OSU Physics Department
Comprehensive Examination #122

Monday, March 30, Tuesday, March 31, 2015

Spring 2015 Comprehensive Examination

PARTS 1, 2, 3 & 4

General Instructions

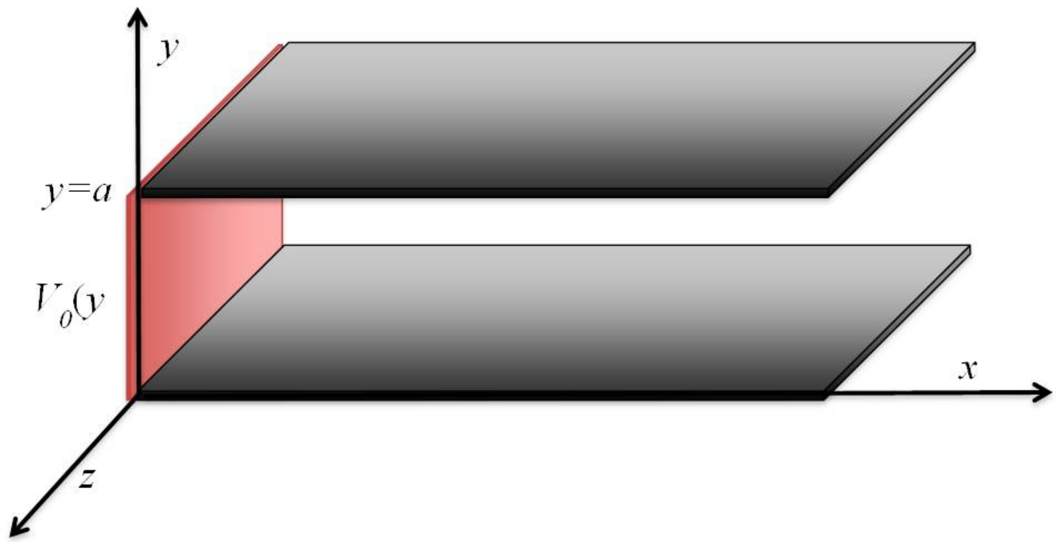
This Spring 2015 Comprehensive Examination consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, March 30, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:00 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, March 31, at 9:00 am and 1:00 pm, respectively. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please return all bluebooks and formula sheets at the end of the exam. Use the last pages of your bluebooks for "scratch" work, separated by at least one empty page from your solutions. "Scratch" work will not be graded.

The task is to find the electric potential everywhere inside a semi-infinite box where the electric potential on some walls is specified. The configuration is described and depicted below.

Two very large, flat metal plates are located parallel to the $x - z$ plane for $x > 0$ and $-\infty < z < \infty$. One plate is located at $y = 0$ and the other at $y = a$. The plates are grounded.

In the $y - z$ plane at $x = 0$ and between $0 < y < a$, the electric potential is specified by an external control, somehow. Call it $V_0(y)$ - it could be constant or it could be a function of y.



(a) Set up the defining equation for the electric potential everywhere between the plates and carefully state the appropriate boundary conditions.

(b) Solve the equation for the electric potential subject to the boundary conditions.

(c) Find the coefficients in the series solution for the particular case where the externally controlled potential is

$$V_0(y) = \begin{cases} V_0, & a/2 < y < a \\ -V_0, & 0 < y < a/2 \end{cases} \tag{1}$$

(d) How, in a real laboratory, might you approximate the situation described in (c)?

Comprehensive Exam, Winter 2015 E&M Undergraduate (Solution)

(a) The defining equation in the absence of external charges is the **Laplace equation** $\nabla^2 V(x,y) = 0$.

The potential cannot be a function of z because the plates extend to infinity in the $\pm z$ directions, and the externally controlled potential is explicitly not a function of z. The appropriate coordinate system is rectangular. **The boundary conditions are**:

(i) $V = 0$ when $y = 0$ (metal plate is an equipotential and is grounded)
(ii) $V = 0$ when $y = a$ (metal plate is an equipotential and is grounded)
(iii) $V = V_0(y)$ when $x = 0$ (externally specified)
(iv) $V \to 0$ as $x \to \infty$ on physical grounds – further from the "hot" strip, the potential must become constant, and that constant must be zero because of the zero-voltage condition on the plates.

The solution is unique because the potential is specified on all boundaries.

(b) Solve for $V(x,y)$:

$$\frac{d^2V(x,y)}{dx^2} + \frac{d^2V(x,y)}{dy^2} = 0$$

Assume separable solutions: $V(x,y) = X(x)Y(y)$ and plug into the Laplace equation:

$$Y(y)\frac{d^2X(x)}{dx^2} + X(x)\frac{d^2Y(y)}{dy^2} = 0 \Rightarrow \underbrace{\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}}_{f(x)} + \underbrace{\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}}_{g(y)} = 0$$

f and g must be constant (and equal and opposite), otherwise infinitesimal changes in x with y constant or *vice versa* lead to a violation of the statement that the sum is constant.

Let the separation constant be k^2.

$$\frac{d^2X(x)}{dx^2} = k^2X(x) \Rightarrow X(x) = Ce^{-kx} + De^{kx}$$

$$\frac{d^2Y(y)}{dy^2} = -k^2Y(y) \Rightarrow Y(y) = A\sin(ky) + B\cos(ky)$$

$$\Rightarrow V(x,y) = \left(Ce^{-kx} + De^{kx}\right)\left(A\sin(ky) + B\cos(ky)\right)$$

BC (iv) requires that $D = 0$, or else the potential becomes infinite as $x \to \infty$.
BC (i) requires that $B = 0$ because cos(k.0) = 1 at $y = 0$ and the potential is zero at $y = 0$.
BC (ii) requires that $k = n\pi/a$ (n =1,2,3…) because then $\sin(ka) = 0$. $n = 0$ is trivial and excluded.
Thus

$$V(x,y) = Ae^{-k_n x}\sin(k_n y)$$

Now the general solution to Laplace's equation is a superposition of all possible solutions with different k_n, :

$$V(x,y)=\sum_{n=1}^{\infty}c_n e^{-k_n x}\sin(k_n y)$$

We are left with BC (iii), and must match the coefficients c_n to the particular $V_0(y)$.

(c) The general solution is a Fourier sine series in y, with an overall multiplier that depends on x. In particular at $x = 0$,

$$V(0,y)=\sum_{n=1}^{\infty}c_n\sin(k_n y) \text{ and}$$

$$V_0(y)=\begin{cases} V_0 & a/2<y<a \\ -V_0 & 0<y<a/2 \end{cases}$$

To find the coefficients, use the orthogonality property of sine functions (excluding $m=n=0$):

$$\frac{2}{a}\int_0^a \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{a}y\right)dy=\delta_{n,m}$$

Then

$$\frac{2}{a}\int_0^a V(0,y)\sin\left(\frac{m\pi}{a}y\right)dy=\frac{2}{a}\int_0^a\sum_{n=1}^{\infty}c_n\sin(k_n y)\sin\left(\frac{m\pi}{a}y\right)dy$$

$$\frac{2}{a}\int_0^a V(0,y)\sin\left(\frac{m\pi}{a}y\right)dy=c_n\delta_{n,m}$$

$$\frac{2}{a}\left(-V_0\int_0^{a/2}\sin\left(\frac{m\pi}{a}y\right)dy+V_0\int_{a/2}^a\sin\left(\frac{m\pi}{a}y\right)dy\right)=c_m$$

$$c_m=\frac{2V_0}{a}\frac{a}{m\pi}\left(\cos\left(\frac{m\pi}{a}y\right)\Big|_0^{a/2}-\cos\left(\frac{m\pi}{a}y\right)\Big|_{a/2}^{a}\right)$$

$$c_m=\frac{2V_0}{m\pi}\left(\cos\left(\frac{m\pi}{2}\right)-1-\left\{\cos(m\pi)-\cos\left(\frac{m\pi}{2}\right)\right\}\right)$$

$$c_m=\frac{2V_0}{m\pi}\left(2\cos\left(\frac{m\pi}{2}\right)-1-(-1)^m\right)$$

For m odd, $\cos\left(\frac{m\pi}{2}\right)=0;\quad (-1)^m=-1$ so that $c_m = 0$

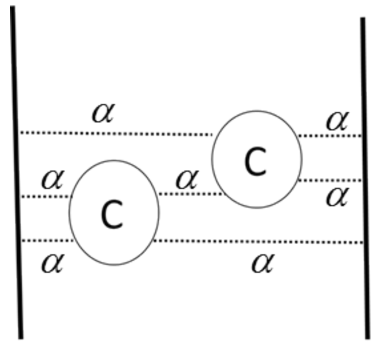
For m even, $\cos\left(\frac{m\pi}{2}\right)=\begin{cases}-1; & m=2,6.. \\ +1; & m=4,8..\end{cases}$; $(-1)^m=+1$ so that $c_m=\begin{cases}-\frac{8V_0}{m\pi}; m=2,6,.. \\ 0; m=4,8,..\end{cases}$

$$V(x,y)=\sum_{n=2,6,10...}^{\infty}-\frac{8V_0}{n\pi}\sin\left(\frac{n\pi}{a}y\right)e^{-n\pi x/a}$$ Various index relabelings are possible.

