OSU Physics Department
Comprehensive Examination #123

Thursday, September 24 & Friday, September 25, 2015

Fall 2015 Comprehensive Examination

PARTS 1, 2, 3 & 4

General Instructions

This Fall 2015 Comprehensive Examination consists of four separate parts of two problems each. Each problem caries equal weight (20 points each). The first part (Quantum Mechanics) is handed out at 9:00 am on Thursday, September 24 , and lasts three hours. The second part (Electricity and Magnetism) will be handed out at 1:00 pm on the same day and will also last three hours. The third (Statistical Mechanics) and fourth (Classical Mechanics) parts will be administered on Friday, September 25, at 9:00 am and 1:00 pm, respectively. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Use the last pages of your bluebooks for "scratch" work, separated by at least one empty page from your solutions. "Scratch" work will not be graded.

If you submit blue books for any given section, that section will be graded as part of your cumulative score. Unless you are taking the exam for practice, all sections not previously passed need to be attempted and submitted.

A quantum mechanical particle of mass m moves in the potential:

$$V(z) = \begin{cases} mgz, & \text{if } z > 0. \\ +\infty, & \text{if } z < 0. \end{cases}$$

where z is the position coordinate (height) and g is the acceleration due to gravity.

A trial wavefunction

$$\psi(z) = \begin{cases} Cze^{-az}, & \text{if } z > 0. \\ 0, & \text{if } z < 0. \end{cases}$$

has the correct qualitative shape to be the ground state wavefunction, although it does not exactly describe the ground state wavefunction.

(a) List three features of $\psi(z)$ that make this function a physically reasonable choice.

(b) Use $\psi(z)$ and the variational principle to estimate the ground state energy of the particle in terms of $\hbar$, m and g.

a) $\psi = 0$ for $z < 0$; this is the forbidden region

$\psi \to 0$ for $z \to +\infty$; this is also a forbidden region for large z.

The wavefn has ~~a single peak and~~ no nodes (minimal curvature) as expected for ground state.

b) First normalize the trial wavefn.

$$\int_{-\infty}^{\infty} |\psi|^2 dz = \int_0^{\infty} |C|^2 z^2 e^{-2az} dz = \frac{|C|^2}{4a^3} = 1$$

$$\Rightarrow |C|^2 = 4a^3$$

set $C = \sqrt{4a^3}$

Energy expectation value

$$\langle E \rangle = \langle \psi | H | \psi \rangle$$

$$= \int_0^{\infty} \psi^*(z) \left[-\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + mgz \right] \psi(z)\, dz$$

$$= 4a^3 \int_0^{\infty} z e^{-az} \left(-\frac{\hbar^2}{2m} \frac{d^2}{dz^2} \left(z e^{-az} \right) + mgz^2 e^{-az} \right) dz$$

$$= 4a^3 \int_0^{\infty} z e^{-az} \left(-\frac{\hbar^2}{2m} \left(-2a e^{-az} + a^2 z e^{-az} \right) + mgz^2 e^{-az} \right) dz$$

$$= 4a^3 \int_0^\infty \left(\frac{\hbar^2}{2m} \left(2aze^{-2az} - a^2z^2e^{-2az} \right) + mgz^3e^{-2az} \right) dz$$

$$= 4a^3 \left[\frac{\hbar^2}{2m} \left(\frac{2a}{(2a)^2} - \frac{2a^2}{(2a)^3} \right) + mg\frac{6}{(2a)^4} \right]$$

$$= 4a^3 \left[\frac{\hbar^2}{8ma} + \frac{3mg}{8a^4} \right]$$

$$\boxed{\langle E \rangle = \frac{\hbar^2 a^2}{2m} + \frac{3mg}{2a}}$$

Now we determine the value of a that minimizes $\langle E \rangle$

Extrema occurs when $\frac{d\langle E \rangle}{da} = 0$

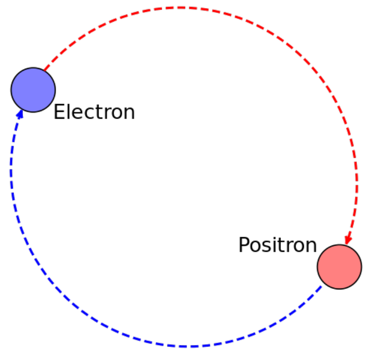
$$\frac{d\langle E \rangle}{da} = \frac{\hbar^2 a}{m} - \frac{3mg}{2a^2} = 0$$

$$\Rightarrow \quad \frac{\hbar^2 a}{m} = \frac{3mg}{2a^2}$$

$$\boxed{a = \left(\frac{3m^2 g}{2\hbar^2} \right)^{1/3}}$$

The estimated value of the ground state energy is thus

$$\boxed{\langle E \rangle_{min} = \frac{\hbar^2}{2m} \left(\frac{3m^2 g}{2\hbar^2} \right)^{2/3} + \frac{3mg}{2} \left(\frac{2\hbar^2}{3m^2 g} \right)^{1/3}}$$



Positronium is a bound state of an electron $\left(s_1 = \frac{1}{2}\right)$ and a positron $\left(s_2 = \frac{1}{2}\right)$. The Hamiltonian for the system in a magnetic field B can be written as

$$H = H_o + \frac{A}{\hbar^2}\left(\mathbf{S_1} \cdot \mathbf{S_2}\right) + \frac{\mu_B B}{\hbar}\left(S_{1,z} - S_{2,z}\right)$$

where $\mathbf{S_1}$ and $\mathbf{S_2}$ are spin operators for the electron and positron respectively. For example, the eigenstates of $S_{1,z}$ are $|\uparrow_1\rangle$ and $|\downarrow_1\rangle$ such that

$$S_{1,z}\,|\uparrow_1\rangle = +\frac{\hbar}{2}\,|\uparrow_1\rangle$$
$$S_{1,z}\,|\downarrow_1\rangle = -\frac{\hbar}{2}\,|\downarrow_1\rangle$$

(a) At $B = 0$ there is an energy difference between different spin configurations of positronium. Spectroscopy experiments have measured this splitting to be 200 GHz. Find the value of A in units of Joules or electron Volts.

(b) Consider the spin configurations of positronium that are energy eigenstates when $B > 0$. Find the energies of these states (relative to E_o) when $B > 0$. Express your answers in terms of A, $\hbar$, μ_B and B.

Useful information: $\hbar \cong 10^{-34} J \cdot s$

a) When $B=0$ $\qquad H = H_0 + \frac{A}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2$

Define the total spin of positronium $\vec{S} = \vec{S}_1 + \vec{S}_2$

$$|\vec{S}|^2 = |\vec{S}_1|^2 + |\vec{S}_2|^2 + \vec{S}_1 \cdot \vec{S}_2$$

$$\Rightarrow \quad \vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(S^2 - S_1^2 - S_2^2\right)$$

$$= \frac{1}{2}\left(S^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2\right)$$

(all spin configurations are eigenstates of $|\vec{S}_1|^2$ and $|\vec{S}_2|^2$, the eigenvalues are always $\hbar^2 s(s+1)$ where $s = 1/2$.)

