OSU PHYSICS DEPARTMENT
COMPREHENSIVE EXAMINATION #119

Monday, March 31 and Tuesday, April 1, 2014

Spring 2014 Comprehensive Examination

PART 1, Monday, March 31, 9:00am

General Instructions

This Spring 2014 Comprehensive Examination consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, March 31, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:00 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, April 1, at 9:00 am and 1:00 pm, respectively. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please return all bluebooks and formula sheets at the end of the exam. Use the last pages of your bluebooks for "scratch" work, separated by at least one empty page from your solutions. "Scratch" work will not be graded.

A mass undergoes vertical motion under the influence of the gravitational force $F_g = -mg$, and a retarding/drag force proportional to, and directed opposite to, its velocity v, $F_r = -m\gamma v$. Here $g > 0$ and the positive direction is upward from the earth so gravity acts downward, and the direction of v determines the direction of the retarding/drag force.
Note: It is not necessary to have (b) completely correct to be able to make sensible attempts at the other parts of the problem. Conversely, even if you have (b) completely correct, you must still give good, physical interpretations to answer the other parts.

(a) If a mass is projected upwards from a certain height, is the time for upward travel larger, equal to, or smaller than that for downward travel back to the same height? You may be qualitative, but your reasoning must be rigorous.

(b) Find the expression for the position $z(t)$ as a function of time if the initial velocity is v_0 and the initial position is z_0. Show that the standard form

$$z(t) = z_0 + v_0 t - \frac{1}{2} g t^2$$

is recovered if the drag force goes to zero.

(c) What does terminal speed mean? Give an expression for the terminal speed of this mass in terms of the parameters of the problem, explaining how it arises from the equations describing this physical situation.

(d) Describe the motion of the mass if it is

(i) projected downward with an initial speed that is small compared to the terminal speed,

(ii) projected downward with an initial speed that is large compared to the terminal speed,

(iii) projected upward with an initial speed that is small compared to the terminal speed,

(iv) projected upward with an initial speed that is large compared to the terminal speed.

"Describe" means use words to point out the interesting phases of the motion; such descriptions are usually best accompanied by illustrative graphs and limiting forms for equations. It is not necessary to solve for the times for the upward and downward motion, for example.

Solutions to problem 1

A mass undergoes vertical motion under the influence of the gravitational force $F_g = -mg$, and a retarding/drag force proportional to, and directed opposite to, its velocity v, $F_r = -m\gamma v$. Here $g > 0$ and the positive direction is upward from the earth so gravity acts downward, and the direction of v determines the direction of the retarding/drag force.

It is not necessary to have (b) completely correct to be able to make sensible attempts at the other parts of the problem. Conversely, even if you have (b) completely correct, you must still give good, physical interpretations to answer the other parts.

(a) If a mass is projected upwards from a certain height, is the time for upward travel larger, equal to, or smaller than that for downward travel back to the same height? You may be qualitative, but your reasoning must be rigorous.

On the upward journey, the gravitational force and the drag force both act downwards (so a large downward force). A large force means a large momentum change, so the velocity decreases a large amount in a given time increment. On the downward journey at the same velocity, the gravitational acts downwards and the drag force acts upwards (so a smaller net force downward). The downward speed increases less in the same time increment and therefore the mass does not travel as far as it did on the upward journey when it was at the same speed. The upward journey includes at least all the speeds on the downward journey. We therefore expect the time for the downward journey back to the same height to be longer.
(If you prefer, you can simply solve part (b) and demonstrate mathematically.)

(b) Find the expression for the position z(t) as a function of time if the initial velocity is v_0 and the initial position is z_0. Show that the standard form $z = z_0 + v_0 t - \frac{gt^2}{2}$ is recovered if the retarding force goes to zero.

Let the *z* direction be vertical, and define the positive direction to be positive *z*, away from the earth. Let the particle mass be *m*, the gravitational constant $g > 0$, and there is a friction coefficient $\gamma > 0$. The force is then

$$F = -mg - \gamma mv$$

The gravitational force is always in the $-z$ (down) direction, regardless of the particle motion. The retarding force is in the $-z$ (down) direction if the velocity is in the $+z$ direction ($v > 0$, upward). The retarding force is in the $+z$ (up) direction if the velocity is in the $+z$ direction ($v < 0$, downward).

Newton's law: $\frac{dp}{dt} = F$

$$m\frac{dv}{dt} = -mg - \gamma mv \Rightarrow \frac{dv}{g+\gamma v} = -dt$$

Integrate (primes are dummy variables):

$$\int_{v(t=0)}^{v} \frac{dv'}{g+\gamma v'} = -\int_0^t dt'$$

$$\frac{1}{\gamma}\ln\left(\frac{g+\gamma v}{g+\gamma v_0}\right) = -t$$

$$g+\gamma v = \left(g+\gamma v_0\right)e^{-\gamma t}$$

$$v = -\frac{g}{\gamma}+\left(\frac{g}{\gamma}+v_o\right)e^{-\gamma t}$$

Now integrate again:

$$v = \frac{dz}{dt} = -\frac{g}{\gamma}+\left(\frac{g}{\gamma}+v_o\right)e^{-\gamma t}$$

$$\int_{z(t=0)}^{z} dz' = \int_0^t\left[-\frac{g}{\gamma}+\left(\frac{g}{\gamma}+v_o\right)e^{-\gamma t'}\right]dt'$$

$$z-z_0 = \int_0^t -\frac{g}{\gamma}dt' + \int_0^t\left(\frac{g}{\gamma}+v_o\right)e^{-\gamma t'}dt'$$

$$z-z_0 = -\frac{g}{\gamma}t - \frac{1}{\gamma}\left(\frac{g}{\gamma}+v_0\right)\left(e^{-\gamma t}-1\right)$$

$$z = z_0 - \frac{g}{\gamma}t - \left(\frac{g+v_0\gamma}{\gamma^2}\right)\left(e^{-\gamma t}-1\right)$$

