OSU Physics Department
Comprehensive Examination #120

Monday, September 29 and Tuesday, September 30, 2014

Fall 2014 Comprehensive Examination

PART 1, Monday, September 29, 9:00pm

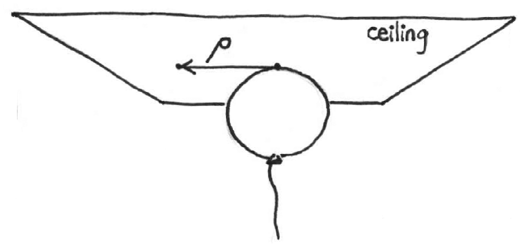
General Instructions

This Fall 2014 Comprehensive Examination consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, September 29, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:00 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, September 30, at 9:00 am and 1:00 pm, respectively. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please return all bluebooks and formula sheets at the end of the exam. Use the last pages of your bluebooks for "scratch" work, separated by at least one empty page from your solutions. "Scratch" work will not be graded.

A ceiling is covered by a horizontal foil of aluminum (a conducting metal) that is insulated by a thin plastic coating. The plastic has a relative dielectric constant of 1, i.e. $\epsilon = \epsilon_o$.

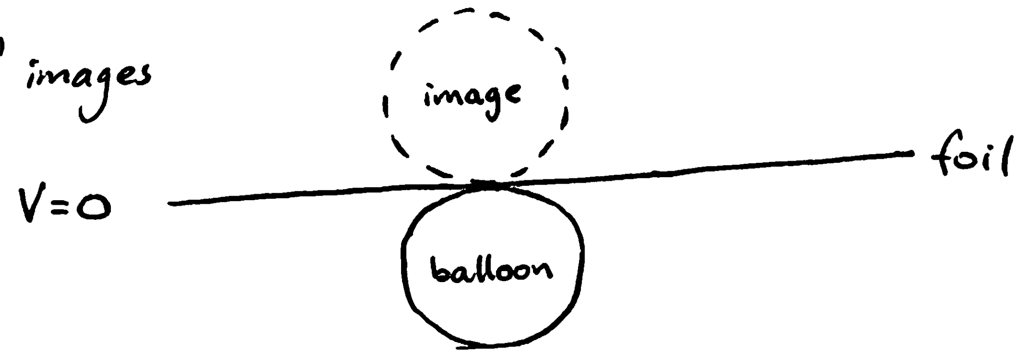
A spherical balloon of mass m and radius a is given an electric charge Q by rubbing it against a wool sweater. When placed in contact with the ceiling, the balloon remains suspended. Assume the charge is, and remains, distributed uniformly on the balloon‘s surface and the balloon remains spherical.



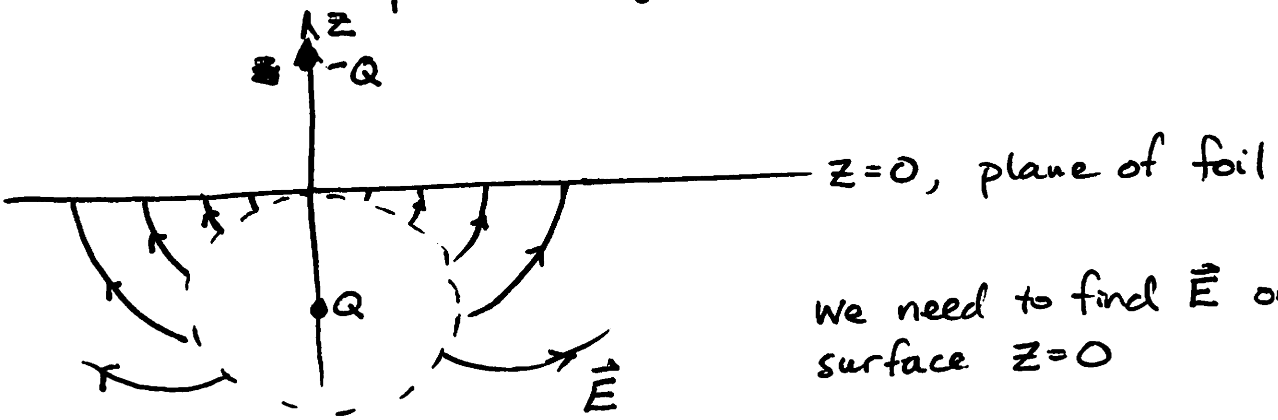
(a) Find the distribution of surface charge density in the foil, $\sigma(\rho)$, as a function of the distance ρ from the point of contact.

(b) What is the minimum value of Q necessary so the balloon will not fall?

a) Using method of images

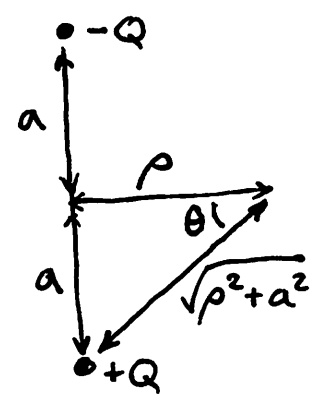


E-field in region $r > a$ can be described by E-field around a point charge, Q.



We need to find $\vec{E}$ on the surface $z=0$

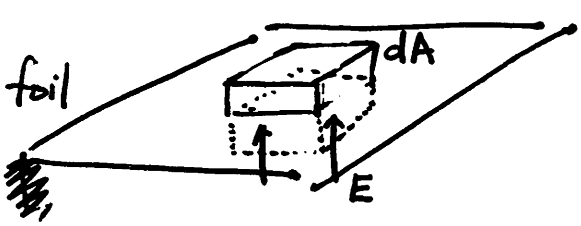
Coordinate system:
ρ is the distance from origin to a point on the foil.



$$E(\rho) = \left(\frac{Q}{4\pi\epsilon_0(\rho^2+a^2)} \underbrace{\frac{a}{\sqrt{\rho^2+a^2}}}_{=\sin\theta} + \frac{Q}{4\pi\epsilon_0(\rho^2+a^2)} \underbrace{\frac{a}{\sqrt{\rho^2+a^2}}}_{=\sin\theta} \right) \hat{z}$$

Note: The $\hat{\rho}$ components cancel.

Use Gauss's Law to calculate surface charge density, σ.



$$\vec{E} \cdot d\vec{A} = \frac{\sigma \, dA}{\epsilon_0}$$

$$-E = \frac{\sigma}{\epsilon_0}$$

$$\sigma = -\epsilon_0 E_z(\rho)$$

$$= \frac{-Qa}{2\pi(a^2+\rho^2)^{3/2}}$$

Note that σ has opposite sign as Q.

b) $$F = \frac{Q^2}{4\pi\epsilon_0 (2a)^2} \geq mg$$

$$\Rightarrow \quad Q > \sqrt{16\pi\epsilon_0 a^2 mg}$$

$$Q > 4a\sqrt{\pi\epsilon_0 mg}$$

Bead in a bowl. Chester the cat watches the motion of a bead in a bowl. A bead of mass m moves under the influence of gravity on the inner frictionless surface of a paraboloid bowl of revolution described by $x^2 + y^2 = az$. The z-axis is vertical upwards and there is a uniform gravitational field.