(d) To achieve the approximate configuration in the lab, use two large metal plates connected by metal wires to the electrical ground in the supply in the lab, or to a copper pipe driven into the earth outside. "Large" means length and width much larger than the separation. (These are the plates parallel to the *xz* plane. To achieve the configuration described for the *yz* plane, one has to deal with the discontinuities in the voltage at one has to deal with the discontinuities in the voltage at $y = 0$, $a/2$, a described in the mathematically tractable case in the problem. One could use two metal plates insulated from each other and from the grounded plates by as-thin-as-possible insulators – skinny ceramic tubes, perhaps, or maybe just an air gap. One plate could be charged to $+V_0$, and the other to $-V_0$ with independent voltage supplies (or batteries). Deviations from the ideal configuration described would come from the finite size of the insulators, and from the finite lengths of the grounded plates relative to their separation.

Crazy carbons coupled in one-dimension

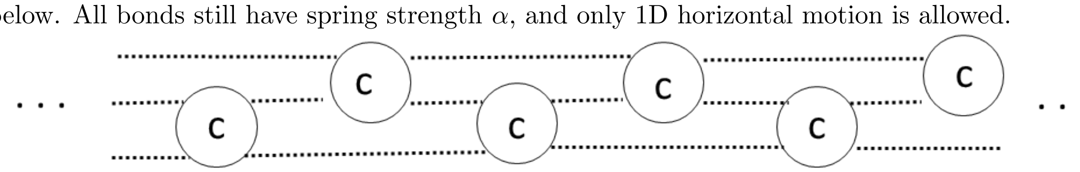
(a) The circles in the diagram below represent a *spring-ball* model of two carbon atoms constrained to move in the horizontal direction only (without twisting). Here, each carbon atom is modeled as a point mass sphere with mass m, and contains four bonds (dashed lines) that can be approximated as springs. Assume all seven bonds shown have the same constant strength (in N/m), α. The sides of the system (thick black lines) are immovable walls. Neglect any gravitational or quantum effects.



(i) Find the characteristic (or natural) frequencies of this system in terms of known constants.

(ii) Evaluate mathematically and describe the physical motion associated with the normal mode(s) of this system.

(b) Assume now the walls are removed to give the infinitely long system of *springs & balls* shown below. All bonds still have spring strength α, and only 1D horizontal motion is allowed.

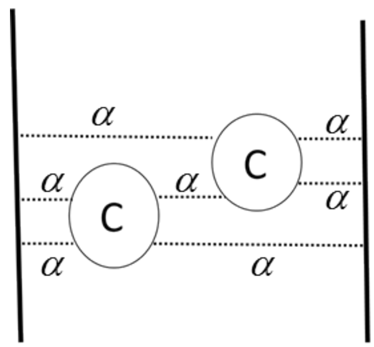


(i) Write down a (differential) 1D-equation of motion for n^{th} carbon mass, m in terms of the neighboring masses.

(ii) Quantitatively determine and qualitatively discuss the allowed characteristic frequencies, ω for this 1D infinite system. What are the minimum and maximum characteristic (or natural) frequencies one might excite in this 1D system?

Crazy carbons coupled in one-dimension

(a) The circles in the diagram below represent a *spring-ball* model of two carbon atoms constrained to move in the horizontal direction only (without twisting). Here, each carbon atom is modeled as a point mass sphere with mass m, and contains four bonds (dashed lines) that can be approximated as springs. Assume all seven bonds shown have the same constant strength (in N/m), α. The sides of the system (thick black lines) are immovable walls. Neglect any gravitational or quantum effects.



(i) Find the characteristic (or natural) frequencies of this system in terms of known constants.

Solution:
(7-8 pts) Using Newtonian mechanics (i.e. $F = m\ddot{x}$) we can readily write a coupled differential equation system, where x_1 and x_2 are the respective motions away from equilibrium. The characteristic frequencies are independent of spring length, and depends only on spring force ($-\alpha x_1$ and $-\alpha x_2$). By inspection, we find that for the left carbon mass (x_1) we get:

$$m\ddot{x}_1 = -4\alpha x_1 + \alpha x_2 \quad (3)$$

For the right mass we get:

$$m\ddot{x}_2 = -4\alpha x_2 + \alpha x_1 \quad (4)$$

Let's define $\omega_o^2 = \alpha/m$.Assume the solution may be obtained with an exponential of form $x_1(t) = A_1 \exp i\omega t$ and $x_2(t) = A_2 \exp i\omega t$, and we may solve the coupled ODEs by method of determinants where the matrix equation is $\ddot{x} = Cx$ or more explicitly
$C = \begin{pmatrix} -4\omega_o^2 & \omega_o^2 \\ \omega_o^2 & -4\omega_o^2 \end{pmatrix}$
We can now solve for the eigenvalues (or characteristic frequencies) of matrix C by,

$$\det\left(C - \omega^2 I\right) = \left(-4\omega_o^2 - \omega^2\right)^2 - \omega_o^4 = 0 \quad (5)$$

(6)

Hence characteristic frequencies are $\omega_1 = \sqrt{5}\omega_o = \sqrt{5\alpha/m}$ and $\omega_2 = \sqrt{3}\omega_o = \sqrt{3\alpha/m}$

(ii) Evaluate mathematically and describe the physical motion associated with the normal mode(s) of this system.

Solution:
(4-5 pts) Explicitly solving for the two normal modes (eigenvectors), we obtain

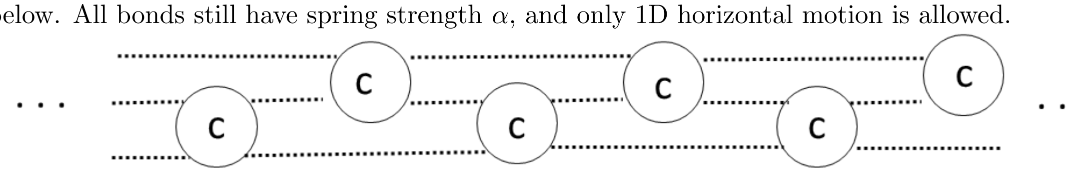
$$\left(C - 5\omega_o^2 I\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

AND

$$\left(C - 3\omega_o^2 I\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Solving the above eigenequations, we see by inspection that two modes $x_2 = -x_1$ or $x_2 = x_1$. These modes physically correspond to in-phase symmetric oscillation for the two masses, and anti-phase oscillations (scissors motion) of the two masses.

(b) Assume now the walls are removed to give the infinitely long system of *springs & balls* shown below. All bonds still have spring strength α, and only 1D horizontal motion is allowed.



(i) Write down a (differential) 1D-equation of motion for n^{th} carbon mass, m in terms of the neighboring masses.

Solution:
(2 pts) Using Newtonian mechanics (i.e. $F = m\ddot{x}$) we can readily write a coupled differential equation system. By inspection that for the n^{th} carbon mass (x_n) we get:

$$m\ddot{x}_n = -4\alpha x_n + \alpha x_{n-2} + \alpha x_{n-1} + \alpha x_{n+1} + \alpha x_{n+2} \tag{7}$$

(ii) Quantitatively determine and qualitatively discuss the allowed characteristic frequencies, ω for this 1D infinite system. What are the minimum and maximum characteristic (or natural) frequencies one might excite in this 1D system?