The eigenstates of $|\vec{S}|^2$ are the spin singlet and spin triplet configurations, $s=0$ and $s=1$ respectively.

singlet

$s=0$ $\qquad$ Eigen energy is $\frac{A}{\hbar^2}\frac{1}{2}\left(-\frac{6}{4}\hbar^2\right) = -A\left(\frac{3}{4}\right)$

triplet

$s=1$ $\qquad$ Eigen energy is $\frac{A}{\hbar^2}\frac{1}{2}\left(2\hbar^2 - \frac{6}{4}\hbar^2\right) = \frac{A}{4}$

The energy difference is $A = hf = (h)(200\,\text{GHz})$

$$= (6\times10^{-34})(2\times10^{11})$$

$$= 12\times10^{-23}\,\text{J}$$

$$= 0.8\,\text{meV}$$

b) At finite B, the spin-dependent part of the Hamiltonian is

$$H_{spin} = \frac{A}{\hbar^2}\frac{1}{2}\left(S^2 - \frac{3}{2}\hbar^2\right) + \frac{\mu_B B}{\hbar}\left(S_{1z} - S_{2z}\right)$$

I want to express H_{spin} as a 4×4 matrix.
The 4 basis states can be described using
s ; the total spin quantum number
s_z ; the z-projection of total spin

Find the matrix elements

$$\langle 1,1|\ \frac{A}{\hbar^2}\frac{1}{2}\left(S^2 - \frac{3}{2}\hbar^2\right) + \frac{\mu_B B}{\hbar}\left(S_{1z} - S_{2z}\right)\ |1,1\rangle$$

$$= \frac{A}{\hbar^2}\frac{1}{2}\left(\hbar^2(1)(2) - \frac{3}{2}\hbar^2\right) + \frac{\mu_B B}{\hbar}\left(\frac{\hbar}{2} - \frac{\hbar}{2}\right)$$

$$= \frac{A}{2}\left(2 - \frac{3}{2}\right)$$

$$= A\left(1 - \frac{3}{4}\right) = A/4$$

Similarly

$$\langle 1,-1|\ H_{spin}\ |1,-1\rangle = A/4$$

There are two states with $s_z = 0$. These are more tricky.
Recall that

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$\langle 1,0|\ \frac{A}{\hbar^2}\frac{1}{2}\left(S^2-\frac{3}{2}\hbar^2\right)+\frac{\mu_B B}{\hbar}\left(S_{1z}-S_{2z}\right)\Big|1,0\rangle$$

$$=\frac{A}{\hbar^2}\frac{1}{2}\left(\hbar^2(1)(2)-\frac{3}{2}\hbar^2\right)$$

$$=\frac{A}{2}\left(2-\frac{3}{2}\right)=\frac{A}{4}$$

$$\langle 0,0|\,H_{spin}\,|0,0\rangle=\frac{A}{\hbar^2}\frac{1}{2}\left(0-\frac{3}{2}\hbar^2\right)=-\frac{3}{4}A$$

$$\langle 1,0|\,H_{spin}\,|0,0\rangle=\langle 1,0|\,\frac{\mu_B B}{\hbar}\left(S_{1z}-S_{2z}\right)\Big|0,0\rangle$$

$$=\frac{\mu_B B}{\hbar}\left(\frac{1}{\sqrt{2}}\left(\langle\uparrow\downarrow|+\langle\downarrow\uparrow|\right)\left(S_{1z}-S_{2z}\right)\left(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle\right)\frac{1}{\sqrt{2}}\right)$$

$$=\frac{\mu_B B}{2\hbar}\left(\left(\langle\uparrow\downarrow|+\langle\downarrow\uparrow|\right)\left(\frac{\hbar}{2}|\uparrow\downarrow\rangle+\frac{\hbar}{2}|\uparrow\downarrow\rangle+\frac{\hbar}{2}|\downarrow\uparrow\rangle+\frac{\hbar}{2}|\downarrow\uparrow\rangle\right)\right)$$

$$=\frac{\mu_B B}{2\hbar}\left(\hbar+\hbar\right)=\mu_B B$$

Similarly

$$\langle 0,0|\,H_{spin}\,|1,0\rangle=\mu_B B$$

The 4x4 matrix is

$$H_{spin}=\begin{bmatrix}\frac{A}{4} & 0 & 0 & 0\\ 0 & \frac{A}{4} & 0 & \mu_B B\\ 0 & 0 & A/4 & 0\\ 0 & \mu_B B & 0 & -\frac{3A}{4}\end{bmatrix}$$

The states $|1,1\rangle$ and $|1,-1\rangle$ remain eigenstates at any B-field and have eigen energy $\frac{A}{4}$.

The states $|1,0\rangle$ and $|0,0\rangle$ are connected by off-diagonal elements

$$\begin{matrix} |1,0\rangle \\ |0,0\rangle \end{matrix} \quad \begin{bmatrix} A/4 & \mu_B B \\ \mu_B B & -\frac{3A}{4} \end{bmatrix}$$

Diagonalize this 2×2 matrix to find the eigen-energies, E_i

$$\left(\frac{A}{4} - E_i\right)\left(\frac{-3A}{4} - E_i\right) - \mu_B^2 B^2 = 0$$

$$E_i^2 + \frac{AE_i}{2} - \frac{3}{16}A^2 - \mu_B^2 B^2 = 0$$

$$E_i = \frac{-\frac{A}{2} \pm \sqrt{\frac{A^2}{4} - 4\left(-\frac{3}{16}A^2 - \mu_B^2 B^2\right)}}{2}$$

$$= -\frac{A}{4} \pm \frac{1}{2}\sqrt{A^2 + 4\mu_B^2 B^2}$$

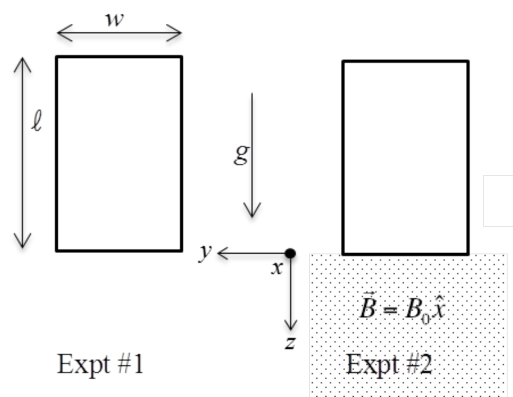
One eigenstate has energy

$$\boxed{-\frac{A}{4} + \frac{1}{2}\sqrt{A^2 + 4\mu_B^2 B^2}}$$

The other eigenstate has energy

$$\boxed{-\frac{A}{4} - \frac{1}{2}\sqrt{A^2 + 4\mu_B^2 B^2}}$$

A rigid metal wire frame in the plane of the paper (yz plane) is released from rest and falls to the ground. It is close to earth, so the gravitational acceleration, g, is constant. In a second experiment, there is a constant magnetic field in the positive $\hat{x}$ direction (out of the page) in the semi-infinite halfspace $z > 0$. The situation is depicted below.



Parameters that may be relevant are:
B_0, magnitude of the magnetic field
w, width of the frame
ℓ, length of frame
m, mass of the frame
R, electrical resistance of the frame
g, acceleration due to gravity

(a) Describe, qualitatively but carefully, how and why the motion of the wire frame is different in experiment 2 from the motion in experiment 1. Be sure to discuss the situations where the frame is entirely outside the field region, partially in the field region and completely within the field region.

(b) Would your answer to (a) be different if the field were in the $-\hat{x}$ direction? Why or why not?

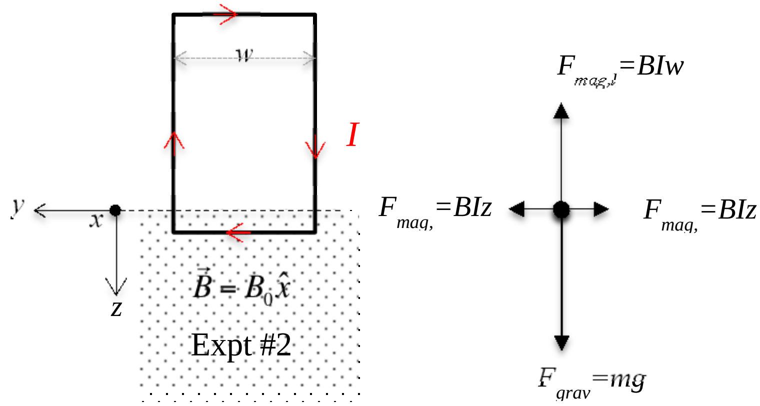
(c) Now be quantitative.

(i) Define the forces on the frame in experiment 2.

(ii) Suppose the frame is released from rest at $t = 0$ with its bottom horizontal wire just at the border of the field region ($z = 0$). Find the velocity of the frame as a function of time.