Assume the bead can be treated as a point-particle (and neglect the rotation of the bead).

(a) Chester imposes arbitrary initial conditions on the bead. He then desires to catch and kill the bead at some later time.

Help Chester catch the bead by deriving the general equation of motion for the bead in the paraboloid. Do not solve.

(You may assume cats are adept at solving differential equations, enabling him to follow the bead's three-dimensional trajectory perfectly once you derive the simplest differential equation(s) describing the bead's motion.)

(b) Suppose the bead's motion is now constrained to a circle of fixed height $z = h$. Find the bead's angular velocity and show it is independent of the height, h.

(c) Chester get restless, and he decides to perturb the bead slightly when it is traveling in a purely circular path (as in part b). In the limit that $h \ll a$, show that Chester observes the bead's new time-dependent frequency of oscillation to be approximately twice the original angular velocity of the unperturbed problem.

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(You may assume cats are adept at solving differential equations, enabling him to follow the bead's three-dimensional trajectory perfectly once you derive the simplest differential equation(s) describing the bead's motion.)

Solution:
This a three-dimensional problem in which gravity in the only potential force. So in Cartesian coordinates we get the following kinetic (T) and potential energy (U);

$$T = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) \tag{1}$$

$$U = mgz \tag{2}$$

Cylindrical coordinates are optimal for this problem, it is difficult to find out what the conserved quantity is otherwise. Let $x = r\cos\phi$, $y = r\sin\phi$ such that the bead is on the paraboloid $r^2 = za$. Thus, for T and U we get,

$$T = \frac{mr^2}{2}\dot{\phi}^2 + \frac{m\dot{r}^2}{2} + \frac{m\dot{z}^2}{2} \tag{3}$$

$$U = mgz \tag{4}$$

OR

$$T = \frac{mr^2}{2}\dot{\phi}^2 + \frac{m\dot{r}^2}{2}\left(1 + \frac{4r^2}{a^2}\right) \tag{5}$$

$$U = \frac{mg}{a}r^2 \tag{6}$$

Hence, the bead's motion may be fully described by $r(t)$ and $\phi(t)$.

We then use the Euler-Lagrange equation to get the equation for motion, where our Lagrangian is $L(\phi, r, \dot{\phi}, \dot{r}) = T - U$. With respect to ϕ we get,

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0 \tag{7}$$

$$\frac{d}{dt}\left(mr^2\dot{\phi}\right) = 0 \tag{8}$$

This differential equation is solved trivially, and suggests the existence of constant conserved quantity, $J = mr^2\dot{\phi}$.

Evaluating the Euler-Lagrange equations for r, we get $\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0$ or

$$mr\dot{\phi}^2 + \frac{4mr\dot{r}^2}{a^2} - \frac{2mg}{a}r - m\left(1 + \frac{4r^2}{a^2}\right)\ddot{r} - \frac{8mr}{a^2}\dot{r}^2 = 0 \tag{9}$$

Lastly, we must note that we can effectively eliminated the $\dot{\phi}$ contribution by subbing in our conserved quantity, J. So our final differential equation of motion for the bead becomes,

$$\left(1 + \frac{4r^2}{a^2}\right)\ddot{r} = \frac{J^2}{m^2r^3} - \frac{4r}{a^2}\dot{r}^2 - \frac{2g}{a}r \tag{10}$$

OR, alternatively Chester the cat can simply simplify equation 9 (taking note of the conserved quantity) to get ,

$$\left(1 + \frac{4r^2}{a^2}\right)\ddot{r} = \left(\dot{\phi}^2 - \frac{4}{a^2}\dot{r}^2 - \frac{2g}{a}\right)r \tag{11}$$

to catch the bead in bowl, where $\dot{\phi} = J/mr^2$.

(b) Suppose the bead's motion is now constrained to a circle of fixed height $z = h$. Find the bead's angular velocity and show it is independent of the height, h.

Solution:
When confined to a circle the bead motion will have a constant radius, $r_o = \sqrt{ha}$. Furthermore, we note that $\dot{r} = 0$ and $\ddot{r} = 0$.

Evaluating our differential equation of motion under these conditions we obtain,

$$0 = \frac{J^2}{m^2 r_o^3} - \frac{2g}{a} r_o \tag{12}$$

$$J^2 = \frac{2m^2 g r_o^4}{a} \tag{13}$$

Our angular velocity, ω (i.e. $\dot{\phi}$) can be then solved for directly using $J = mr^2\dot{\phi}$,

$$\dot{\phi} = \pm\sqrt{2g/a} \tag{14}$$

$$= \pm\omega \tag{15}$$

$\dot{\phi} = \omega$ is independent of h, as required. Hence, Chester the cat will observe a constant angular velocity of the bead everywhere in the bowl (providing its motion is confined to a perfect circle). This will make Chester's catch much easier!

(c) Chester get restless, and he decides to perturb the bead slightly when it is traveling in a purely circular path (as in part *b*). In the limit that $h \ll a$, show that Chester observes the bead's new time-dependent frequency of oscillation to be approximately twice the original angular velocity of the unperturbed problem.

Solution:
Chester perturbs both the constant radius, r_o and angular velocity ω to give a new trajectory described by time-dependent perturbation r_1 and ω_1, i.e.

$$\dot{\phi}(t) = \omega + \omega_1(t) \tag{16}$$

$$r(t) = r_o + r_1(t) \tag{17}$$

We now sub this into the Euler-Lagrange equations and linearize each equation. Combining with the first Euler-Lagrange equation we get,

$$\frac{d}{dt}\left(mr^2\dot{\phi}\right) = 0 \tag{18}$$

$$2r\dot{r}\dot{\phi} + r^2\ddot{\phi} = 0 \tag{19}$$

$$2(r_o + r_1)\dot{r_1}(\omega + \omega_1) + (r_o + r_1)^2\dot{\omega_1} = 0 \tag{20}$$

Since the perturbation is small we can now linearize the above equation with respect to $\dot{r}_1$, neglecting all quadratic and higher order perturbative terms, i.e. equation 20 can be reduced to,

$$2r_o\dot{r_1}\omega + r_o^2\dot{\omega_1} = 0 \tag{21}$$

$$\dot{\omega_1} = \frac{-2\dot{r}_1}{r_o}\omega \tag{22}$$

Likewise for the second Euler-Lagrange equation we desire to get equation of motion for $\ddot{r}_1(t)$, we sub in our purturbative approximation from equation 11, obtaining

$$\left(1 + \frac{4(r_o + r_1)^2}{a^2}\right)\ddot{r_1} = \left((\omega + \omega_1)^2 - \frac{4}{a^2}\dot{r_1}^2 - \frac{2g}{a}\right)(r_o + r_1) \tag{23}$$