Solution:
(6 pts) Assume a solution of the form $x_n(t) = Ae^{-i\omega t}e^{ikna}$ where $k = 2\pi/\lambda$, and λ is the wavelength of the normal mode excited, and a is the 1D translation distance. Subbing in part (i) we get:

$$-m\omega^2 e^{ikna} = -4\alpha e^{ikna} + \alpha e^{ik(n-2)a} + \alpha e^{ik(n-1)a} + \alpha e^{ik(n+1)a} + \alpha e^{ik(n+2)a} \tag{8}$$

$$-m\omega^2 = -4\alpha + \alpha(e^{-i2ka} + e^{i2ka}) + \alpha(e^{-ika} + e^{ika}) \tag{9}$$

$$\omega^2 = \frac{\alpha}{m}(4 - 2\cos(2ka) - 2\cos(ka)) \tag{10}$$

Hence we get a continuous dispersion relation, with the following values of characteristic frequencies. $\omega(k) = \sqrt{\alpha/m}\sqrt{4 - 2\cos(2ka) - 2\cos(ka)}$. The minimum value approaches zero, and the maximum value approaches $\sqrt{6\alpha/m}$.
An alternate (less ideal) approach is to solve the characteristic frequencies of a larger, but finite mass problems (e.g. a 5×5), and then to graphically extrapolate to infinite limit of allowed characteristic frequencies.

An ideal paramagnet satisfies the equation of state (Curie's law),

$$M = \frac{D}{T} H,$$

where M is the magnetization, H the magnetic field, T the absolute temperature, and D a constant. An internal energy U is independent of M, following that $dU = C_M dT$, where C_M is a constant heat capacity. Assume that the paramagnet is used to create a Carnot engine and that the engine operates between temperatures T_h and T_c such that $T_h > T_c$.

(a) The first law of Thermodynamics for the magnetic system is written as

$$dU = dQ - dW = TdS + HdM.$$

For an adiabatic process, show that

$$\frac{1}{2}(M_h^2 - M_c^2) = C_M D \ln \frac{T_h}{T_c} = C_M D \ln \frac{H_h M_c}{H_c M_h},$$

where M_h (M_c) is the magnetization at T_h (T_c), when the magnetic field is H_h (H_c).

The Carnot cycle takes the following steps:

- $1 \to 2$ isothermal demagnetization at $T = T_h$, $M_2 < M_1$
- $2 \to 3$ adiabatic demagnetization, $T_h \to T_c$, $M_3 < M_2$
- $3 \to 4$ isothermal magnetization at $T = T_c$, $M_4 > M_3$
- $4 \to 1$ adiabatic magnetization, $T_c \to T_h$, $M_1 > M_4$.

(b) Determine the heat transfer ΔQ and the work performed by the system ΔW for each of the four steps.

(c) Sketch the Carnot cycle in the (M, T)-plane and in the (M, H)-plane.

(d) Prove that the engine has efficiency

$$\eta = 1 - \frac{T_c}{T_h}.$$

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where M_h (M_c) is the magnetization at T_h (T_c), when the magnetic field is H_h (H_c).

Solution:
For an adiabatic process, $dQ = 0$, therefore $dU = -dW$, where $dU = C_M dT$ and $-dW = HdM$. Using the equation of state, we obtain

$$C_M dT = HdM = \frac{T}{D} MdM \tag{11}$$

$$\Rightarrow \quad C_M D \frac{dT}{T} = MdM \tag{12}$$

$$\Rightarrow \quad \int_{T_c}^{T_h} C_M D \frac{dT}{T} = \int_{M_c}^{M_h} MdM \tag{13}$$

$$\Rightarrow \quad C_M D \ln \frac{T_h}{T_c} = \frac{1}{2}(M_h^2 - M_c^2) \tag{14}$$

The equation of state leads to

$$\frac{T_h}{T_c} = \frac{DH_h/M_h}{DH_c/M_c} = \frac{H_h M_c}{H_c M_h}. \tag{15}$$

Therefore,

$$\frac{1}{2}(M_h^2 - M_c^2) = C_M D \ln \frac{T_h}{T_c} = C_M D \ln \frac{H_h M_c}{H_c M_h} \tag{16}$$

The Carnot cycle takes the following steps:

- $1 \rightarrow 2$ isothermal demagnetization at $T = T_h$, $M_2 < M_1$
- $2 \rightarrow 3$ adiabatic demagnetization, $T_h \rightarrow T_c$, $M_3 < M_2$
- $3 \rightarrow 4$ isothermal magnetization at $T = T_c$, $M_4 > M_3$
- $4 \rightarrow 1$ adiabatic magnetization, $T_c \rightarrow T_h$, $M_1 > M_4$.

(b) Determine the heat transfer ΔQ and the work performed by the system ΔW for each of the four steps.

Solution:

- $1 \to 2$ isothermal demagnetization at $T = T_h$, $M_2 < M_1$

For an isothermal process, $dU = C_M dT = 0$, therefore $dQ = dW = -H dM$. Using the equation of state, we obtain

$$dQ = dW = -\frac{T_h}{D} M dM \tag{18}$$

$$\Rightarrow \quad \Delta Q_{12} = \Delta W_{12} = -\frac{T_h}{D} \int_{M_1}^{M_2} M dM = \frac{T_h}{2D}(M_1^2 - M_2^2) > 0 \tag{19}$$

- $2 \to 3$ adiabatic demagnetization, $T_h \to T_c$, $M_3 < M_2$

For an adiabatic process, $dQ = 0$, therefore

$$\Delta Q_{23} = 0. \tag{20}$$

Since $dW = -dU = -C_M dT$,

$$\Delta W_{23} = -C_M \int_{T_h}^{T_c} dT = C_M (T_h - T_c). \tag{21}$$

- $3 \to 4$ isothermal magnetization at $T = T_c$, $M_4 > M_3$

$$dQ = dW = -\frac{T_c}{D} M dM \tag{22}$$

$$\Rightarrow \quad \Delta Q_{34} = \Delta W_{34} = -\frac{T_c}{D} \int_{M_3}^{M_4} M dM = -\frac{T_c}{2D}(M_4^2 - M_3^2) < 0 \tag{23}$$

- $4 \to 1$ adiabatic magnetization, $T_c \to T_h$, $M_1 > M_4$.

Since the step is an adiabatic process,

$$\Delta Q_{41} = 0. \tag{24}$$

Since $dW = -dU = -C_M dT$,

$$\Delta W_{41} = -C_M \int_{T_c}^{T_h} dT = -C_M (T_h - T_c). \tag{25}$$

(c) Sketch the Carnot cycle in the (M, T)-plane and in the (M, H)-plane.

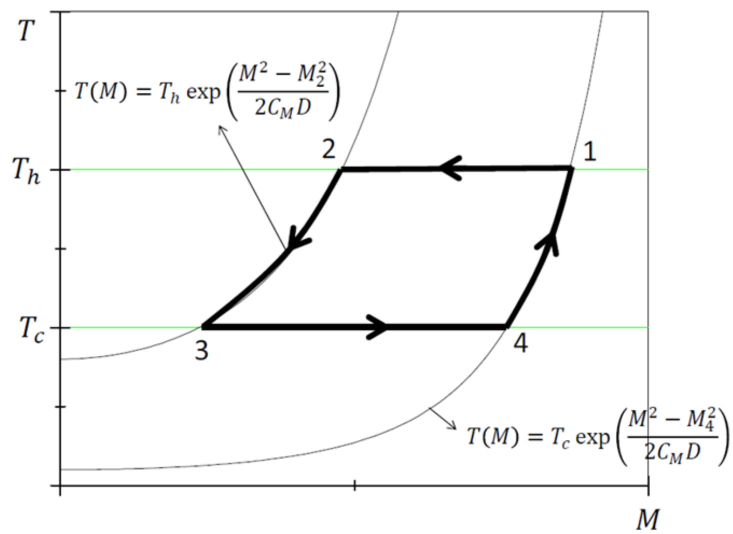
Solution:
In the (M,T)-plane,

- $2 \to 3$ adiabatic demagnetization, $T_h \to T_c$, $M_3 < M_2$

$$\frac{1}{2}(M_2^2 - M^2) = C_M D \ln \frac{T_h}{T} \tag{25}$$
$$\Rightarrow \quad \frac{T_h}{T} = \exp\left(\frac{M_2^2 - M^2}{2C_M D}\right) \tag{26}$$
$$\Rightarrow \quad T(M) = T_h \exp\left(\frac{M^2 - M_2^2}{2C_M D}\right) \tag{27}$$

- $4 \to 1$ adiabatic magnetization, $T_c \to T_h$, $M_1 > M_4$.