(iii) Show that the *difference* in distance fallen between experiment 1 and experiment 2 in a time t is $|\Delta z| = \frac{B_0^2 w^2 g}{6mR} t^3$ in the short-time limit of $\frac{B_0^2 w^2}{mR} t \ll 1$.

Comprehensive Exam, Fall 2015 E&M (Solution)



(a) In experiment #1, the only force acting is the constant force *mg* due to gravity, and the frame falls to earth just as a point mass *m*. That is, its velocity increases linearly, displacement quadratically in time. $v_{B=0}(t) = v_0 + gt$; $z_{B=0}(t) = z_0 + v_0 t + \frac{1}{2}gt^2$. The right-handed system of axes above is set up so that increasing *z* is in the direction of travel.

• In experiment #2, the situation is the same as in experiment #1 while the frame is completely outside the field region *i.e.* until the lowest wire of the frame reaches $z = 0$.

• While the frame enters the region with the magnetic field, the magnetic flux $\Phi = \oint \vec{B} \bullet d\vec{A}$ threading the metal frame increases. This changing Φ induces an electromotive force $\varepsilon = d\Phi/dt$ that generates a current in the wire frame that produces a field in the direction *opposite* to the applied field by Lenz' law. If the applied field is in the +*x* direction, the induced current is clockwise, inducing an opposing magnetic field in the –*x* direction. The induced current is therefore in the $+\hat{y}$ direction in the lower (horizontal) arm of the frame. With a current in the wire, the moving charges in the lowest part of the frame experience a force $\vec{F} = w\vec{I} \times \vec{B} = |wIB_0|(\hat{y} \times \hat{x}) = -|wIB_0|\hat{z}$ (upward). So the frame accelerates less than if there were no field.*

* In the left (right) vertical arm of the frame, the magnetic force tends to cause the frame to move in the –*y* (+*y*) direction, producing no net motion in the horizontal direction, but rather a slight constriction, which we ignore if the frame is rigid enough.

• Once the frame is entirely within the magnetic field region, the magnetic flux no longer changes, and there is no longer any effect of the magnetic field. The frame continues to accelerate with acceleration *g* from whatever velocity it had reached.

(b) Field reversal has no effect – this experiment cannot distinguish the field direction. If the field is reversed, then the induced current is reversed, and the magnetic force on the lower arm is $\vec{F} = \vec{I}w \times \vec{B} = |IBw|(-\hat{y} \times (-\hat{x})) = -|IBw|\hat{z}$ is the same (upward), so the frame still accelerates less than in experiment #1.

(c) Quantitative:

(i) Forces

Gravitational force $\boxed{\vec{F}_g = mg\hat{z} \quad (down)}$

Magnetic force on a straight, current-carrying wire of length *w* is $\vec{F}_{mag} = \vec{I}w \times \vec{B}$

Magnetic force (on lower wire) $\vec{F}_{mag,l} = \vec{I}w \times \vec{B} = |IB_0w|(\hat{y}\times\hat{x}) = -|IBw|\hat{z} \quad (up)$

Net horizontal magnetic force (on vertical wires) is zero as discussed in (a)

Find the current. In the field region, the field is perpendicular to the frame, so $\Phi = \oint \vec{B}\bullet d\vec{A} = B_0\int dA = B_0A = B_0wz$, with *w* the width of the frame and *z* the position of the lower wire of the frame.

The induced EMF is $\varepsilon = \frac{d\Phi}{dt} = B_0w\left|\frac{dz}{dt}\right|$

The modulus reminds that it is the magnitude that is important; the current direction is determined by Lenz' law and is different in the 4 sections of the frame. The induced current magnitude is $I = \frac{\varepsilon}{R} = \frac{B_0w}{R}\frac{dz}{dt}$ where the modulus has been removed and *dz/dt* is assumed positive (which is why the axes are set up with *z* downwards).

Use value of *I* in magnetic force expression for the lower wire:

$$\boxed{\vec{F}_{mag,l} = -|IBw|\hat{z} = -\frac{B_0^2w^2}{R}\left|\frac{dz}{dt}\right|\hat{z} = -\alpha|v_z|\hat{z} \quad \text{with} \quad \alpha \equiv \frac{B_0^2w^2}{R}}$$

(ii) Newton

$$\vec{F}_g + \vec{F}_{mag,l} = m\frac{dv_z}{dt}\hat{z}$$

$$mg\hat{z} - \frac{B_0^2w^2}{R}v_z\hat{z} = m\frac{dv_z}{dt}\hat{z}$$

$$g\hat{z} - \frac{\alpha}{m}v_z\hat{z} = \frac{dv_z}{dt}\hat{z}$$

$$\frac{dv_z}{dt} + \frac{\alpha}{m}v_z = g$$

with solution satisfying $v_z(0) = 0$,

$$\boxed{v_z(t) = \frac{mg}{\alpha}\left(1 - e^{-\frac{\alpha}{m}t}\right)}$$

(iii) Position:
Integrate, and note $z(0) = 0$:

$$\int_0^t v_z(t')dt' = \int_0^t \frac{dz}{dt'}dt' = \int_0^t dz = z(t) - z(0)$$

$$z(t) = \int_0^t \frac{mg}{\alpha}\left(1 - e^{-\frac{\alpha}{m}t'}\right)dt' = \frac{mg}{\alpha}t + \left(\frac{m}{\alpha}\right)^2 g\left(e^{-\frac{\alpha}{m}t} - 1\right)$$

Assume $\frac{\alpha}{m}t = \frac{B_0^2 w^2}{mR}t << 1$

$$z(t) \approx \frac{mg}{\alpha}t + \left(\frac{m}{\alpha}\right)^2 g\left(1 - \frac{\alpha}{m}t + \frac{1}{2}\left(\frac{\alpha}{m}t\right)^2 - \frac{1}{6}\left(\frac{\alpha}{m}t\right)^3 - 1\right).$$

$$z(t) \approx \left(\frac{1}{2}gt^2 - \frac{1}{6}\left(\frac{\alpha}{m}\right)gt^3\right)$$

Field free with same initial conditions: $z_{B=0}(t) = \frac{1}{2}gt^2$

Subtract to find leading o $\Delta z(t) = z(t) - z_{B=0}(t)$

$$\Delta z(t) \approx -\frac{\alpha}{6m}gt^3 = -\frac{B_0^2 w^2}{6mR}gt^3$$

The magnitude is the required value, and the negative sign means that the frame falls less distance in time *t* when the *B* field is present, consistent with the discussion in (a) above.

Propagation of electromagnetic waves:

(a) Use the Maxwell equations to show that in an insulator (linear, homogenous, dielectric permittivity ϵ and magnetic permeability μ), monochromatic electromagnetic waves with electric field $\vec{E} = \hat{x}E_0 e^{-i(wt-kz)}$ propagate with a phase velocity v where $\frac{1}{v^2} = \mu\epsilon$.

(b) Now let the material have an electrical conductivity σ, so that it supports a current density $\vec{J} = \sigma\vec{E}$, and you may assume that any free charge density $\rho = 0$. Extend the analysis above to show that in this case, the wave propagation is governed by the dispersion relation

$$k^2 = \mu\epsilon \left(1 + i\frac{\sigma}{\epsilon\omega}\right)\omega^2$$

(c) Show that a consequence of the non-zero conductivity is that the amplitude of the electric field is attenuated and find the attenuation length in the limit of small conductivity $\sigma/\epsilon\omega \ll 1$.

(d) Show that another consequence of the non-zero conductivity is that the electric and magnetic fields of the electromagnetic wave are not in phase (as they are in a pure insulator) and that the phase difference between them is $\phi = \tan^{-1}(\sigma/2\epsilon\omega)$ in the same small conductivity limit.

Comprehensive Exam, Fall 2015 E&M (Solution)

(a) The Maxwell equations are (equation sheet):

$$\nabla \bullet \vec{D} = \frac{\rho_f}{\varepsilon} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + J_f \qquad (1)$$

where, in a linear medium, $\vec{D} = \varepsilon \vec{E}; \quad \vec{H} = \frac{\vec{B}}{\mu}$.