Since the perturbation is small we again linearize the above equations this time with respect to ω_1, neglecting all quadratic and higher order perturbative terms. We reatain only the lowest order terms (with respect to r and ω) on both sides of the equality. Solving for $\ddot{r}_1$ we obtain,

$$\ddot{r_1} = \frac{2r_o\omega\omega_1}{1 + 4r_o^2/a^2} \tag{24}$$

Furthermore we note that since $h \ll a$, $4r_o^2/a^2 = 4ha/a^2 = 4h/a \ll 1$ hence, $\ddot{r_1} \cong 2r_o\omega\omega_1$. We can now decouple the ODEs by subbing in (from equation 22) that

$$\ddot{\omega_1} = \frac{-2\ddot{r}_1}{r_o}\omega \tag{25}$$

$$\ddot{\omega_1} = -(2\omega)^2\omega_1 \tag{26}$$

Hence the approximate solution to the resulting frequency of oscillation is form, $\omega_1(t) = A\cos(2\omega t + B)$, consequently the observed frequency of oscillation (ω_1) is twice the beads angular velocity ($\dot{\phi} = \omega$), as required.

Note: this last part was intended to be challenging. Any reasonable approach using perturbation theory and sensible approximations will get full points.

N particles of spin 1/2 form a 1D lattice, lining up on a straight line. Every spin interacts with its nearest neighbors. The Hamiltonian of this 1D Ising model is

$$H = -J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}, \ J\text{: interaction strength.}$$

The spin variables σ_i can only take the values +1 (spin-up) and -1 (spin-down).

(a) Show that the partition function Z of the assembly at temperature T is

$$Z = 2^N \left[\cosh(\beta J)\right]^{N-1} \ \text{with } \beta = \frac{1}{kT}.$$

(b) Find the energy E of the system as a function of temperature T. What are the energies at two extreme temperatures, $T = 0$ and $T \to \infty$? Justify that your answer is consistent with the spin configurations at $T = 0$ and $T \to \infty$.

(c) Find the entropy S of the system as a function of temperature T. What are the entropies at two extreme temperatures, $T \to 0$ and $T \to \infty$?

Useful formula: Binomial expansion

$$(1 + x)^n = \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} x^k$$

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Solution:
As there are N particles, $(N-1)$ interacting pairs. Of theses, N_p is the number of parallel spins and N_a the number of antiparallel spins. Since

$$N_p + N_a = N - 1, \tag{27}$$

the energy of a given configuration is

$$E_{N_p,N_a} = J(N_p - N_a) = 2N_p + 1 - N. \tag{28}$$

The partition function is defined as

$$Z = \sum_{i,j} e^{-E/kT}. \tag{29}$$

There are $(N-1)!$ permutations of $N-1$ pairs, but only $(N-1)!/N_a!N_p!$ are distinguishable. Hence

$$\begin{aligned} Z &= 2\sum_{N_p=0}^{N-1} \frac{(N-1)!}{N_a!N_p!} \exp\left[-\frac{J(2N_p+1-N)}{kT}\right] && (30) \\ &= 2\exp\left[-\frac{J(N-1)}{kT}\right] \sum_{N_p=0}^{N-1} \frac{(N-1)!}{((N-1)-N_p)!N_p!} \exp\left[-\frac{2JN_p}{kT}\right] && (31) \end{aligned}$$

The overall factor of 2 arises because reversing the direction of all spins does not change N_p or N_a but does give rise to a new configuration. In the above, the sum is a binomial expansion. The partition function then becomes

$$\begin{aligned} Z &= 2\exp\left[-\frac{J(N-1)}{kT}\right] \left[1 + \exp\left(-\frac{2J}{kT}\right)\right]^{N-1} && (32) \\ &= 2^N \left[\cosh\left(\frac{J}{kT}\right)\right]^{N-1} && (33) \\ &= 2^N \left[\cosh\left(\beta J\right)\right]^{N-1}. && (34) \end{aligned}$$

(b) Find the energy E of the system as a function of temperature T. What are the energies at two extreme temperatures, $T = 0$ and $T \to \infty$? Justify that your answer is consistent with the spin configurations at $T = 0$ and $T \to \infty$.

Solution:
The energy of the N particle system is

$$\begin{aligned} E &= -\frac{\partial}{\partial \beta} \ln Z && (35) \\ &= -\frac{\partial}{\partial \beta} \ln \left\{ 2^N \left[\cosh(\beta J)\right]^{N-1} \right\} && (36) \\ &= -(N-1)\frac{\partial}{\partial \beta} \ln\left[\cosh(\beta J)\right] && (37) \\ &= -(N-1) J \tanh(\beta J) && (38) \end{aligned}$$

At $T = 0$ ($\beta \to \infty$), $\tanh(\beta J) = 1$, thus, $E(T = 0) = -(N-1)J$. All the spins must be parallel at $T = 0$ for the system to be in the ground state. The energy of the $N-1$ spin pairs of the same direction is $E(T = 0) = -(N-1)J$.

For $T \to \infty$ ($\beta = 0$), $E = 0$. Since the spins are randomly oriented for $T \to \infty$, $N_a = N_p$ and $E = 0$.

(c) Find the entropy S of the system as a function of temperature T. What are the entropies at two extreme temperatures, $T \to 0$ and $T \to \infty$?

Solution:
The Helmholtz free energy is

$$F = -kT \ln Z \tag{39}$$

and the entropy is

$$\begin{aligned} S &= -\left(\frac{\partial F}{\partial T}\right) = k \ln Z + kT \frac{\partial}{\partial T} \ln Z && (40) \\ &= k \ln Z + kT \frac{\partial}{\partial \beta} \ln Z \frac{\partial \beta}{\partial T} = k \ln Z + \frac{E}{T} && (41) \\ &= k \ln \left[2^N \left[\cosh(\beta J)\right]^{N-1} \right] - \frac{1}{T}(N-1) J \tanh(\beta J) && (42) \end{aligned}$$

For $T \to 0$ ($\beta \to \infty$), $\cosh(\beta J) \to \frac{1}{2} e^{\beta J}$, thus

$$\begin{aligned} S &\cong k \ln \left[2^N \left(\frac{e^{\beta J}}{2} \right)^{N-1} \right] - \frac{(N-1)J}{T} && (43) \\ &= k \ln 2 + \frac{(N-1)J}{T} - \frac{(N-1)J}{T} && (44) \\ &= k \ln 2 && (45) \end{aligned}$$

For $T \to \infty$ ($\beta \to 0$), $\cosh(\beta J) \to 1$, thus

$$\begin{aligned} S &\cong k \ln \left(2^N \right) + E(\infty)/T && (46) \\ &= kN \ln 2 && (47) \end{aligned}$$

Useful formula: Binomial expansion

$$(1+x)^n = \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} x^k$$

(a) The potential energy of a non-relativistic particle of mass m in 1 dimension is zero everywhere except at $x = 0$, where the potential energy is

$$V(x) = \frac{\hbar^2 \beta}{2m}\delta(x)$$

$\delta(x)$ is the Dirac delta function, and $\beta < 0$ is a parameter that characterizes the potential well strength. Let us first explore bound states of this system.