Similarly,
$$T(M) = T_c \exp\left(\frac{M^2 - M_4^2}{2C_M D}\right) \tag{28}$$



In the (M,H)-plane,

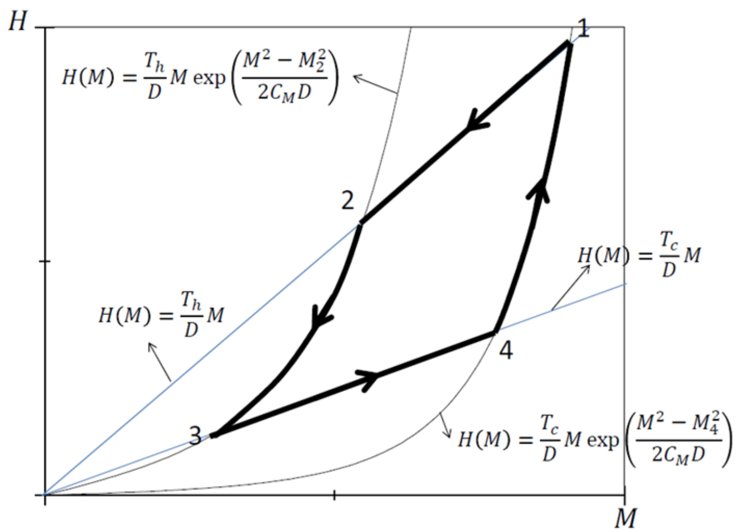
- $2 \to 3$ adiabatic demagnetization, $T_h \to T_c$, $M_3 < M_2$

$$\frac{1}{2}(M_2^2 - M^2) = C_M D \ln\left(\frac{T_h}{D}\frac{M}{H}\right) \tag{29}$$
$$\Rightarrow \quad \frac{T_h}{D}\frac{M}{H} = \exp\left(\frac{M_2^2 - M^2}{2C_M D}\right) \tag{30}$$
$$\Rightarrow \quad H(M) = \frac{T_h}{D} M \exp\left(\frac{M^2 - M_2^2}{2C_M D}\right) \tag{31}$$

- $4 \to 1$ adiabatic magnetization, $T_c \to T_h$, $M_1 > M_4$.

Similarly,

$$H(M) = \frac{T_c}{D} M \exp\left(\frac{M^2 - M_4^2}{2C_M D}\right) \tag{32}$$



(d) Prove that the engine has efficiency

$$\eta = 1 - \frac{T_c}{T_h}.$$

Solution:
The total work performed by the system is

$$\Delta W = \Delta W_{12} + \Delta W_{23} + \Delta W_{34} + \Delta W_{41}. \tag{33}$$

Since $\Delta W_{12} = \Delta Q_{12}$, $\Delta W_{34} = \Delta Q_{34}$, and $\Delta W_{23} = -\Delta W_{41}$,

$$\Delta W = \Delta Q_{12} + \Delta Q_{34}. \tag{34}$$

The engine efficiency is

$$\eta = \frac{\Delta W}{\Delta Q_{12}} \tag{35}$$

$$= 1 + \frac{\Delta Q_{34}}{\Delta Q_{12}} \tag{36}$$

$$= 1 + \frac{\frac{T_c}{2D}(M_3^2 - M_4^2)}{\frac{T_h}{2D}(M_1^2 - M_2^2)} \tag{37}$$

$$= 1 - \frac{T_c}{T_h}\frac{M_4^2 - M_3^2}{M_1^2 - M_2^2} \tag{38}$$

From Eq.(15), we obtain

$$M_1^2 - M_2^2 = M_4^2 - M_3^2 = 2C_M D \ln \frac{T_h}{T_c} \tag{39}$$

Thus,

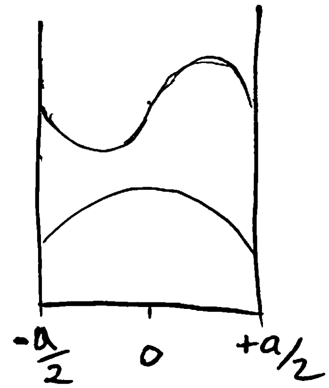
$$\eta = 1 - \frac{T_c}{T_h} \tag{40}$$

An electron (charge e and mass m) is trapped in an infinite potential well of width a. At $t = 0$, the electron is prepared in the state $\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$ where $|1\rangle$ is the ground state and $|2\rangle$ is the first excited state.

(a) Find the expectation value for the electron energy as a function of time in terms of a and fundamental constants.

(b) Find the expectation value for the electron position as a function of time in terms of a and fundamental constants.

4. QM



$$\sqrt{\frac{2}{a}} \sin \frac{2\pi x}{a} \, e^{i\omega_2 t}$$

$$\sqrt{\frac{2}{a}} \cos \frac{\pi x}{a} \, e^{i\omega_1 t}$$

where $\omega_1 = \frac{E_1}{\hbar}$

$\omega_2 = \frac{E_2}{\hbar}$

and $E_1 = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{2ma^2}$

$E_2 = 4 \frac{\hbar^2 \pi^2}{2ma^2}$

(a) $$\langle E \rangle = \frac{1}{2}\left(\langle 1| + \langle 2|\right) H \left(|1\rangle + |2\rangle\right)$$

$$= \frac{1}{2}\left(E_1 + E_2\right) = \frac{1}{2}(1+4)\frac{\hbar^2\pi^2}{2ma^2}$$

$$= \frac{5}{2}\frac{\hbar^2\pi^2}{2ma^2}$$ independent of t.

(b) $$\langle x \rangle = \frac{1}{2}\left(\langle 1| + \langle 2|\right) x \left(|1\rangle + |2\rangle\right)$$

$$= \frac{1}{2}\left[\int \psi_1^* x \psi_1 \, dx + \int \psi_1^* x \psi_2 \, dx + \int \psi_2^* x \psi_1 \, dx + \int \psi_2^* x \psi_2 \, dx\right]$$

There are four integrals to evaluate

$$\int \psi_1^* x \, \psi_1 \, dx = 0$$ (Expectation value of position for gnd state)

$$\int \psi_2^* x \, \psi_2 \, dx = 0$$ (Expectation value of position for $|2\rangle$)

$$\int \psi_1^* x \, \psi_2 \, dx = \int_{-a/2}^{a/2} \sqrt{\frac{2}{a}} \cos \frac{\pi x}{a} e^{-i\omega_1 t} \, x \, \sqrt{\frac{2}{a}} \sin \frac{2\pi x}{a} e^{i\omega_2 t} \, dx$$

$$= \frac{2}{a} e^{i(\omega_2 - \omega_1)t} \underbrace{\int_{-a/2}^{a/2} x \cos \frac{\pi x}{a} \sin \frac{2\pi x}{a} \, dx}_{\text{call this } \beta}$$

$$= \frac{2\beta}{a} e^{i(\omega_2 - \omega_1)t}$$

$$\int \psi_2^* x \, \psi_1 \, dx = \frac{2\beta}{a} e^{-i(\omega_2 - \omega_1)t}$$

Then we have $\langle x \rangle = \frac{2\beta}{a} \frac{1}{2} \left(e^{i(\omega_2 - \omega_1)t} + e^{-i(\omega_2 - \omega_1)t} \right)$

$$= \frac{2\beta}{a} \cos(\omega_2 - \omega_1)t$$

Final step is evaluating β

$$\beta = \int_{-a/2}^{a/2} x \cos \frac{\pi x}{a} \sin \frac{2\pi x}{a} \, dx = 2\int_{-a/2}^{a/2} x \cos^2 \frac{\pi x}{a} \sin \frac{\pi x}{a} \, dx$$

Use integration by parts

$$\int u v' \, dx = uv - \int u' v \, dx$$

$$\beta = 2\left[x\left(\frac{-a}{3\pi}\cos^3\frac{\pi x}{a}\right)\right]_{-a/2}^{a/2} + 2\int_{-a/2}^{a/2} \frac{a}{3\pi}\cos^3\frac{\pi x}{a}\,dx$$

$$= \frac{2a}{3\pi}\int_{-a/2}^{a/2} \cos^3\frac{\pi x}{a}\,dx$$

$$= \frac{2a}{3\pi}\left[\frac{3\sin\frac{\pi x}{a}}{4\,\pi/a} + \frac{\sin\frac{3\pi x}{a}}{12\,\pi/a}\right]_{-a/2}^{a/2}$$ from Table of integrals provided.