In the absence of free charge and current $\rho_f = 0; \quad J_f = 0$ (2)

and in a homogeneous medium where ε and μ do not depend on position,

$$\nabla \bullet \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t} \quad . \qquad (3)$$

Take the curl of the curl equations:

$$\nabla \times \nabla \times \vec{E} = -\frac{\partial\left(\nabla \times \vec{B}\right)}{\partial t}$$
$$\nabla \times \nabla \times \vec{B} = \mu\varepsilon \frac{\partial\left(\nabla \times \vec{E}\right)}{\partial t} \qquad (4)$$

Use standard vector identities (equation sheet) on the LHS, and use the relation of the curl of one field to the time derivative of the other (Eqs 3) on the RHS:

$$\nabla\left(\nabla \bullet \vec{E}\right) - \nabla^2 \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$
$$\nabla\left(\nabla \bullet \vec{B}\right) - \nabla^2 \vec{B} = -\mu\varepsilon \frac{\partial^2 \vec{B}}{\partial^2 t} \qquad (5)$$

Use zero divergence of both fields (3), rearrange to get decoupled wave equations for *E* and *B*:

$$\boxed{\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}; \qquad \nabla^2 \vec{B} = \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2}} \qquad (6)$$

Now assume monochromatic waves $\vec{E} = \hat{x} E_0 \, \mathrm{e}^{-i(\omega t - kz)}$ and use in (6) to find

$-k^2 \vec{E} = -\omega^2 \mu\varepsilon \vec{E}$ or $\mu\varepsilon = \frac{k^2}{\omega^2}$. (7)

In $\vec{E} = \hat{x} E_0 \, \mathrm{e}^{-i(\omega t - kz)}$ the condition of constant phase $d\phi = \omega dt - k dz = 0$ identifies the phase velocity $v = \frac{dz}{dt} = \frac{\omega}{k}$. (8)

(8) in (7) gives the required result: $\frac{1}{v^2} = \mu\varepsilon$. (9)

(b) With conductivity, $\vec{J} = \sigma \vec{E}$ but free charge density $\rho = 0$.

$$\nabla \bullet \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \bullet \vec{B} = 0 \quad \nabla \times B = \mu\varepsilon \frac{\partial \vec{E}}{\partial t} + \mu\sigma \vec{E} \qquad (10)$$

Same principle as in (a), but now

$$\nabla\left(\nabla \bullet \vec{E}\right) - \nabla^2 \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t}$$

$$\nabla\left(\nabla \bullet \vec{B}\right) - \nabla^2 \vec{B} = -\mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} - \mu\sigma \frac{\partial \vec{B}}{\partial t}$$

and with zero divergence:

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}$$

$$\nabla^2 \vec{B} = \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} + \mu\sigma \frac{\partial \vec{B}}{\partial t} \qquad (11)$$

Use monochromatic form

$$\vec{E}_0 \frac{\partial^2 e^{i(kz-\omega t)}}{\partial z^2} = \mu\varepsilon \vec{E}_0 \frac{\partial^2 e^{i(kz-\omega t)}}{\partial t^2} + \mu\sigma \vec{E}_0 \frac{\partial e^{i(kz-\omega t)}}{\partial t}$$

$$-\vec{E}_0 k^2 e^{i(kz-\omega t)} = -\omega^2 \mu\varepsilon \vec{E}_0 e^{i(kz-\omega t)} - i\omega\mu\sigma \vec{E}_0 e^{i(kz-\omega t)}$$

$$k^2 = \omega^2 \mu\varepsilon + i\omega\mu\sigma$$

So now

$\frac{k^2}{\omega^2} = \mu\varepsilon\left(1 + i\frac{\sigma}{\varepsilon\omega}\right)$ which gives the result required: $k^2 = \mu\varepsilon\left(1 + i\frac{\sigma}{\varepsilon\omega}\right)\omega^2$. (12)

(c) To show the wave is attenuated, notice that with ω, σ, μ, ε real, k must be complex: $k = k_r + ik_i$, so the E field has the form

$$\vec{E} = \hat{x}E_0 \, \mathrm{e}^{-i(\omega t - k_r z - ik_i z)} = \hat{x}E_0 e^{-k_i z} \, \mathrm{e}^{-i(\omega t - k_r z)} \qquad (13)$$

and the wave is attenuated with 1/*e* attenuation length (you can use another if you like, just define it)

$$\lambda = 1/k_i \qquad (13)$$

so we must find the imaginary part of k in the low conductivity limit.

$$\frac{k^2}{\omega^2} = \mu\varepsilon\left(1 + i\frac{\sigma}{\varepsilon\omega}\right)$$

$$k = \omega\sqrt{\mu\varepsilon}\left(1 + i\frac{\sigma}{\varepsilon\omega}\right)^{1/2} \approx \omega\sqrt{\mu\varepsilon}\left(1 + i\frac{\sigma}{2\varepsilon\omega}\right)$$

$$k_i \approx \sqrt{\frac{\mu}{\varepsilon}}\frac{\sigma}{2}$$

Hence

$$\lambda^{-1} = \sqrt{\frac{\mu}{\varepsilon}}\frac{\sigma}{2} \qquad (14)$$

(d) The magnetic field *B* obeys the same type of equations as the *E* field, so it propagates with the same velocity (Eqs 11).

So we have $\vec{E} = \hat{x}E_0\,\mathrm{e}^{-i(\omega t - kz)}, \vec{B} = \vec{B}_0\,\mathrm{e}^{-i(\omega t - kz)}$.

Because $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$,

then $\hat{y}\frac{\partial E_x}{\partial z} = -(-i\omega)\vec{B}$

and therefore

$$\hat{y}(ik)E_0 e^{-i(\omega t - kz)} = i\omega\vec{B}$$

$$\vec{B} = \frac{k}{\omega}E_0 e^{-i(\omega t - kz)}\hat{y} = \frac{1}{v}E_0 e^{-i(\omega t - kz)}\hat{y}$$

We have just shown that *v* is complex (Eqn 12): $\frac{1}{v} = \sqrt{\mu\varepsilon}\left(1 + i\frac{\sigma}{\varepsilon\omega}\right)^{1/2}$

Plugging in, we see that the amplitude of *B* is complex and therefore introduces an additional phase ϕ relative to *E*:

$$\vec{B} = \frac{1}{v}E_0 e^{-i(\omega t - kz)}\hat{y} = E_0\sqrt{\mu\varepsilon}\left(1 + i\frac{\sigma}{\varepsilon\omega}\right)^{1/2} e^{-i(\omega t - kz)}\hat{y}$$

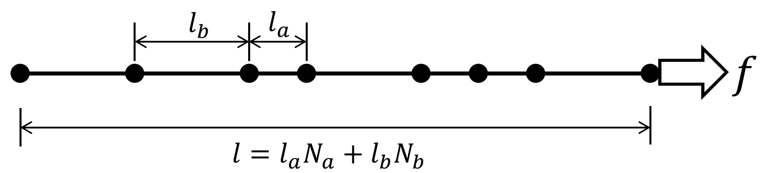
$$\approx E_0\sqrt{\mu\varepsilon}\left(1 + i\frac{\sigma}{2\varepsilon\omega}\right)e^{-i(\omega t - kz)}\hat{y}$$

$$= E_0\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma^2}{4\varepsilon^2\omega^2}\right)^{1/2} e^{i\phi}e^{-i(\omega t - kz)}\hat{y}$$

With $\tan\phi = \frac{\sigma}{2\varepsilon\omega}$

Linear polymer chain

We model the elasticity of fibrous proteins with a linear polymer chain. Consider a single linear polymer chain composed of units each of which can be in a short state a of length l_a or a long state b of length l_b ($l_b > l_a$). If a pulling force f is applied to the chain, some a units are converted into b units and the chain will lengthen. Neighboring units in the chain are independent of each other.



Let $N = N_a + N_b$ ($\gg 1$) be the total number of units, with N_a and N_b of the two types, and μ be a single unit energy (either a or b), that is, an energy required to add a single unit to the system.