(i) Integrate the eigenvalue equation $H\phi(x) = E\phi(x)$ to show that the derivative of the wave function $\phi(x)$ is not continuous at $x = 0$, and that the change in the derivative at $x = 0$ is equal to $\beta\phi(0)$.

(ii) Calculate the bound state energies and wave functions of the system.

(iii) How many are states are there - zero? a finite number (how many)? an infinite number?

(b) Now let $\beta > 0$, turning the well into a barrier (very thin, very high at $x = 0$), again with zero potential everywhere else. We will explore the scattering of unbound states. The non-relativistic particle is now incident from the left ($x = -\infty$) and travels in the positive x direction with momentum p, total energy $E > 0$.

(i) Calculate the reflection and transmission coefficients. The result in (a)(i) will be useful.

(ii) Show that the sum of the transmission probability and the reflection probability is 1.

Undergraduate quantum mechanics

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Undergraduate quantum mechanics - SOLUTION

(a) (i) Integrate Hφ = Eφ across the origin (ε>0)

Eigenalue eqn : $-\frac{\hbar}{2m}\frac{d^2}{dx^2}\varphi(x)+\frac{\hbar^2\beta}{2m}\delta(x)\varphi(x)=E\varphi(x)$

Rearrange : $\frac{d^2}{dx^2}\varphi(x)-\beta\delta(x)\varphi(x)=-k^2\varphi(x)$

Integrate: $\int_{-\varepsilon}^{\varepsilon}\frac{d^2}{dx^2}\varphi(x)dx-\beta\int_{-\varepsilon}^{\varepsilon}\delta(x)\varphi(x)dx=-k^2\int_{-\varepsilon}^{\varepsilon}\varphi(x)dx$

$$\left.\frac{d\varphi(x)}{dx}\right|_{x=-\varepsilon}^{x=+\varepsilon}-\beta\varphi(0)=-k^2\left.\varphi(x)\right|_{x=-\varepsilon}^{x=+\varepsilon}$$

Take the limit that ε->0. The wave function is continuous at the origin, but the derivative is not necessarily. 0+(-) means "very close to zero, but on the positive (negative) side".

$$\underbrace{\left.\frac{d\varphi(x)}{dx}\right|_{x=0+}}_{\varepsilon\to 0}-\underbrace{\left.\frac{d\varphi(x)}{dx}\right|_{x=0-}}_{\varepsilon\to 0}-\beta\varphi(0)=-k^2\underbrace{\left(\left.\varphi(x)\right|_{x=0+}-\left.\varphi(x)\right|_{x=0-}\right)}_{\to 0\ as\ \varepsilon\to 0}$$

The required result follows:

$$\Delta\left.\frac{d\varphi(x)}{dx}\right|_{x=0}=\beta\varphi(0)$$

(a) (ii) For bound states, $E < 0$.

Away from the origin, $V(x) = 0$.

Eigenvalue equation: $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi(x)=E\varphi(x)$

$$\frac{d^2}{dx^2}\varphi(x)=-\frac{2mE}{\hbar^2}\varphi(x)\equiv-k^2\varphi(x)$$

Solutions: $\varphi(x)=Ae^{ikx}+Be^{-ikx}\quad ;k=\sqrt{\frac{2mE}{\hbar^2}}$

For $E<0,\ k=\sqrt{\frac{2mE}{\hbar^2}}$ is imaginary. Define real $\kappa\equiv ik=\sqrt{\frac{2m|E|}{\hbar^2}}$ and rewrite:

$\varphi(x)=Ae^{\kappa x}+Be^{-\kappa x}$

a superposition of growing and decaying exponentials. The growing exponential becomes infinite and therefore not normalizable as x goes to ±infinity, which is unphysical for a bound state. So we have wave function: $\varphi(x)=\begin{cases} Ae^{\kappa x} & x<0 \\ Be^{-\kappa x} & x>0 \end{cases}$ where $\kappa=\sqrt{\frac{2m|E|}{\hbar^2}}$

derivative: $\frac{d\varphi(x)}{dx}=\begin{cases} \kappa Ae^{\kappa x} & x<0 \\ -\kappa Be^{-\kappa x} & x>0 \end{cases}$

At the origin, continuity of the wave function at $x = 0$ requires
$A=B$

Normalize the wave function:

$$\int_{-\infty}^{\infty}|\varphi(x)|^2\,dx=1\Rightarrow\int_{-\infty}^{0}A^2e^{2\kappa x}\,dx+\int_{0}^{\infty}A^2e^{-2\kappa x}\,dx=1$$

$$A^2\left[\left.\frac{e^{2\kappa x}}{2\kappa}\right|_{-\infty}^{0}+\left.\frac{e^{-2\kappa x}}{-2\kappa}\right|_{0}^{\infty}\right]=1\Rightarrow A^2\left[(1-0)-(0-1)\right]=2\kappa$$

$$A^2=\sqrt{2\kappa}$$

wave function: $\varphi(x)=\begin{cases} \sqrt{2\kappa}e^{\kappa x} & x<0 \\ \sqrt{2\kappa}e^{-\kappa x} & x>0 \end{cases}$ where $\kappa=\sqrt{\frac{2m|E|}{\hbar^2}}$

(Still need E)

Using the result from part (a)(i) to evaluate the discontinuity of the derivative:

$$\left.\frac{d\varphi(x)}{dx}\right|_{x=-\varepsilon}^{x=+\varepsilon}-\beta\varphi(0)=-\kappa^2\,\varphi(x)\Big|_{x=-\varepsilon}^{x=+\varepsilon}$$

$$\Rightarrow\underbrace{\left.\frac{d\varphi(x)}{dx}\right|_{x=0+}}_{-\kappa B\ as\ \varepsilon\to 0}-\underbrace{\left.\frac{d\varphi(x)}{dx}\right|_{x=0-}}_{\kappa A\ as\ \varepsilon\to 0}-\beta\underbrace{\varphi(0)}_{A(orB)}=\kappa^2\underbrace{\left(\varphi(x)\big|_{x=\varepsilon}-\varphi(x)\big|_{x=-\varepsilon}\right)}_{\to 0\ as\ \varepsilon\to 0}$$

$\Rightarrow -\kappa B-\kappa A-\beta A=0$ and recall $A=B$ $\quad\Rightarrow\kappa=-\frac{\beta}{2}$

$$\kappa^2\equiv\frac{2m|E|}{\hbar^2}=\frac{\beta^2}{4}\Rightarrow|E|=\frac{\hbar^2\beta^2}{8m}$$

Energy:

$$E=-\frac{\hbar^2\beta^2}{8m}$$

(a)(iii)

There is only one bound state solution. Never zero. In a finite square well potential (of which the delta function is the infinitely narrow, infinitely deep limit) there are a finite number of solutions. In the infinite square well, there are an infinite number.

