvectors. We note that the amplitude (A and B) of oscillation are related by (equation 16):

$$-AL^2\omega^2 - BL\omega^2 + AL^2\omega_o^2 = 0 \tag{20}$$

$$\Rightarrow A = \frac{B}{L}\left(\frac{\omega^2}{\omega_o^2 - \omega^2}\right) \tag{21}$$

Subbing in eigenvalues $\omega^2 = \frac{1}{2}\left(3 \pm \sqrt{5}\right)\omega_o^2$ in the above equation we get,

$$A = \frac{B}{2L}\frac{\sqrt{5}+3}{(-2-\sqrt{5})} \tag{22}$$

$$\Rightarrow A = -\frac{1}{L}\frac{\sqrt{5}+1}{2}B \tag{23}$$

when $\omega^2 = \frac{1}{2}\left(3 + \sqrt{5}\right)\omega_o^2$. Likewise when $\omega^2 = \frac{1}{2}\left(3 - \sqrt{5}\right)\omega_o^2$, we get

$$A = \frac{1}{L}\frac{\sqrt{5}-1}{2}B \tag{24}$$

Hence the normal modes of the system will be anti-symmetric (scissor-like motion of mass and block) and symmetric (block and masse motion together). The general solution for the motion will, in general, be a superposition of the these normal modes (full equations are determined by the system initial conditions).

Rail car jumpin'. A railroad car of mass M is initially at rest on a frictionless track. Imagine there are N people each of mass m that are initially standing at rest on the car.

(a) Case 1: All N people run to the end of the railroad car in unison and reach a speed, relative to the car of v_p. At that point, they all jump off at once. Calculate the resulting velocity of the car relative to the ground after they have jumped off.

(b) Case 2: N people jump off one at a time. Specifically, the people remain at rest relative to the car, while one of them runs to the end, attains a velocity v_p and jumps off. Then the next person starts running, attains the same speed v_p relative to the cart, and jumps off. This continues until all N people have jumped off.
Derive an expression for the final velocity of the railroad car relative to the ground.

(c) Under which case will the railroad car attain a greater velocity? (case 1, case 2 or both are equal).
Use both your equations from parts a & b, AND a qualitative physical description to support your answer.

Rail car jumpin'. A railroad car of mass M is initially at rest on a frictionless track. Imagine there are N people each of mass m that are initially standing at rest on the car.

(a) Case 1: All N people run to the end of the railroad car in unison and reach a speed, relative to the car of v_p. At that point, they all jump off at once. Calculate the resulting velocity of the car relative to the ground after they have jumped off.

Solution:
This problem is best solved using conservation of linear momentum and choosing a reference frame at rest with respect to the ground. Clearly, the car and people will move in opposite direction to conserve momentum, let's define the velocity the car moves be the positive direction, v_c. Initially, the train and the people are at rest and the momentum is zero. After all N people initially at speed v_p jump off, the velocity of rail car is $+v_c$ and the speed of the N people relative to the ground in $-v_p + v_c$. Hence, conservation of momentum gives,

$$p_{initial} = p_{final} \tag{25}$$
$$0 = Mv_c + Nm(v_c - v_p) \tag{26}$$

Solving for v_c we get the resulting rail car speed relative the ground,

$$v_c = \frac{Nm}{M + Nm} v_p \tag{27}$$

(b) Case 2: N people jump off one at a time. Specifically, the people remain at rest relative to the car, while one of them runs to the end, attains a velocity v_p and jumps off. Then the next person starts running, attains the same speed v_p relative to the cart, and jumps off. This continues until all N people have jumped off.
Derive an expression for the final velocity of the railroad car relative to the ground.

Solution:

Lets define v_n to be the velocity of the car relative to the ground when there are n people of mass m aboard. A general solution for this sequential jumping scenario, can be obtained by applying momentum conservation to the event when the car transitions from n people aboard to $n-1$ people. Hence when there are n people aboard the initial momentum, p_n is

$$p_n = Mv_n + nmv_n \tag{28}$$

Once the n^{th} jumps off, the total momentum (p_{n-1}) is that of the rail car (same expression as above for $n-1$) plus momentum of the jumper $m(v_{n-1} - v_c)$,

$$p_{n-1} = Mv_{n-1} + m(n-1)v_{n-1} + m(v_{n-1} - v_p) \tag{29}$$
$$p_{n-1} = (M + nm)v_{n-1} - mv_p \tag{30}$$

Since total momentum in conserved in each step (there are no external forces outside our system), $p_{final} = p_n = p_{n-1} = p_{initial} = 0$, and we can can evaluate for the resulting speed (v_{n-1}) added the car after each jump,

$$p_n = p_{n-1} \tag{31}$$
$$Mv_n + nmv_n = (M + nm)v_{n-1} - mv_p \tag{32}$$
$$\Rightarrow 0 = (M + nm)v_{n-1} - mv_p \tag{33}$$
$$v_{n-1} = \frac{mv_p}{M + nm} \tag{34}$$

Hence the final velocity of the rail car (v_c) is just the summation of the velocities given to the cart (relative the ground) by each of the N jumpers, specifically

$$v_c = \sum_{n=1}^{N} \frac{mv_p}{M + nm} \tag{35}$$

Note: alternatively, one can arrive at a solution by analyzing the cases of 1, 2 and 3 jumpers separately and rigorously using the principle of mathematical induction to prove the case for N jumpers. If doing it this way, you may likely arrive at the mathematically equivalent expression:

$$v_c = \sum_{n=0}^{N-1} \frac{mv_p}{M + (N - n)m} \tag{36}$$

(c) Under which case will the railroad car attain a greater velocity? (case 1, case 2 or both are equal).
Use both your equations from parts a & b, AND a qualitative physical description to support your answer.

Solution:
In Case 2 the rail cars attains the larger final speed. Mathematically this can be seen by inspection of,

$$\sum_{n=1}^{N} \frac{mv_p}{M + nm} > \frac{Nm}{M + Nm} v_p \tag{37}$$

since for small $n < N$ the denominator in case 2 is smaller. The physical explanation is that when the jump off one-at-a-time, each successive person is imparting a momentum impulse on a slightly lighter cart, which corresponds to a faster cart recoil. (In case 1, the all N people have to impart an impulse which has to push the cart and all N people, hence the cart recoil is less than case 2)