(a) The fundamental thermodynamics equation of the energy conservation is expressed as

$$dE = TdS - dW,$$

where E is the internal energy of the system, T is its temperature, S is the entropy, and W is the work done by the system. Elaborating dW, rewrite the equation for the two different sets of independent variables:

(i) S, l, and N, where $l = l_a N_a + l_b N_b$ is the length of the chain

(ii) S, N_a, and N

(b) We let q_a and q_b represent the partition functions of one a and one b unit, respectively. What is the canonical ensemble partition function $\Omega(N_a, N, T)$ of the chain in terms of q_a and q_b?

(c) Using your answers in (a) and (b), find the relation:

$$\frac{f(l_b - l_a)}{k_B T} = \ln\left(\frac{1-r}{r}\frac{q_a}{q_b}\right),$$

where $r = N_a/N$ and f is the pulling force. Note that Helmholtz free energy is $A = E - TS = -k_B T \ln \Omega$.

(d) Find the ratio N_a/N_b at zero force, $f = 0$.

(e) Show that l and r are related by

$$1 - r = \frac{l - Nl_a}{N(l_b - l_a)}$$

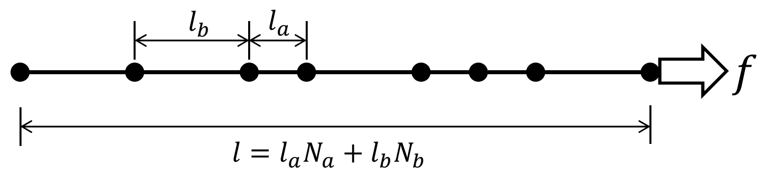
and that, for a small force f, the change in the length of the chain is proportional to the force: $f = \alpha \Delta l$. What is the spring constant α?

(f) Find r and l when $f \to \infty$ using the equation in (c) and justify your answer.

Useful formula: $\ln n! \cong n \ln n - n$ for $n \gg 1$

Linear polymer chain

We model the elasticity of fibrous proteins with a linear polymer chain. Consider a single linear polymer chain composed of units each of which can be in a short state a of length l_a or a long state b of length l_b ($l_b > l_a$). If a pulling force f is applied to the chain, some a units are converted into b units and the chain will lengthen. Neighboring units in the chain are independent of each other.



Let $N = N_a + N_b$ ($\gg 1$) be the total number of units, with N_a and N_b of the two types, and μ be a single unit energy (either a or b), that is, an energy required to add a single unit to the system.

(a) The fundamental thermodynamics equation of the energy conservation is expressed as

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where E is the internal energy of the system, T is its temperature, S is the entropy, and W is the work done by the system. Elaborating dW, rewrite the equation for the two different sets of independent variables:

(i) S, l, and N, where $l = l_a N_a + l_b N_b$ is the length of the chain

(ii) S, N_a, and N

Solution:

(i) The energy conservation yields

$$dE = TdS + fdl + \mu dN \tag{1}$$

(ii) Because $dl = l_a dN_a + l_b dN_b = l_a dN_a + l_b(dN - dN_a) = -(l_b - l_a)N_a + l_b dN$,

$$dE = TdS - f(l_b - l_a)dN_a + (\mu + fl_b)dN \tag{2}$$

(b) We let q_a and q_b represent the partition functions of one a and one b unit, respectively. What is the canonical ensemble partition function $\Omega(N_a, N, T)$ of the chain in terms of q_a and q_b?

Solution:

The number of ways of distributing N_a a units among a total of N possible positions in the chain is

$$\frac{N!}{N_a!(N - N_a)!}; \tag{3}$$

therefore, the canonical ensemble partition function is

$$\Omega(N_a, N, T) = \frac{N!}{N_a!(N - N_a)!} q_a^{N_a} q_b^{N - N_a}. \tag{4}$$

(c) Using your answers in (a) and (b), find the relation:

$$\frac{f(l_b - l_a)}{k_B T} = \ln\left(\frac{1-r}{r}\frac{q_a}{q_b}\right),$$

where $r = N_a/N$ and f is the pulling force. Note that Helmholtz free energy is $A = E - TS = -k_B T \ln \Omega$.

Solution:
Because an infinitesimal change in Helmholtz free energy is expressed as

$$dA = -SdT - dW = -SdT - f(l_b - l_a)dN_a + (\mu + f l_b)dN, \tag{5}$$

we can write

$$\left(\frac{\partial A}{\partial N_a}\right)_{N,T} = -f(l_b - l_a). \tag{6}$$

Furthermore, using $A = -k_B T \ln \Omega$ and $\ln n! \cong n \ln n - n$, we get

$$\begin{aligned}
\left(\frac{\partial A}{\partial N_a}\right)_{N,T} &= -k_B T \left(\frac{\partial \ln \Omega}{\partial N_a}\right)_{N,T} \\
&= -k_B T \frac{\partial}{\partial N_a}\left[\ln N! - \ln N_a! - \ln(N - N_a)! + N_a \ln q_a + (N - N_a)\ln q_b\right] \\
&\cong -k_B T \frac{\partial}{\partial N_a}\left[-N_a \ln N_a - (N - N_a)\ln(N - N_a) + N_a \ln q_a + (N - N_a)\ln q_b\right] \\
&= -k_B T\left(-\ln N_a + \ln(N - N_a) + \ln q_a - \ln q_b\right) \\
&= -k_B T \ln\left(\frac{N - N_a}{N_a}\frac{q_a}{q_b}\right) = -k_B T \ln\left(\frac{1-r}{r}\frac{q_a}{q_b}\right)
\end{aligned} \tag{7}$$

From Eqs. 6 and 7, we obtain

$$\frac{f(l_b - l_a)}{k_B T} = \ln\left(\frac{1-r}{r}\frac{q_a}{q_b}\right). \tag{8}$$

(d) Find the ratio N_a/N_b at zero force, $f = 0$.

Solution:
When $f = 0$,

$$\ln\left(\frac{1-r}{r}\frac{q_a}{q_b}\right) = 0 \rightarrow \frac{1-r}{r}\frac{q_a}{q_b} = 1, \tag{9}$$

where

$$\frac{1-r}{r} = \frac{N - N_a}{N_a} = \frac{N_b}{N_a}. \tag{10}$$

Thus, the stability ratio is

$$\frac{N_a}{N_b} = \frac{q_a}{q_b}. \tag{11}$$

(e) Show that l and r are related by

$$1 - r = \frac{l - Nl_a}{N(l_b - l_a)}$$

and that, for a small force f, the change in the length of the chain is proportional to the force: $f = \alpha \Delta l$. What is the spring constant α?

Solution:
Because $l = N_a l_a + (N - N_a) l_b$,

$$l - N_a l_a = N(1-r) l_b \to 1 - r = \frac{l - Nl_a}{N(l_b - l_a)}. \tag{12}$$

For a small force f,

$$\frac{1-r}{r}\frac{q_a}{q_b} \cong 1 \to \frac{1-r}{r}\frac{q_a}{q_b} = 1 + x \text{ where } x \ll 1. \tag{13}$$

We define $r = r_0$ at $f = 0$ such as

$$\frac{1-r_0}{r_0}\frac{q_a}{q_b} = 1 \to \frac{1}{r_0} = 1 + \frac{q_a}{q_b} \tag{14}$$

and Δr as a small deviation for r_0, $r = r_0 + \Delta r$. Then,

$$\begin{aligned} x &= \left(\frac{1}{r} - 1\right)\frac{q_a}{q_b} - 1 \\ &= \left(\frac{1}{r_0 + \Delta r} - 1\right)\frac{q_a}{q_b} - 1 \\ &\cong \left[\frac{1}{r_0}\left(1 - \frac{\Delta r}{r_0}\right) - 1\right]\frac{q_a}{q_b} - 1 \\ &= \left(\frac{1}{r_0} - 1\right)\frac{q_a}{q_b} - 1 - \frac{\Delta r}{r_0^2}\frac{q_a}{q_b} \\ &= -\frac{\Delta r}{r_0^2}\frac{q_a}{q_b}. \end{aligned} \tag{15}$$