(b)((i)

As before, <u>away from the origin</u>, $V(x) = 0$.

Eigenvalue equation: $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi(x) = E\varphi(x)$

$$\frac{d^2}{dx^2}\varphi(x) = -\frac{2mE}{\hbar^2}\varphi(x) \equiv -k^2\varphi(x)$$

The situation described is represented by a piecewise wave function

$$\varphi(x) = \begin{cases} e^{ikx} + re^{-ikx} & x < 0 \\ te^{ikx} & x > 0 \end{cases}.$$

For $E > 0$, k is real, so this is a superposition of a positive-traveling (e^{ikx}) incident wave (amplitude 1) and a negative-traveling (e^{-ikx}) reflected wave (amplitude $|r|<1$) on the left of the barrier and a positive-traveling transmitted wave (amplitude $|t|<1$) on the right.

Continuity of the wave function at $x = 0$: $\quad 1 + r = t$

The derivative is

$$\frac{d\varphi(x)}{dx} = \begin{cases} ike^{ikx} + -ikre^{-ikx} & x < 0 \\ ikte^{ikx} & x > 0 \end{cases}$$

Use the same technique as in (a)(i) to evaluate the discontinuity in the derivative.

Eigenalue eqn : $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi(x) + \frac{\hbar^2\beta}{2m}\delta(x)\varphi(x) = E\varphi(x)$

Rearrange : $\frac{d^2}{dx^2}\varphi(x) - \beta\delta(x)\varphi(x) = -k^2\varphi(x)$

Integrate: $\int_{-\varepsilon}^{\varepsilon}\frac{d^2}{dx^2}\varphi(x)dx - \beta\int_{-\varepsilon}^{\varepsilon}\delta(x)\varphi(x)dx = -k^2\int_{-\varepsilon}^{\varepsilon}\varphi(x)dx$

Take the limit ε -> 0. The wave function is continuous at x=0, but the derivative is not.

$$\int_{-\varepsilon}^{\varepsilon}\frac{d^2\varphi(x)}{dx^2}dx - \beta\int_{-\varepsilon}^{\varepsilon}\delta(x)\varphi(x)dx = -k^2\int_{-\varepsilon}^{\varepsilon}\varphi(x)dx$$

$$\left.\frac{d\varphi(x)}{dx}\right|_{x=-\varepsilon}^{x=+\varepsilon} - \beta\varphi(0) = -k^2\left.\varphi(x)\right|_{x=-\varepsilon}^{x=+\varepsilon}$$

$$\underbrace{\left.\frac{d\varphi(x)}{dx}\right|_{x=\varepsilon}}_{ikt\ as\ \varepsilon\to 0} - \underbrace{\left.\frac{d\varphi(x)}{dx}\right|_{x=-\varepsilon}}_{ik(1-r)\ as\ \varepsilon\to 0} - \beta\underbrace{\varphi(0)}_{(1+r)} = -k^2\underbrace{\left(\left.\varphi(x)\right|_{x=\varepsilon} - \left.\varphi(x)\right|_{x=-\varepsilon}\right)}_{\to 0\ as\ \varepsilon\to 0}$$

$$ikt - ik(1-r) = \beta(1+r)$$

Solve simultaneously:

$$1 + r = t$$
$$ikt - ik(1-r) = \beta(1+r)$$

$$ik(1+r)-ik(1-r)=\beta(1+r)$$
$$\Rightarrow(2ik-\beta)r=\beta$$
$$\Rightarrow r=\frac{\beta}{2ik-\beta}=\frac{\beta}{\sqrt{4k^2+\beta^2}}e^{i\arctan\left(\frac{2k}{\beta}\right)}$$

and

$$t=1+r=\frac{2ik}{2ik-\beta}=\frac{4k^2}{\sqrt{4k^2+\beta^2}}e^{i\arctan\left(-\frac{2k}{\beta}\right)}$$

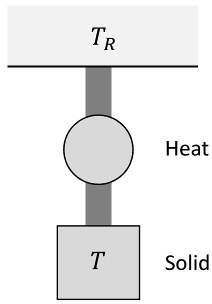
(b)(ii)

r and *t* (the amplitude reflection and transmission coefficients) are complex, so there is a phase change upon reflection that is in general different from 0 or π. They do *not* sum to 1. It is R and *T*, the intensity reflection and transmission coefficients, that should sum to 1 to preserve particle number. Check that this is indeed so. With

$$R\equiv|r|^2;T\equiv|t|^2$$
$$R+T=\left|\frac{\beta}{2ik-\beta}\right|^2+\left|\frac{2ik}{2ik-\beta}\right|^2=\frac{\beta^2}{4k^2+\beta^2}+\frac{4k^2}{4k^2+\beta^2}=1$$

Note that the above definition of *T* is special to this case of a symmetric potential and is defined more generally if the potential is asymmetric. The result $R+T=1$ is general, however.

Consider a reversible refrigerator consisting of a solid at temperature T, a heat engine, and a high temperature reservoir at room temperature T_R.



Goal is to cool the solid from T_R to some lower temperature. The efficiency of the heat engine at the solid temperature T is

$$\eta = \frac{-dW}{-dQ'} = \frac{T_R - T}{T_R},$$

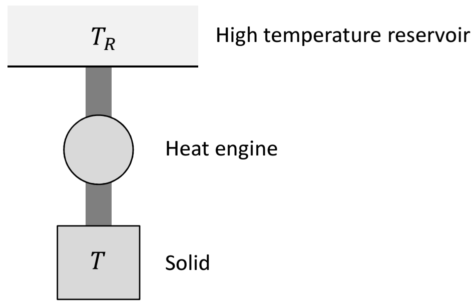
where W is the work done on the refrigerator and Q' is the heat flowing from the engine into the room temperature reservoir. The heat capacity of the solid is $C = AT^3$, where A is a constant.

(a) Find the work W required to cool this solid from room temperature T_R to $T \cong 0$.

(b) What are the changes in entropy and internal energy of the solid for the cooling?

(c) Show that the changes in internal energy and entropy of the high temperature reservoir satisfy the relation, $\Delta U_R = T_R \Delta S_R$.

Consider a reversible refrigerator consisting of a solid at temperature T, a heat engine, and a high temperature reservoir at room temperature T_R.