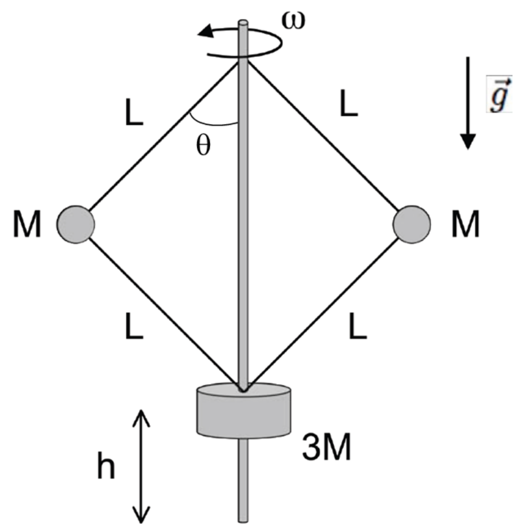
$$= \frac{2a}{3\pi}\left[\frac{3}{4\pi/a} - \frac{1}{12\pi/a} + \frac{3}{4\pi/a} - \frac{1}{12\pi/a}\right]$$

$$= \frac{2a^2}{3\pi^2}\left[\frac{6}{4} - \frac{2}{12}\right] = \frac{2a^2}{3\pi^2}\left[\frac{18}{12} - \frac{2}{12}\right]$$

$$= \frac{2a^2}{3\pi^2}\,\frac{16}{12} = \frac{2a^2}{3\pi^2}\,\frac{4}{3} = \frac{8a^2}{9\pi^2}$$

Thus $\langle x \rangle = \frac{2}{a}\,\frac{8a^2}{9\pi^2}\cos\left(\frac{3\hbar\pi^2}{2ma^2}t\right)$

$$= \frac{16a}{9\pi^2}\cos\left(\frac{3\hbar\pi^2}{2ma^2}t\right)$$

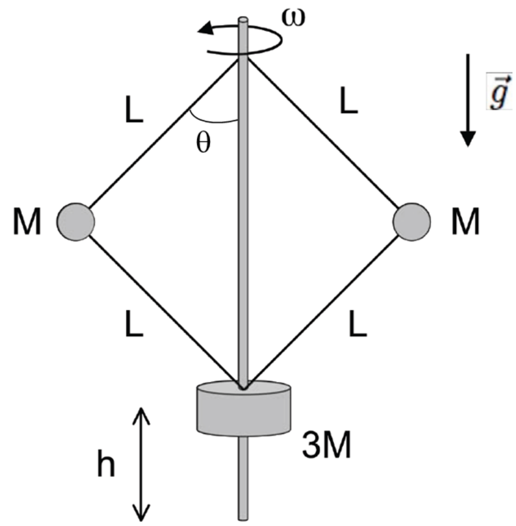


A flyball governor is a device commonly used in steam engines to control the flow of steam. In the simplified version shown above, a rotating shaft is connected to two hinges of mass M through rigid, massless rods of length L. The rods are also attached at the bottom of the device to a larger block of mass 3M which can slide freely up and down the shaft. The shaft rotates at constant angular rate ω.

You may assume masses are all coplanar and can be treated as point masses. There is a constant gravitational acceleration g acting downward g. Let θ be the angle between the hinged rod and the vertical rod.

(a) Assume the you observe a constant time-independent angle, θ (i.e. $\theta(t) = \theta_o$). What is the angular rate of rotation, ω?

(b) For a given angular rotation rate ω, how high (h) does the lower 3M mass rise above its lowest (*rest*) position?



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(a) Assume the you observe a constant time-independent angle, θ (i.e. $\theta(t) = \theta_o$). What is the angular rate of rotation, ω?

Solution:
Write the Lagrangian, $\mathcal{L} = T - U$.
First we need to define a coordinate system, in terms the rod angle, θ and the rotation angle ϕ. Let's make the top point where the rods are all anchored our origin point. Accordingly, the location of a masses m can be given by the coordinates, $x_1 = L\sin\theta\cos\phi$, $y_1 = L\sin\theta\sin\phi$ and $z_1 = -L\cos\theta$.

Likewise, the location of the lower mass is $x_2 = 0$, $y_2 = 0$ and $z_2 = -2L\cos\theta$.

The only potential energy is gravitational, so $U = 2Mgz_1 + 3Mgz_2 = -8MgL\cos\theta$.

Similarly the kinetic can be expressed as,

$$T = 2\frac{1}{2}M(\dot{x_1}^2 + \dot{y_1}^2 + \dot{z_1}^2) + \frac{3}{2}M(\dot{z_2}^2) \tag{41}$$

$$= ML^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + 6ML^2\sin^2\theta\dot{\theta}^2 \tag{42}$$

Since $\dot{\phi} = \omega$. The system Lagrangian is then:

$$\mathcal{L} = T - U \tag{43}$$

$$= L^2(M + 6M\sin^2\theta)\dot{\theta}^2 + ML^2\omega^2\sin^2\theta + 8MgL\cos\theta \tag{44}$$

We then use the Euler-Lagrange equations to get the equation for motion. With respect to θ we get,

$$0 = \frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \tag{45}$$

$$0 = -L^2(M + 6M\sin^2\theta)\ddot{\theta}^2 - 6ML^2\sin\theta\cos\theta\dot{\theta}^2 + ML^2\omega^2\sin\theta\cos\theta - 4MgL\sin\theta \tag{46}$$

For equilibrium we can simplify by noting that, $\ddot{\theta} = \dot{\theta} = 0$ giving, $ML^2\omega^2\sin\theta\cos\theta - 4MgL\sin\theta = 0$ or

$$\omega = \sqrt{\frac{4g}{L}\frac{1}{\cos\theta}} \tag{47}$$

(b) For a given angular rotation rate ω, how high (h) does the lower 3M mass rise above its lowest (*rest*) position?

Solution:
The central mass height is given by $h = 2L + z_1$. At a fixed rotation rate our answer in part b can be rewritten as,

$$\omega = \sqrt{4g\frac{1}{-z_1}} \tag{48}$$

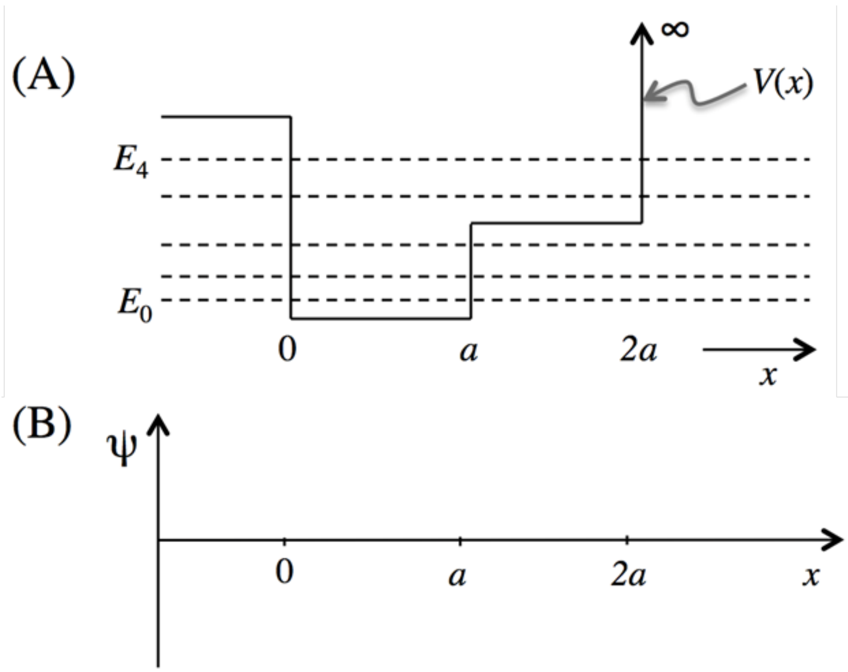
$$z_1 = -\frac{4g}{\omega^2} \tag{49}$$

or $h = 2L - \frac{4g}{\omega^2}$

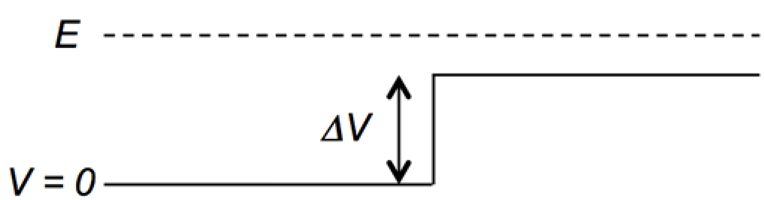
(a) Consider the potential well shown below (Fig. 1A). The potential profile has three steps, located at $x = 0$, a and $2a$. For $x > 2a$, the potential goes to infinity. An electron is trapped in this well.