From Eq. 12, we can obtain the relation between Δr and the infinitesimal change in the length, Δl:

$$-\Delta r = \frac{1}{N(l_b - l_a)}\Delta l \tag{16}$$

Inserting Eq.16 into Eq. 15, we get

$$x = \frac{1}{r_0^2}\frac{q_a}{q_b}\frac{1}{N(l_b - l_a)}\Delta l = \frac{1}{N(l_b - l_a)}\frac{q_a}{q_b}\left(1 + \frac{q_a}{q_b}\right)^2 \Delta l. \tag{17}$$

From Eqs.8 and 17, we obtain the relation between f and Δl:

$$\begin{aligned} f &= \frac{k_B T}{l_b - l_a} \ln\left(\frac{1-r}{r}\frac{q_a}{q_b}\right) \\ &= \frac{k_B T}{l_b - l_a} \ln(1+x) \\ &\cong \frac{k_B T}{l_b - l_a} x \\ &= \frac{k_B T}{N(l_b - l_a)^2}\frac{q_a}{q_b}\left(1+\frac{q_a}{q_b}\right)^2 \Delta l \\ &= \alpha \Delta l, \end{aligned} \tag{18}$$

where the spring constant,

$$\alpha = \frac{k_B T}{N(l_b - l_a)^2}\frac{q_a}{q_b}\left(1+\frac{q_a}{q_b}\right)^2 \tag{19}$$

(f) Find r and l when $f \to \infty$ using the equation in (c) and justify your answer.

Solution:
When $f \to \infty$, $r \to 0$; therefore, from the r and l relation in (e),

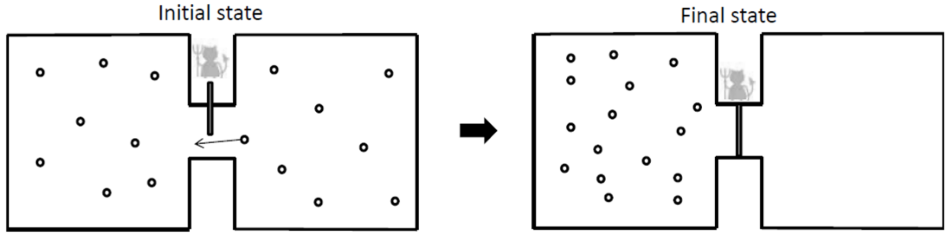
$$l = Nl_a + N(l_b - l_a) = Nl_b, \tag{20}$$

that is, all units are in the longer b state. This is reasonable because, if a very strong force is applied, the polymer chain should be stretched to the maximum length.

Useful formula: $\ln n! \cong n \ln n - n$ for $n \gg 1$

Maxwell's demon

Consider a chamber filled with an ideal monoatomic gas (temperature T, volume V, pressure P, number of atoms $N \gg 1$). Suppose that a partition is placed across the middle of the chamber separating the two sides into left and right. Maxwell imagined a trap door in the partition with an imaginary creature poised at the door who is observing the molecules. The demon only opens the door if a molecule is approaching the trap door from the right. Assume that the trap door is massless and no energy is required to operate the door. This would result in all the molecules ending up on the left side.



(a) It appears that this thought experiment might violate the Second law of thermodynamics. Explain why a physicist might mistakenly claim the Second law is violated.

(b) Find the temperature, energy, and pressure of the ideal gas in the final state.

(c) What is the work done by the ideal gas during the process?

(d) What is the change in the entropy of the ideal gas during the process?

(e) We now consider the entropy associated with information. The demon acquires information about the state of the system via measurements on the atoms. More information means more entropy (for example, blank computer memory consists of all zeros, while full computer memory consists of zeros and ones).

(i) First, we consider a case when the demon makes a decision on one atom and assume only two possible measurement outcomes (Obviously, this is the simplest case.), where w_1 and w_2 are the probabilities of getting outcomes 1 and 2, respectively. Let S_1 and S_2 be the entropies associated with outcomes 1 and 2. Show that a lower bound on S_1 and S_2 is given by

$$e^{-S_1/k_B} + e^{-S_2/k_B} \leq 1.$$

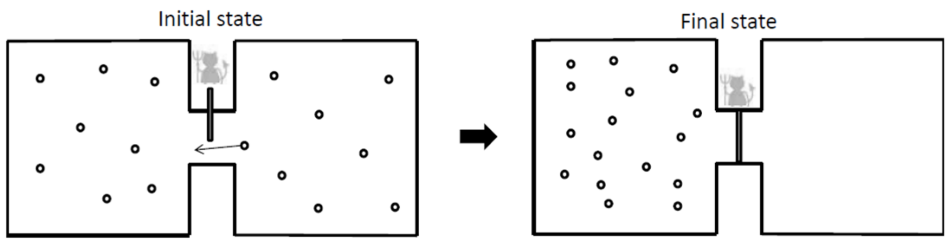
Note: When w_i is the probability of getting outcome i and S_i is the entropy associated with the outcome, entropy and probability has an inequality relation, $S_i \geq -k_B \ln w_i$.

(ii) The average entropy cost of measurement per cycle is

$$S_a = w_1 S_1 + w_2 S_2.$$

Show that for any values of S_1 and S_2 that satisfy the lower-bound constraint, the resulting value for S_a is no less than the entropy decrease that violates the Second Law, and hence, on average, the entropy increase due to measurement is no less than the entropy decrease of the ideal gas.

Maxwell's demon
Consider a chamber filled with an ideal monoatomic gas (temperature T, volume V, pressure P, number of atoms $N \gg 1$). Suppose that a partition is placed across the middle of the chamber separating the two sides into left and right. Maxwell imagined a trap door in the partition with an imaginary creature poised at the door who is observing the molecules. The demon only opens the door if a molecule is approaching the trap door from the right. Assume that the trap door is massless and no energy is required to operate the door. This would result in all the molecules ending up on the left side.



(a) It appears that this thought experiment might violate the Second law of thermodynamics. Explain why a physicist might mistakenly claim the Second law is violated.

Solution:
In the final state, the ideal gas occupies a reduced volume and therefore the gas becomes more ordered. It appears that the Second law is violated.

(b) Find the temperature, energy, and pressure of the ideal gas in the final state.

Solution:
(i) There is no change in temperature during the process because the temperature of an ideal gas is determined only by its kinetic energy and the kinetic energy of each atom is conserved during the process, that is, $T_f = T$.

(ii) For an ideal monoatomic gas the entire energy is kinetic and the mean kinetic energy of an atom is $\frac{3}{2}k_BT$. Therefore, the energy of the ideal gas is

$$E_f = E_i = \frac{3}{2}Nk_BT$$

(iii) The volume of the final state is $V/2$; therefore, the pressure of the final state should be doubled,

$$P_f = \frac{Nk_BT_f}{V_f} = 2\frac{Nk_BT}{V} = 2P.$$

(c) What is the work done by the ideal gas during the process?

Solution:
the work done by the ideal gas (ΔW) is

$$\Delta W = \int_{V_i}^{V_f} P dV = \int_{V_i}^{V_f} \frac{N k_B T}{V} dV = N k_B T \ln \frac{V_f}{V_i} = -N k_B T \ln 2. \tag{21}$$

(d) What is the change in the entropy of the ideal gas during the process?

Solution:
Because there is no change in the energy of the ideal gas, the heat absorbed by the ideal (ΔQ) gas must be equal to the work done by the ideal gas (ΔW),

$$\Delta Q = \Delta W = -N k_B T \ln 2; \tag{22}$$

therefore, the change in entropy is

$$\Delta S = \int_i^f \frac{1}{T} dQ = \frac{\Delta Q}{T} = -N k_B \ln 2. \tag{23}$$

Entropy of the system decreases during the process!

(e) We now consider the entropy associated with information. The demon acquires information about the state of the system via measurements on the atoms. More information means more entropy (for example, blank computer memory consists of all zeros, while full computer memory consists of zeros and ones).