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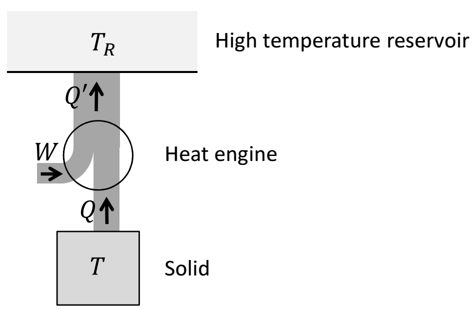
(a) Find the work W required to cool this solid from room temperature T_R to $T \cong 0$.

Solution:
An amount of work W is done on the refrigerator, while an amount of heat Q flows from the solid into the heat engine and an amount of Q' flows from the engine into the room temperature reservoir. Since the energy must be conserved, $Q' = Q + W$.



The efficiency of the heat engine at the solid temperature T is

$$\eta = \frac{-dW}{-dQ'} = \frac{T_R - T}{T_R}. \tag{48}$$

This gives

$$\Delta Q = \Delta Q' - \Delta W = \left(\frac{1}{\eta} - 1\right)\Delta W = \frac{T}{T_R - T}\Delta W. \tag{49}$$

The amount of heat required for an infinitesimal temperature change from T to $T - \Delta T$ in the solid is $\Delta Q = C\Delta T = AT^3 \Delta T$ and hence the amount of work for this cooling is

$$\Delta W = \frac{T_R - T}{T}\Delta Q = (T_R - T)AT^2 \Delta T \tag{50}$$

$\Delta T (> 0)$ is defined in the direction of cooling, and hence

$$\frac{dW}{dT} = -A(T_R - T)T^2. \tag{51}$$

Therefore the amount of work needed to cool the solid to $T \cong 0$ is

$$W = -A \int_{T_R}^{0} (T_R - T)T^2 dT = \frac{1}{12} AT_R^4. \tag{52}$$

(b) What are the changes in entropy and internal energy of the solid for the cooling?

Solution:
Using

$$\left(\frac{\partial S}{\partial T}\right)_V = \frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_V = \frac{C}{T} = AT^2, \tag{53}$$

we find the change in entropy of the solid

$$\Delta S_s = \int_{T_R}^{0} AT^2 dT = -\frac{1}{3} AT_R^3. \tag{54}$$

The change in internal energy of the solid is

$$\Delta U_s = \int_{T_R}^{0} C dT = \int_{T_R}^{0} AT^3 dT = -\frac{1}{4} AT_R^4. \tag{55}$$

(c) Show that the changes in internal energy and entropy of the high temperature reservoir satisfy the relation, $\Delta U_R = T_R \Delta S_R$.

Solution:
The change in internal energy of the high temperature reservoir is

$$\Delta U_R = -\Delta U_S + W = \frac{1}{3} AT_R^4 \tag{56}$$

and the entropy change is

$$\Delta S_R = -\Delta S_S = \frac{1}{3} AT_R^3 \tag{57}$$

Therefore,

$$\Delta U_R = T_R \Delta S_R. \tag{58}$$

Consider two non-relativistic, identical, non-interacting spin-1/2 particles of mass m that are confined in a 1-dimensional infinite square well of width L.

(a) Construct the two-particle singlet eigenstates and give the energy eigenvalues. What is the lowest possible energy? Discuss any degeneracy.

(b) Repeat (a) for the two-particle triplet eigenstates.

Now assume that an interaction between the particles is turned on of the form
$V_{int}(x) = V_o b\delta(x_1 - x_2)$ where b is a characteristic length and V_o is the interaction strength.

(c) Calculate the first-order correction to the energy of the states found in (a) and (b). You do not need to evaluate integrals, but you must identify whether they are zero or non-zero.

(d) Provide a simple physical interpretation of the result in (c).

Helpful information:

Please assume you know the solution to the one-particle problem:

$\phi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) |\pm\rangle$, where $|+\rangle$ means spin-up and $|-\rangle$ means spin down.

$E_n = n^2 \frac{\hbar^2\pi^2}{2mL^2}$, n=1, 2, 3, . . .

Graduate quantum mechanics

Consider **two** non-relativistic, **identical**, **non-interacting** spin-1/2 particles of mass m that are confined in a 1-dimensional infinite square well of width L.

(a) Construct the two-particle singlet eigenstates and give the energy eigenvalues. What is the lowest possible energy? Discuss any degeneracy.
(b) Repeat (a) for the two-particle triplet eigenstates.

Assume that an interaction between the particles is turned on of the form $V_{\text{int}}(x) = V_0 b\delta(x_1 - x_2)$ where b is a characteristic length and V_0 is the interaction strength.

(c) Calculate the first-order correction to the energy of the states found in (a) and (b). You don't need to evaluate integrals, but you must identify whether they are zero or non-zero.

(d) Provide a simple physical interpretation of the result in (c).

Hint:

Assume you know the solution to the one-particle problem:

$\varphi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) |\pm\rangle$ where $|+\rangle$ means spin-up and $|-\rangle$ means spin down.

$E_n = n^2 \frac{\hbar^2 \pi^2}{2mL^2}$ $n = 1,2,3 \ldots$

Graduate quantum mechanics -SOLUTION

Let the position and spin coordinates of particle 1 be x_1 and σ_1 and those of particle 2, x_2 and σ_2.

The Hamiltonian is spin independent, so the spin and space parts of the two-particle wave function separate. $\psi(x_1,x_2;\sigma_1,\sigma_2)=\psi_{space}(x_1,x_2)\psi_{spin}(\sigma_1,\sigma_2)$.

A two-particle eigenstate for identical fermions must be antisymmetic under interchange of the coordinates of particle 1 with those of particle 2. That is

$$\psi(x_1,x_2;\sigma_1,\sigma_2)=-\psi(x_2,x_1;\sigma_2,\sigma_1)$$

If the total wave function is antisymmetric under particle interchange and it separates into two functions, then one of the functions must be symmetric and the other antisymmetric.

(a) The **singlet** state has the spin part antisymmetric (AS) under particle interchange:

$\psi_{spin}^{AS}(\sigma_1,\sigma_2)=\frac{1}{\sqrt{2}}\left(|+-\rangle-|-+\rangle\right)$ where the notation is $|particle1, particle2\rangle$.

The space part of the function must be symmetric (S) under particle interchange.

$$\psi_{space}^{S}(x_1,x_2)=\frac{1}{\sqrt{2}}\left(\varphi_n(x_1)\varphi_m(x_2)+\varphi_n(x_2)\varphi_m(x_1)\right)$$

The lowest energy state is n=1, m=1:

$$\psi_{space}^{S}(x_1,x_2)=\varphi_{n=1}(x_1)\varphi_{m=1}(x_2)$$

Both particles are in the (spatial) single particle ground state. The energy is:

$$\boxed{E=E_n+E_{m=n}=E_1+E_1=\left(1^2+1^2\right)\frac{\hbar^2\pi^2}{2mL^2}=2\frac{\hbar^2\pi^2}{2mL^2}}$$

The singlet state is **non-degenerate**. There is only one possible AS spin wave function, and one possible S space wave function.