(i) Copy Figure 1B into your blue book. Draw a plausible electron wavefuntion, $\psi_o(x)$, for the ground state (the eigenstate with energy E_o). Explain the important features of your wavefunction.

(ii) Make another copy of Figure 1B in your blue book. Draw a plausible electron wavefuntion, $\psi_4(x)$, for the 4^{th} excited state (the eigenstate with energy E_4). Explain the important features of your wavefunction.

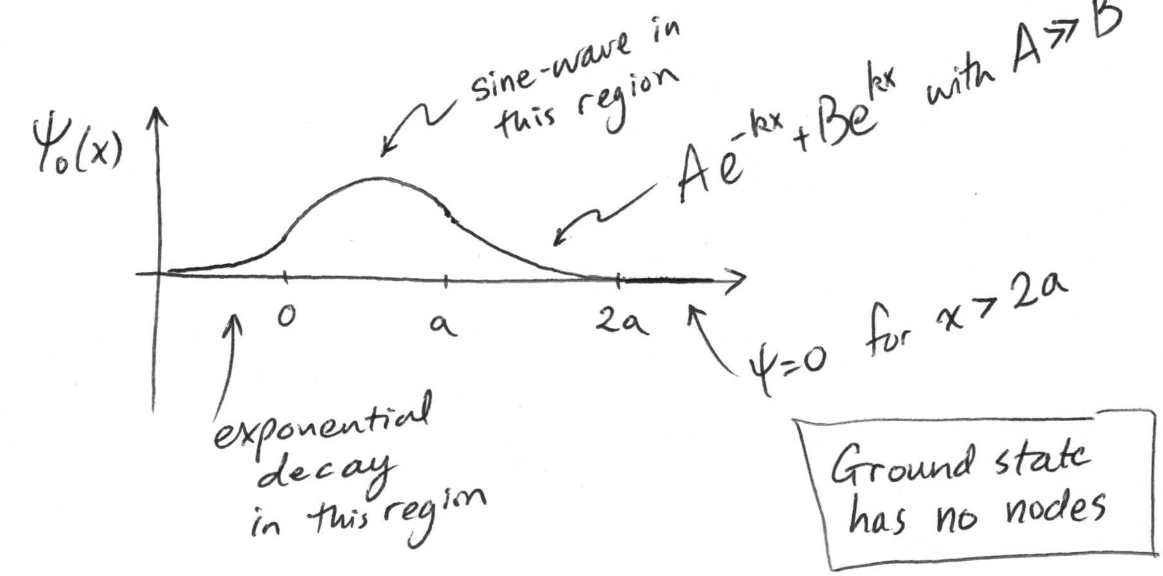


(b) Calculate the probability of reflection and probability of transmission for an electron incident from the left onto a potential step (see figure below). The step height ΔV is less than the electron energy E. Give your answer in terms of E, ΔV, and fundamental constants.



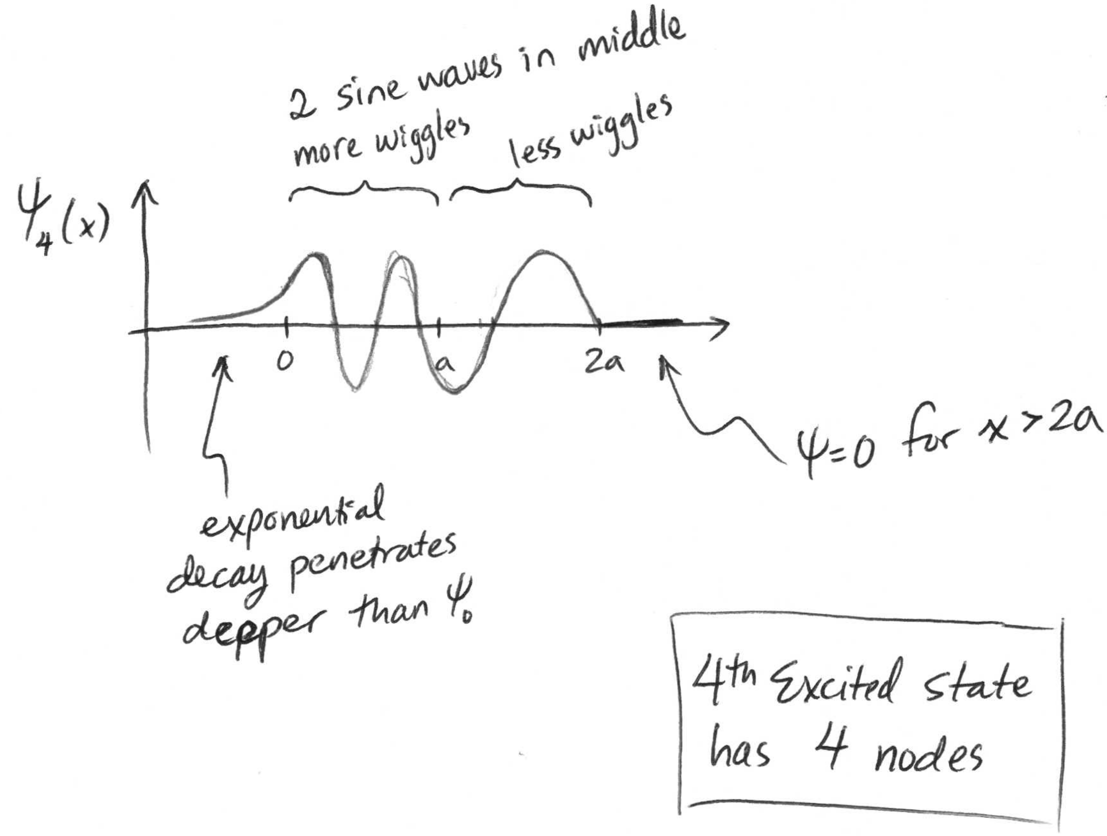
6. QM

(a) (i)



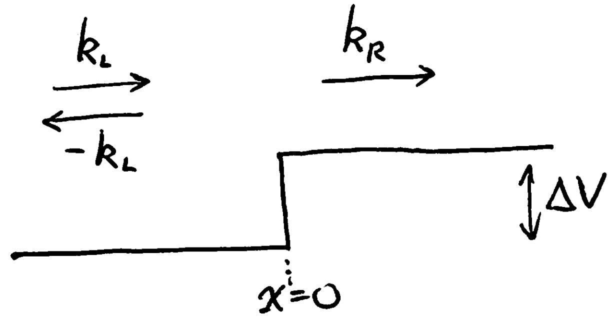
$\psi_0(x)$
sine-wave in this region
$Ae^{-kx} + Be^{kx}$ with $A \gg B$
0
a
2a
exponential decay in this region
$\psi = 0$ for $x > 2a$
Ground state has no nodes

(ii)



$\psi_4(x)$
2 sine waves in middle
more wiggles
less wiggles
0
a
2a
exponential decay penetrates depper than ψ_0
$\psi = 0$ for $x > 2a$
4th Excited state has 4 nodes

(b)