(i) First, we consider a case when the demon makes a decision on one atom and assume only two possible measurement outcomes (Obviously, this is the simplest case.), where w_1 and w_2 are the probabilities of getting outcomes 1 and 2, respectively. Let S_1 and S_2 be the entropies associated with outcomes 1 and 2. Show that a lower bound on S_1 and S_2 is given by

$$e^{-S_1/k_B} + e^{-S_2/k_B} \leq 1.$$

Note: When w_i is the probability of getting outcome i and S_i is the entropy associated with the outcome, entropy and probability has an inequality relation, $S_i \geq -k_B \ln w_i$.

Solution:
Lower bounds for S_1 and S_2 are given by

$$S_1 \geq -k_B \ln w_1 \text{ and } S_2 \geq -k_B \ln w_2 \tag{24}$$

which leads to

$$w_1 \geq e^{-S_1/k_B} \text{ and } w_2 \geq e^{-S_2/k_B} \tag{25}$$

Using $w_1 + w_2 = 1$, we obtain the lower-bound constraint,

$$e^{-S_1/k_B} + e^{-S_2/k_B} \leq w_1 + w_2 = 1. \tag{26}$$

(ii) The average entropy cost of measurement per cycle is

$$S_a = w_1 S_1 + w_2 S_2.$$

Show that for any values of S_1 and S_2 that satisfy the lower-bound constraint, the resulting value for S_a is no less than the entropy decrease that violates the Second Law, and hence, on average, the entropy increase due to measurement is no less than the entropy decrease of the ideal gas.

Solution:
If we choose $S_1 = S_2 = k_B \ln 2$, these satisfy the lower-bound constraint:

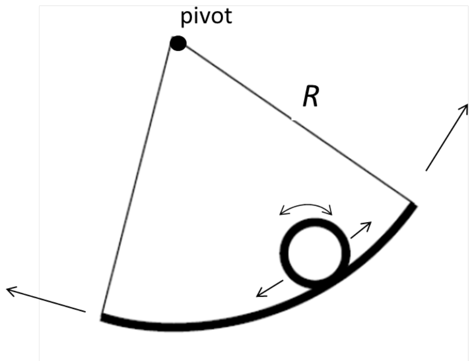
$$e^{-S_1/k_B} + e^{-S_2/k_B} = 2e^{-\ln 2} = 1 \leq 1 \tag{27}$$

and then

$$S_a = w_1 S_1 + w_2 S_2 = k_B \ln 2. \tag{28}$$

Thus, the entropy increase due to measurements on N atoms, $NS_a = Nk_B \ln 2$, is no less than the entropy decrease in the ideal gas, $Nk_B \ln 2$. Hew! The Second law is saved.

A swing of mass m is made from an arc section of radius of R is suspended from a pivot by (massless) ropes at both ends. A hoop, also of mass m, and radius a rolls without slipping on the swing. The swing and the hoop move without dissipative friction subject to a constant gravitational force $F_g = -mg$.



(a) Find the differential equations of motions that describe the angular displacement of the hoop and the swing. You may assume $a/R \ll 1$ and small displacements from equilibrium.

(b) Find all possible frequencies of oscillation of the system for **small** displacements from equilibrium.

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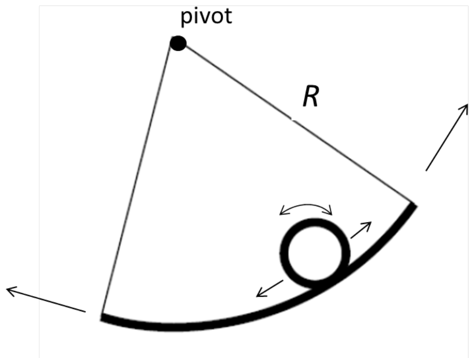


(a) Find the differential equations of motions that describe the angular displacement of the hoop and the swing. You may assume $a/R \ll 1$ and small displacements from equilibrium.

Solution:
Question Credit: 2014 Columbia University Comprehensive Exam

The angular displacement of the system is in terms of two angles that represent the angular displacements of the swing and hoop from equilibrium, ϕ and θ, respectively. The rolling of the hoop can be described by an angular velocity ω_{hoop} which will be taken to be positive for the hoop rolling counter-clockwise. The rolling of the hoop depends on the difference between $\dot{\phi}$ and $\dot{\theta}$ since if θ and ϕ increase or decrease together, the hoop remains in the same position with respect to the swing. Then, the rolling without slipping condition is $a\omega_{hoop} = R\left(\dot{\phi} - \dot{\theta}\right)$. The kinetic energy (e.g. $\frac{1}{2}I\dot{\theta}^2$)of the system is then the sum of the swinging energy of the swing and hoop(about it center of mass $(R-a)$, and the rolling energy of the hoop, i.e.

$$T = \frac{1}{2}m\left(R^2\dot{\phi}^2 + (R-a)^2\dot{\theta}^2\right) + \frac{1}{2}I_{hoop}\omega^2 \tag{29}$$

with $I_{hoop} = ma^2$. Substituting, $a^2\omega^2 = R^2\left(\dot{\phi} - \dot{\theta}\right)^2$ and taking $R - a \cong R$

$$T = mR^2\left(\dot{\phi}^2 + \dot{\theta}^2 - \dot{\theta}\dot{\phi}\right) \tag{30}$$

By the circular arc symmetry, the potential energy can be expressed in terms of a simple pendulum for the swing the hoop(i.e. the height potential above the center mass of the hoop).

$$U = mgR\,(1-\cos\phi) + mg\,(R-a)\,(1-\cos\theta) \tag{31}$$

Write the Lagrangian, $\mathcal{L} = T - U$.
The resulting Lagrangian is then (under the approximation, $R - a \cong R$),

$$\mathcal{L} = T - U \tag{32}$$
$$= mR^2\left(\dot{\phi}^2 + \dot{\theta}^2 - \dot{\theta}\dot{\phi}\right) - mgR\,(2-\cos\phi-\cos\theta) \tag{33}$$

To get the differential equations of motion we invoke the Euler-Lagrange equations. With respect to ϕ we get,

$$0 = \frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \tag{34}$$
$$0 = mR^2\left(2\ddot{\phi} - \ddot{\theta}\right) + mgR\sin\phi \tag{35}$$

With respect to θ we get,

$$0 = \frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \tag{36}$$
$$0 = mR^2\left(2\ddot{\theta} - \ddot{\phi}\right) + mgR\sin\theta \tag{37}$$

(b) Find all possible frequencies of oscillation of the system for **small** displacements from equilibrium.

Solution:
For small oscillations (and $a/R \ll 1$), the standard Taylor series approximation (as is the case of the simple pendulum) applies, i.e. $\sin\theta \cong \theta$ and $\sin\phi \cong \phi$. The resulting coupled harmonic oscillator equations of motion are:

$$0 = 2\ddot{\phi} - \ddot{\theta} + \frac{g}{R}\phi \tag{38}$$
$$0 = 2\ddot{\theta} - \ddot{\phi} + \frac{g}{R}\theta \tag{39}$$

Assume the solution may be obtained with an exponential of form $\theta(t) = A_1 \exp i\omega t$ and $\phi(t) = A_2 \exp i\omega t$, and we may solve the coupled ODEs,

$$-\frac{g}{R}A_1 = -\omega^2(2A_1 - A_2) \tag{40}$$

$$-\frac{g}{R}A_2 = -\omega^2(2A_2 - A_1) \tag{41}$$

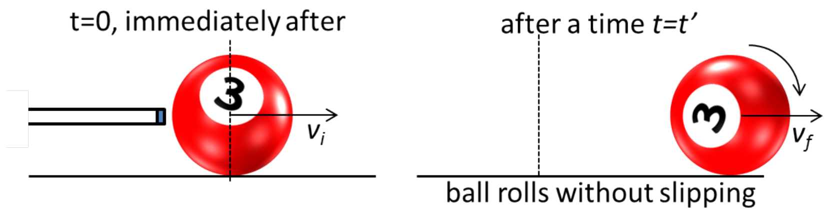
solving by method of determinants let's define $\alpha = g/(\omega^2 R)$ to give a matrix
$C - \alpha I = \begin{pmatrix} 2 - g/(R\omega^2) & -1 \\ -1 & 2 - g/(R\omega^2) \end{pmatrix}$
We can now solve for the eigenvalues (or characteristic frequencies) of matrix C by,

$$\det(C - \alpha I) = 3 - 4\alpha + \alpha^2 = (\alpha - 3)(\alpha - 1) = 0 \tag{42}$$

$$\tag{43}$$

Hence characteristic frequencies are $\omega_1 = \sqrt{\frac{g}{R}}$ and $\omega_2 = \sqrt{\frac{g}{3R}}$

Imagine there is a billiard ball (solid sphere) of mass M and radius R. As illustrated the ball is stationary until hit with a pool cue where is acquires an instantaneous linear velocity, v_i. You may assume the shot was well-centered, such that the ball initially has no angular momentum imparted, and slips over the table. Immediately after, the ball experiences a static coefficient of friction μ with respect to the table.