(b) For the **triplet** state, there are 3 spin states and they are S under particle interchange:

$$\psi_{spin}^{S}(\sigma_1,\sigma_2)=\frac{1}{\sqrt{2}}\left(|+-\rangle+|-+\rangle\right)$$

$$|++\rangle$$

$$|--\rangle$$

The space part of the function must be AS under particle interchange

$$\psi_{space}^{AS}(x_1,x_2)=\frac{1}{\sqrt{2}}\left(\varphi_n(x_1)\varphi_m(x_2)-\varphi_n(x_2)\varphi_m(x_1)\right)$$

If m=n, then the space part is zero, which is unphysical, so we must have n≠m, and the lowest possibility is n=1, m=2.

The energy is $\boxed{E=E_n+E_m=E_1+E_2=\left(1^2+2^2\right)\frac{\hbar^2\pi^2}{2mL^2}=5\frac{\hbar^2\pi^2}{2mL^2}}$

The state is 3-fold degenerate (3 spin states). The n=2, m=1 possibility is not a new space state.

Assume that an interaction between the particles is turned on of the form $V_{\text{int}}(x)=V_0 b\delta(x_1-x_2)$ where b is a characteristic length and V_0 is an interaction strength.

(c) First order perturbation theory: energy correction to a state n is $E_n^{(1)}=\langle\psi_n^{(0)}|H'|\psi_n^{(0)}\rangle$

$|\psi_n^{(0)}\rangle$ is the unperturbed state and H' is the perturbation Hamiltonian.
H' is independent of spin, so the spin contribution to the integral integrates to 1 because it the spin part of the wave function is properly normalized.

Space part for the **singlet** state is $\psi_{space}^{S}(x_1,x_2)=\varphi_{n=1}(x_1)\varphi_{m=1}(x_2)$

$$\begin{aligned}E_n^{(1)}&=\int_{x_1=0}^{L}dx_1\int_{x_2=0}^{L}dx_2\left[\psi_{space}^{S}(x_1,x_2)^* bV_0\delta(x_1-x_2)\psi_{space}^{S}(x_1,x_2)\right]\\&=bV_0\int_{x_1=0}^{L}dx_1\int_{x_2=0}^{L}dx_2\varphi_1^*(x_1)\varphi_1^*(x_2)\delta(x_1-x_2)\varphi_1(x_1)\varphi_1(x_2)\\&=bV_0\int_{x_1=0}^{L}dx_1\varphi_1^*(x_1)\varphi_1^*(x_1)\varphi_1(x_1)\varphi_1(x_1)\\&=bV_0\int_{x_1=0}^{L}dx_1\left|\varphi_1(x_1)\right|^2\left|\varphi_1(x_1)\right|^2\end{aligned}$$

To evaluate the integral, we need the wave functions $\varphi_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$, but it is clear from the positive definite integrand the result is non-zero.

For the **triplet** states, the spin parts again integrate to 1, and the spatial part is:

$$E_n^{(1)} = \int_{x_1=0}^{L} dx_1 \int_{x_2=0}^{L} dx_2 \left[\psi_{space}^{AS}(x_1,x_2)^* \, bV_0 \delta(x_1 - x_2) \psi_{space}^{AS}(x_1,x_2) \right]$$

$$= bV_0 \int_{x_1=0}^{L} dx_1 \int_{x_2=0}^{L} dx_2 \begin{bmatrix} \{\varphi_n(x_1)\varphi_m(x_2) - \varphi_n(x_1)\varphi_m(x_2)\} \\ \delta(x_1 - x_2)\{\varphi_n(x_1)\varphi_m(x_2) - \varphi_n(x_1)\varphi_m(x_2)\} \end{bmatrix}$$

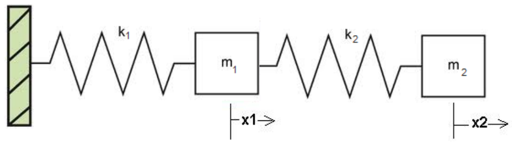
$$= bV_0 \int_{x_1=0}^{L} dx_1 \left[\{\varphi_n(x_2)\varphi_m(x_2) - \varphi_n(x_2)\varphi_m(x_2)\}\{\varphi_n(x_2)\varphi_m(x_2) - \varphi_n(x_2)\varphi_m(x_2)\} \right].$$

$$= bV_0 \int_{x_1=0}^{L} dx_1 \left[\{0\}\{0\} \right]$$

$$= 0$$

(d) Provide a simple physical interpretation of the result in (c).

In the spin singlet state, the particles are both in the spatial ground state, so there is a finite probability to find them at the same location where they interact (the interaction potential is active only when the particles are at the same location). Thus there is a finite interaction energy. However, in the spin triplet state, the particle probability density is zero at $x_1 = x_2$, so the particles <u>never feel the interaction potential</u> and there is no additional energy.



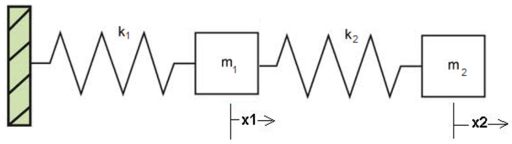
Two masses m_1 and m_2 are connected to each other by a spring of constant k_2, and mass m_1 is connected to a fixed support by another spring with spring constant k_1. Assume motion is allowed in a horizontal straight line from or toward the support, with no friction as shown in the figure. Let x_1 and x_2 represent the position of m_1 and m_2, respectively, relative to the equilibrium of the positions of the masses (i.e. when springs are not stretched).

(a) Derive an expression for the frequencies of characteristic small-amplitude oscillation in terms of the above parameters.

(b) Show that the frequencies are always real, so that the system always has stable oscillations.

(c) Let $k_1 = k_2 = k$ and $m_1 = m_2 = m$. What are both the characteristic frequencies and normal modes in this case?

(d) Using the results from part c, find the resulting trajectories for both masses (i.e. $x_1(t)$ and $x_2(t)$) . Assume that at some initial time, the left mass is at its equilibrium position, but the right mass is pulled out a distance a_o from its equilibrium position.



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(a) Derive an expression for the frequencies of characteristic small-amplitude oscillation in terms of the above parameters.