The wave vector changes when particle crosses the step

$$k_L = \frac{\sqrt{2mE}}{\hbar} \qquad k_R = \frac{\sqrt{2m(E-\Delta V)}}{\hbar}$$

The wave function is a piecewise function

$$\psi(x) = \begin{cases} Ae^{ik_L x} + Be^{-ik_L x} & x<0 \\ Ce^{ik_R x} & x>0 \end{cases}$$

Where A, B & C are constants that must be chosen to satisfy the boundary conditions

① $\psi(0^-) = \psi(0^+)$

② $\left.\frac{d\psi}{dx}\right|_{0^-} = \left.\frac{d\psi}{dx}\right|_{0^+}$

From ① we have $A + B = C$

From ② we have $ik_L A - ik_L B = ik_R C$

$$\Rightarrow A - B = \frac{k_R}{k_L}$$

From these two linear eqns, find B in terms of A:

$$A - B = \frac{k_R}{k_L}(A+B)$$

$$A\left(1 - \frac{k_R}{k_L}\right) = B\left(1 + \frac{k_R}{k_L}\right)$$

$$\frac{B}{A} = \frac{1 - \frac{k_R}{k_L}}{1 + \frac{k_R}{k_L}} = \frac{1 - \sqrt{\frac{E - \Delta V}{E}}}{1 + \sqrt{\frac{E - \Delta V}{E}}}$$

The probability of reflection is

$$\left|\frac{B}{A}\right|^2 = \left(\frac{1 - \sqrt{1 - \frac{\Delta V}{E}}}{1 + \sqrt{1 - \frac{\Delta V}{E}}}\right)^2$$

The Probability of transmission is

$$1 - \left|\frac{B}{A}\right|^2 = 1 - \left(\frac{1 - \sqrt{1 - \frac{\Delta V}{E}}}{1 + \sqrt{1 - \frac{\Delta V}{E}}}\right)^2$$

Electric charge is distributed uniformly with constant volume density inside an infinitely long cylinder, of radius a. The cylinder rotates around its long (z) axis with angular velocity ω. Use cylindrical coordinates, in which ρ is the perpendicular distance to the z axis and ϕ is an azimuthal angle measured from the positive x-axis. In what follows, justify any symmetry arguments carefully and explain why quantities are zero if you assert that they are zero.

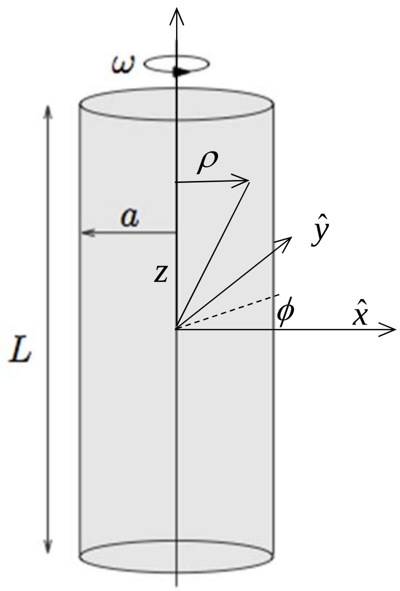
(a) What is the current density J inside the cylinder? Express the result in cylindrical coordinates, *i.e.*, determine J_ρ, J_ϕ, J_z as functions of ρ, ϕ and z.

(b) What is the magnetic field *outside* the cylinder, *i.e.*, at distances ρ, such that $\rho > a$? Express the result in cylindrical coordinates, *i.e.*, determine B_ρ, B_ϕ, B_z as functions of ρ, ϕ and z.

(c) What is the magnetic field *inside* the cylinder? Express the result in cylindrical coordinates, *i.e.*, determine B_ρ, B_ϕ, B_z as functions of ρ, ϕ and z.

Comprehensive Exam, Spring 2015 E&M Graduate (Solution)

Set up the cylinder long axis as the *z*-axis. In cylindrical coordinates, position is specified by $\vec{r} = z\hat{z} + \rho\hat{\rho}$ where *z* measures the distance from the *xy* plane and ρ measures the perpendicular distance to the *z*-axis. $\hat{\rho}$ is a unit vector in the *x*-*y* plane (or parallel to it), and $\hat{\phi}$ is a unit vector in the direction of increasing angle ϕ, which is the angle the position vector (projection) makes with the positive *x*-axis.



$$\hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi$$
$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

(a) When the static charge density α is set into rotation, it generates a current density *J* (charge per unit time per unit area). $\vec{J} = \alpha\vec{v} = \alpha\vec{\omega}\times\vec{r}$

Rotation is along the *z*-axis: $\vec{\omega} = \omega\hat{z}$ so the current is in the azimuthal direction only.

$$\vec{J} = \alpha\omega\hat{z}\times\left(z\hat{z} + \rho\hat{\rho}\right) = \alpha\omega\rho\left(\hat{z}\times\hat{\rho}\right) = \alpha\omega\rho\hat{\phi}$$

$$\boxed{\begin{array}{l} J_\rho = J_z = 0 \\ J_\phi = \alpha\omega\rho \end{array}}$$

The task is to find the magnetic field *B*, which requires Ampere's law. First note that all the possible field components could in principle be functions of *z*, ρ, ϕ:

$$B_z = B_z\left(z,\rho,\phi\right);\quad B_\rho = B_\rho\left(z,\rho,\phi\right);\quad B_\phi = B_\phi\left(z,\rho,\phi\right)$$

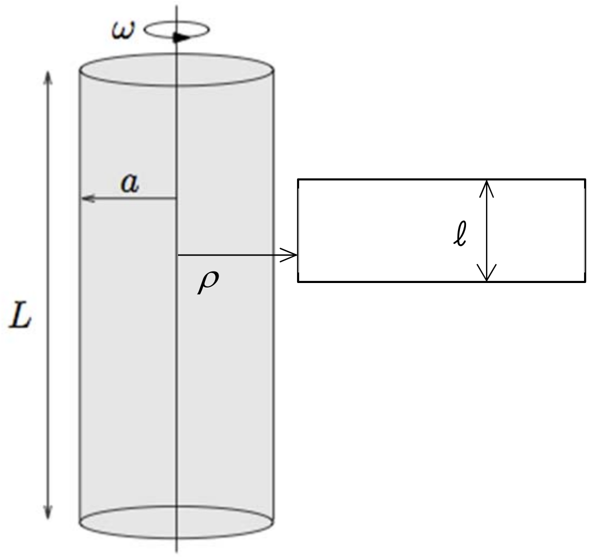
However, NONE can be functions of *z* or ϕ. If they were, that would allow us to determine the vertical or azimuthal position by the *B*-field, and that is impossible given the symmetry of the problem.

Also, the field MUST be in the *z* direction. Consider the cylinder as a superposition of current loops – each producing a dipole field. In the plane of any particular loop, the field is always in the *z* direction (by the Biot Savart Law). The radial component of the field produced by another loop a distance *d* above it is cancelled by the radial component of the field by a corresponding loop a distance *d* below it. There are infinitely many such loops, so there is <u>no radial field</u>.

There is also no azimuthal field. Again the Biot Savart law says $d\vec{B} = id\vec{\ell}\times\vec{r}$, and because the current is azimuthal, there is no azimuthal component of the field.