(a) Calculate the moment of inertia of the rolling billiard ball about its center.
Please show all work.
Hint: consider the moment of inertia of a disc ($\frac{1}{2}MR^2$) or a cylindrical shell (MR^2)

(b) What is the linear velocity of the ball (v_f) when the ball first begins to roll without slipping? Show all work.

(c) What is the numerical fraction of the total initial kinetic energy that gets transferred into heat by the time $t = t'$ (i.e. when the ball rolls without slipping)?

(d) How far does the billiard ball travel before it starts rolling without slipping?

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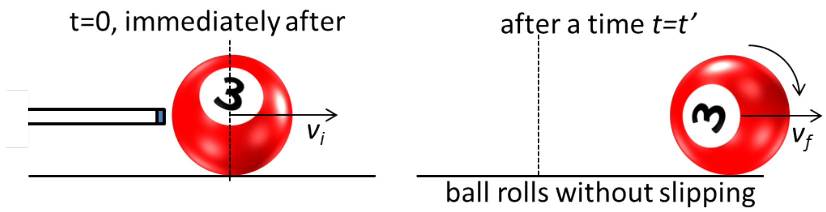

(a) Calculate the moment of inertia of the rolling billiard ball about its center.
Please show all work.
Hint: consider the moment of inertia of a disc ($\frac{1}{2}MR^2$) or a cylindrical shell (MR^2)

Solution:

One approach is to divide the sphere into infinitesimally small discs with axes in the x-direction (alternatively one can divide into cylindrical shells). Let the disc thickness be dz and radii r. The center of each disc will then have radius $r = \sqrt{R^2 - x^2}$ be located at distance x from the origin. The resulting differential inertia is then $dI = \frac{1}{2}r^2 dm$ (using the hint for the disc), where $dm = \rho\pi r^2 dx$ and ρ is the mass density of a sphere. Hence we get,

$$I = \frac{1}{2}\int_{-R}^{R} \rho\pi r^4 dx \tag{44}$$

$$= \frac{1}{2}\int_{-R}^{R} \rho\pi (R^2 - x^2)^2 dx \tag{45}$$

$$= \frac{1}{2}\int_{-R}^{R} \rho\pi (R^4 - 2R^2x^2 + x^2) dx \tag{46}$$

$$= \rho\pi R^5 \left(1 - \frac{2}{3} + \frac{1}{5}\right) \tag{47}$$

Recall the mass density of sphere is $\rho = M/V = M/\frac{4}{3}\pi R^3$. Combining with the above expression we get,

$$I = \frac{2}{5}MR^2 \tag{48}$$

(b) What is the linear velocity of the ball (v_f) when the ball first begins to roll without slipping? Show all work.

Solution:
(*question clarification*: The phrase *Immediately after, the ball experiences a static coefficient of friction* μ was misleading because the ball is now continuously moving and so only kinetic friction matters, the type friction *immediately after* cannot be defined as kinetic or static. Nonetheless, the key to this problem is to recognize the final result is independent of μ, and depends only on the final condition of rolling without slipping.)

There are many way of the solving this problem (torques, angular momentum, etc). It is critical to recognize that the solution is independent of friction and instead depends on the final angular velocity meets the condition of spinning w/o slipping, i.e. $\omega = v_f/R$. The ball will decelerate as the angular velocity accelerates from the point contact of friction. At some point later, the the angular speed and rolling speed will match; the time and distance it take to meeting this condition vary with μ, but the final velocity and energy lost are independent of μ.

Method 1 - angular momentum imparted about single point contact The only unbalanced force in this problem is friction which acts horizontally through point of contact and hence external torque about any axis of rotation passing through the table and normal to the plain of motion, is zero. About the ball's axis, the initial angular momentum of the slipping ball is just, $L_i = Mv_iR$. The final angular momentum includes the slipping and rolling components, i.e.

$$L_f = MRv_f + I\omega \tag{49}$$
$$= MRv_f + \frac{2}{5}MR^2 v_f/R \tag{50}$$
$$= \frac{7}{5}MRv_f \tag{51}$$

Solving for the final velocity of ball v_f we get

$$Mv_iR = \frac{7}{5}MRv_f \tag{52}$$
$$v_f = \frac{5}{7}v_i \tag{53}$$

The final velocity of ball is $v_f = \frac{5}{7}v_i$.

Method 2: kinematics and torques The horizontal deceleration of the ball comes from friction and acts in horizontal direction opposing motion, $a = F/M = \mu g$ and so the velocity at given time t is simply $v(t) = v_i - \mu g t$. Moreover, this force will produce an opposing torque on the ball about its center since the friction only happens at the single point of contact. Since $v_f = \omega R = \alpha t R$, where α is the angular acceleration we can re-write the final velocity as,

$$v_f = v_i - \mu g t = \alpha t R \tag{54}$$

The anti-clockwise angular acceleration of the ball about its center given by the torque, $\tau = R \times F = I\alpha$ or simply $R\mu g M = \frac{2}{5}MR^2\alpha$. Hence the ball experiences angular acceleration

$\alpha = 5\mu g/2R$ or,

$$v_i - \mu g t = 5\mu g t/2 \tag{55}$$

$$\mu g t = \frac{2}{7} v_i \tag{56}$$

or the final velocity $v_f = v_i - \frac{2}{7}v_i = \frac{5}{7}v_i$.

(c) What is the numerical fraction of the total initial kinetic energy that gets transferred into heat by the time $t = t'$ (i.e. when the ball rolls without slipping)?

Solution:
Initially ball has linear and kinetic energy, i.e.

$$KE_{initial} = \frac{1}{2} M v^2 \tag{57}$$

Long after the collision, both balls have linear and angular kinetic energy, hence the same derivation above applies for each ball, i.e.

$$KE_{final} = \frac{1}{2}\frac{2}{5} M v_f^2 + \frac{1}{2} M v_f^2 \tag{58}$$

$$= \frac{7}{10} M \left(\left(\frac{5}{7} v_i \right)^2 \right) \tag{59}$$

$$= \frac{7}{10} M v_i^2 \frac{25}{49} \tag{60}$$

$$= \frac{5}{14} M v_i^2 \tag{61}$$

Therefore the following fraction of initial energy must have been lost to heat,

$$\frac{KE_{initial} - KE_{final}}{KE_{initial}} = 1 - 2 \times \frac{5}{14} \tag{62}$$

$$= \frac{2}{7} \tag{63}$$

(d) How far does the billiard ball travel before it starts rolling without slipping?

Solution:
The ball decelerates due to friction at a rate of $a = F/M = -\mu g$. Recalling the basic kinematic equations for the ball's displacement D we get,

$$v_f^2 = v_i^2 + 2aD \tag{64}$$

$$(\frac{5}{7} v_i)^2 = v_i^2 - 2\mu g D \tag{65}$$

$$D = \frac{12}{49} \frac{v_i^2}{\mu g} \tag{66}$$