Solution:
This problem may solved using a Hamiltonian energy approach or using Newtonian mechanics (i.e. $F = m\ddot{x}$). Using Newtonian mechanics we can readily write by inspection that,

$$m_1\ddot{x}_1 = -(k_1 + k_2)x_1 + k_2x_2 \tag{59}$$

$$m_2\ddot{x}_2 = k_2x_1 - k_2x_2 \tag{60}$$

$$\tag{61}$$

Assume the solution maybe obtained with an exponential of form $x_1(t) = A_1 \exp i\omega t$ and $x_2(t) = A_2 \exp i\omega t$, and we may solve the coupled ODEs by method of determinants where the matrix equation is $\ddot{x} = Cx$ or more explicitly

$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

We can now solve for the eigenvalues (or characteristic frequencies) of matrix C by,

$$\det\left(C - \omega^2 I\right) = \left(\frac{k_1 + k_2}{m_1} - \omega^2\right)\left(\frac{k_2}{m_2} - \omega^2\right) - \frac{k_2^2}{m_1m_2} \tag{62}$$

$$\Leftrightarrow \omega^2 = \frac{1}{2}\left(\frac{k_2m_1 + (k_1 + k_2)m_2}{m_1m_2} \pm \sqrt{\left(\frac{k_2m_1 + (k_1 + k_2)m_2}{m_1m_2}\right)^2 - \frac{4k_1k_2}{m_1m_2}}\right) \tag{63}$$

(b) Show that the frequencies are always real, so that the system always has stable oscillations.

Solution:
By inspection of the solution ω cannot be a negative real number, so we need only show that the discriminant is positive (i.e. the expression in the square root is positive). Expanding the determinant we need to prove the following inequality,

$$\left(k_2^2m_1^2 + k_1^2m_1^2 + k_2^2m_2^2 + 2k_2m_1k_1m_2 + 2k_1k_2m_2^2 + 2k_2^2m_1m_2\right) \geq 4k_1k_2m_1m_2 \tag{64}$$

$$\Leftrightarrow (k_2m_1 - k_1m_2)^2 + k_2^2m_2^2 + 2k_1k_2m_2^2 + 2k_2^2m_1m_2 \geq 0 \tag{65}$$

The last inequality is true by inspection, therefore the frequencies in this system are real and positive. Hence the oscillations must be stable.

(c) Let $k_1 = k_2 = k$ and $m_1 = m_2 = m$. What are both the characteristic frequencies and normal modes in this case?

Solution:

If $k_1 = k_2 = k$ and $m_1 = m_2 = m$, then the characteristic frequencies are $\omega^2 = \frac{1}{2}\left(3 \pm \sqrt{5}\right)\frac{k}{m}$. Solving for the normal modes (eigenvectors), we obtain

$\det\left(C - \omega^2 I\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

Where are matrix C is now simply $\frac{1}{m}\begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}$

Solving for the eigenvectors, we find that the in-phase normal mode is $x_2 = \frac{1}{2}\left(\sqrt{5}+1\right)x_1$ and the out of phase normal mode is $x_2 = -\frac{1}{2}\left(\sqrt{5}-1\right)x_1$. The normal modes Q_1 and Q_2 are $Q_1 = x_1 - \frac{1}{2}(\sqrt{5}-1)x_2$ and $Q_2 = x_1 + \frac{1}{2}(\sqrt{5}+1)x_2$

(d) Using the results from part *c*, find the resulting trajectories for both masses (i.e. $x_1(t)$ and $x_2(t)$) . Assume that at some initial time, the left mass is at its equilibrium position, but the right mass is pulled out a distance a_o from its equilibrium position.

Solution:
At $t = 0$, $x_1 = 0$ and $x_2 = a_o$. The general solution will be a superposition of the time-dependent normal mode motion, $Q_1(t)$, $Q_2(t)$. At $t = 0$, $Q_1 = x_1 - \frac{1}{2}(\sqrt{5}-1)x_2 = -\frac{1}{2}(\sqrt{5}-1)a_o$ and $Q_2 = x_1 + \frac{1}{2}(\sqrt{5}+1)x_2 = \frac{1}{2}(\sqrt{5}+1)a_o$. Solving this system of equations for x_1 and x_2 in general we get,

$$x_1(t) = \frac{1}{2\sqrt{5}}(\sqrt{5}-1)Q_2(t) + \frac{1}{2\sqrt{5}}(\sqrt{5}+1)Q_1(t) \tag{66}$$

$$x_2(t) = \frac{1}{\sqrt{5}}(Q_2(t) - Q_1(t)) \tag{67}$$

Where $Q_1(t)$ and $Q_2(t)$ are just the time dependent normal modes (solved for above using our initial conditions) to obtain,

$$Q_1(t) = -\frac{1}{2}(\sqrt{5}-1)a_o \cos\omega_1 t \tag{68}$$

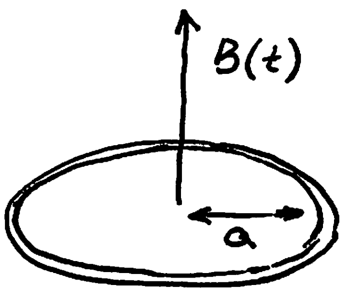
$$Q_2(t) = \frac{1}{2}(\sqrt{5}+1)a_o \cos\omega_2 t \tag{69}$$

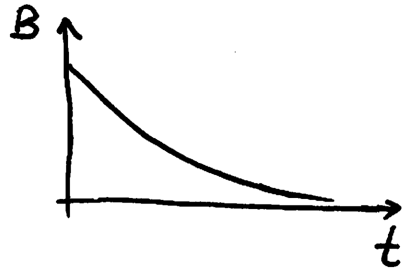
$_1$ and ω_2 are the respective characteristic frequencies of the system.

A non-conducting ring of radius a and mass M lies on a frictionless surface. The ring carries a total charge Q. The charge is immobile and uniformly distributed. Before $t = 0$, the ring is stationary in a magnetic field $B = B_o \hat{z}$, where the z-axis is perpendicular to the plane of the ring. Starting at $t = 0$, the magnetic field is given by $B(t) = B_o \exp(-\alpha t)\hat{z}$.

(a) What is the torque on the ring when $t > 0$?

(b) Find the angular momentum of the ring as $t \to \infty$. Explain the dependence of your result on α.





Changing B-field causes a voltage around the ring

$$\int \vec{E} \cdot d\vec{l} = EMF = \pi a^2 \frac{dB}{dt}$$

$$E \cdot 2\pi a = \pi a^2 \frac{dB}{dt}$$

$$E = \frac{a}{2} \frac{dB}{dt}$$

The torque on the ring is

$$\tau = (\text{Force}) \times (\text{Lever arm})$$

$$= EQa$$

$$= \frac{a^2 Q}{2} \frac{dB}{dt}$$

The change in angular momentum

$$\Delta L = \int_0^\infty \tau \, dt = \int_0^\infty \frac{a^2 Q}{2} \frac{dB}{dt} dt$$

$$= \frac{a^2 Q}{2} \int_0^\infty dB$$

$$= \frac{a^2 Q B_0}{2}$$

The final L does not depend on α.
If B decays slowly there is less force, but it lasts longer.