This leaves us with a *B* field in the *z* direction that, at most, depends on ρ:

$$B_z = B_z\left(\rho\right);\quad B_\rho = B_\phi = 0$$

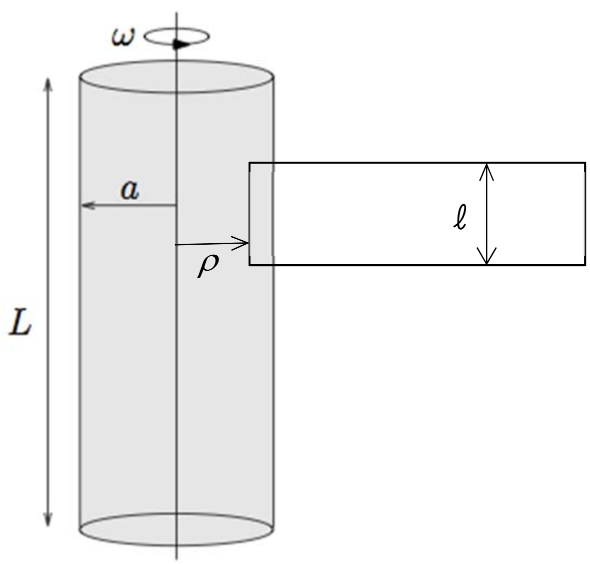


(b) Ampere's law says $\int_{loop} \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc}$ where I_{enc} is the current that threads the loop. Use a rectangular loop *outside* the cylinder, so not current threads the loop. One arm is parallel to, and a distance ρ from, the *z*-axis, and a parallel arm at infinity. The other arms are perpendicular to the *z*-direction. We have already argued that *B* is in the *z*-direction, so there is not contribution to the loop integral along the directions perpendicular to *z* :

$$\int_{loop} \vec{B}\bullet d\vec{\ell} = \mu_0 I_{enc} = 0$$

$$B_z(\rho)\ell + 0 + 0 + B_z(\infty)\ell = 0$$

If the cylinder is infinite, the magnetic field remains confined to the cylinder, which means that $\hat{B}(\infty) = 0$. This in turn implies that $\boxed{B_z(\rho > a) = 0}$.

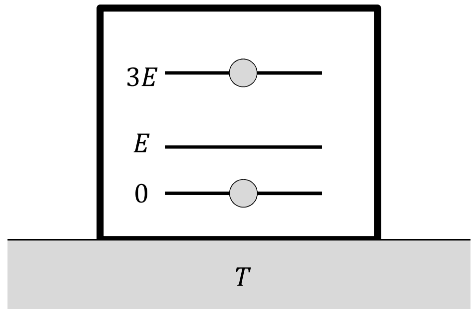


(c) Ampere's law with a rectangular loop with one arm *inside* the cylinder, using the same arguments as in (b), only now there *is* current threading the loop:

$$\int_{loop} \vec{B}\bullet d\vec{\ell} = \mu_0 I_{enc}$$

$$\boxed{\begin{aligned} B_z(\rho)\ell &= \mu_0 \int_0^{\ell}\int_{\rho}^{a} J_{enc}\, d\rho\, dz = \mu_0 \ell \int_{\rho}^{a} \omega\rho\, d\rho = \frac{\mu_0 \ell \omega}{2}\left(a^2 - \rho^2\right) \\ B_z &= \frac{\mu_0 \omega}{2}\left(a^2 - \rho^2\right) \end{aligned}}$$

Consider a quantum system of three energy levels, 0, E, and $3E$. Two identical particles are in the system, and each particle can be in any one of three quantum states. The system is in contact with a heat reservoir at temperature T ($\beta = 1/k_B T$). A possible configuration is shown in the figure below.



Find the partition function Z and the mean energy $\langle E \rangle$, if the particles obey

(a) Fermi-Dirac statistics (Assume that the particles are spin-1/2 Fermions and they are spin up.)

(b) Bose-Einstein statistics (Assume that the particles are spin-0 bosons).

(c) classical Maxwell-Boltzman statistics (Note: Identical particles are distinguishable in classical statistics).

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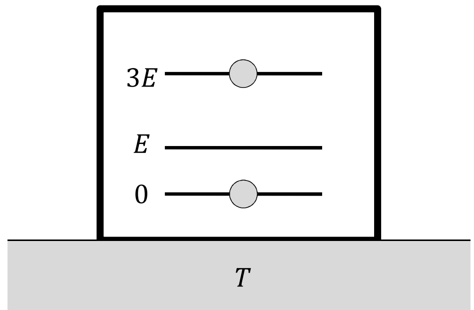


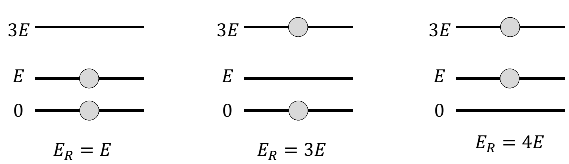
Find the partition function Z and the mean energy $\langle E \rangle$, if the particles obey

(a) Fermi-Dirac statistics (Assume that the particles are spin-1/2 Fermions and they are spin up.)

Solution:
The partition function is

$$Z = \sum_R e^{-\beta E_R}, \tag{50}$$

where the sum is all the possible state R of the whole system. All the possible states of two Fermions are illustrate in the figure below, where $E_R = E$, $3E$, and $4E$.



Therefore,

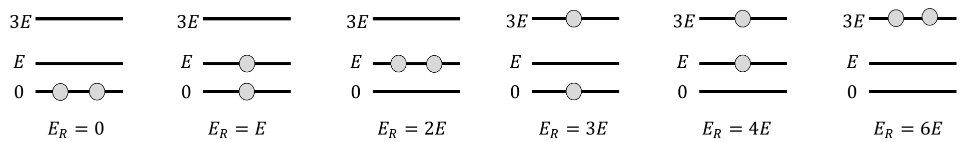
$$Z = e^{-\beta E} + e^{-3\beta E} + e^{-4\beta E}. \tag{51}$$

The mean energy is

$$\begin{aligned}
\langle E \rangle &= -\frac{1}{Z}\frac{\partial Z}{\partial \beta} && (52)\\
&= \frac{Ee^{-\beta E} + 3Ee^{-3\beta E} + 4Ee^{-4\beta E}}{e^{-\beta E} + e^{-3\beta E} + e^{-4\beta E}} && (53)\\
&= E\frac{1 + 3e^{-2\beta E} + 4e^{-3\beta E}}{1 + e^{-2\beta E} + e^{-3\beta E}} && (54)
\end{aligned}$$

(b) Bose-Einstein statistics (Assume that the particles are spin-0 bosons).

Solution:
All the possible states of two Bosons are illustrate in the figure below, where $E_R = 0$, E, $2E$, $3E$, $4E$, and $6E$.



Therefore,

$$Z = 1 + e^{-\beta E} + e^{-2\beta E} + e^{-3\beta E} + e^{-4\beta E} + e^{-6\beta E}. \tag{55}$$

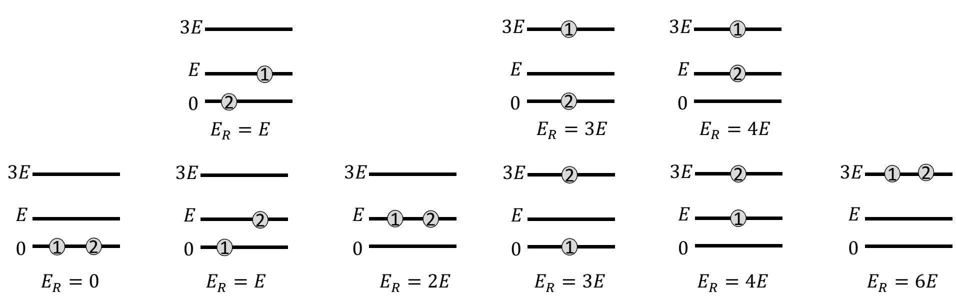
The mean energy is

$$\langle E \rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} \tag{56}$$

$$= E\frac{e^{-\beta E} + 2e^{-2\beta E} + 3e^{-3\beta E} + 4e^{-4\beta E} + 6e^{-6\beta E}}{1 + e^{-\beta E} + e^{-2\beta E} + e^{-3\beta E} + e^{-4\beta E} + e^{-6\beta E}} \tag{57}$$

(c) classical Maxwell-Boltzman statistics (Note: Identical particles are distinguishable in classical statistics).

Solution:
Even though two particles are identical, they are distinguishable in classical MB statistics. All the possible states of two MB particles are illustrate in the figure below, where $E_R = 0$, E, $2E$, $3E$, $4E$, and $6E$.



Therefore,

$$Z = 1 + 2e^{-\beta E} + e^{-2\beta E} + 2e^{-3\beta E} + 2e^{-4\beta E} + e^{-6\beta E}. \tag{58}$$

The mean energy is

$$\langle E \rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} \tag{59}$$

$$= 2E\frac{e^{-\beta E} + e^{-2\beta E} + 3e^{-3\beta E} + 4e^{-4\beta E} + 3e^{-6\beta E}}{1 + 2e^{-\beta E} + e^{-2\beta E} + 2e^{-3\beta E} + 2e^{-4\beta E} + e^{-6\beta E}} \tag{60}$$