The standard result is recovered if $\gamma = 0$ (expand the exponential).

$$z = z_0 - \frac{g}{\gamma}t - \left(\frac{g+v_0\gamma}{\gamma^2}\right)\left(e^{-\gamma t}-1\right)$$

$$z = z_0 - \frac{g}{\gamma}t - \left(\frac{g+v_0\gamma}{\gamma^2}\right)\left(1-\gamma t+\frac{(\gamma t)^2}{2}+...-1\right)$$

$$z = z_0 - \frac{g}{\gamma}t + \left(\frac{g+v_0\gamma}{\gamma^2}\right)(\gamma t) - \left(\frac{g+v_0\gamma}{\gamma^2}\right)\frac{(\gamma t)^2}{2}$$

$$z = z_0 \underbrace{-\frac{g}{\gamma}t+\frac{g}{\gamma}t}_{cancel} + v_0 t - \frac{gt^2}{2} - \underbrace{\frac{v_0\gamma t^2}{2}}_{k=0}$$

$$z = z_0 + v_0 t - \frac{gt^2}{2}$$

(c) What does terminal speed mean? Give an expression for terminal speed in terms of the parameters of the problem, explaining how it arises from the equations describing this physical situation.

When no net force acts on the body, *i.e.* $F = 0$, it no longer accelerates and its velocity is constant (terminal velocity). The magnitude of that terminal velocity is called the terminal speed.

• From the force equation describing this situation, $F = 0 = -mg - \gamma m v_T \Rightarrow v_T = -\frac{g}{\gamma}$, and the terminal speed $|v_T| = \frac{g}{\gamma}$. Notice that the force can be zero only if the velocity is negative (downward).

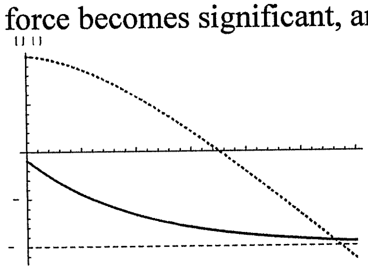
• The terminal velocity and speed can also be found from the velocity equation $v = -\frac{g}{\gamma} + \left(\frac{g}{\gamma} + v_o\right)e^{-\gamma t}$. In the infinite time limit $t >> 1/\gamma$, the exponential term goes to zero, the velocity is directed downward, and its magnitude (terminal speed) is g/γ.

(d) Describe the motion of the mass if it is
(i) projected downward with an initial speed that is small compared to the terminal speed,
(ii) projected downward with an initial speed that is large compared to the terminal speed,
(iii) projected upward with an initial speed that is small compared to the terminal speed,
(iv) projected upward with an initial speed that is large compared to the terminal speed.

"Describe" means use words to point out the interesting phases of the motion, and such descriptions are usually best accompanied by illustrative graphs and limiting forms for equations. It is not necessary to solve for the times for the upward and downward motion, for example.

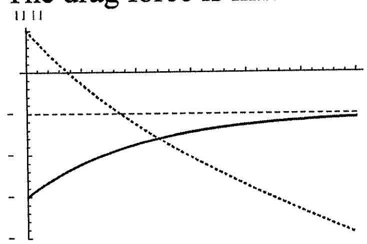
(i) Initial velocity is small (and down), force is down, so the mass accelerates and the magnitude of its (downward) velocity increases to terminal speed. With ε a small positive number, $v = -\frac{g}{\gamma} + \left(\frac{g}{\gamma} - |\varepsilon|\right)e^{-\gamma t} \approx \frac{g}{\gamma}\left(e^{-\gamma t} - 1\right) \xrightarrow[t\to\infty]{} -\frac{g}{\gamma}$

The position decreases (becomes more negative) at first quadratically until the drag force becomes significant, and finally linearly once terminal speed is reached.

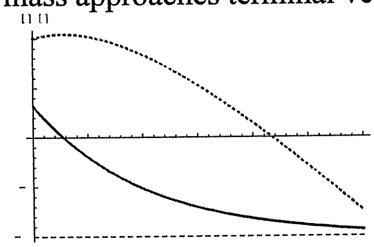


(ii) Initial velocity is large (and down), force is up. The mass decelerates and the magnitude of its downward velocity **de**creases to terminal speed. This perhaps counter-intuitive – or at least not the normal "sky-diver" example – the mass approaches terminal velocity from larger-than-terminal velocity! With Ω a large positive number $v = -\frac{g}{\gamma} + \left(\frac{g}{\gamma} - \frac{\Omega g}{\gamma}\right) e^{-\gamma t} \approx -\frac{\Omega g}{\gamma} e^{-\gamma t} - \frac{g}{\gamma} \xrightarrow[t\to\infty]{} -\frac{g}{\gamma}$

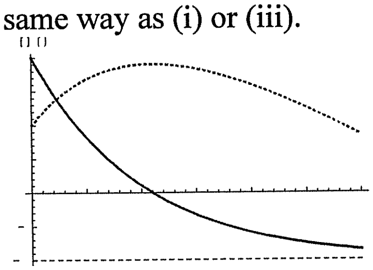
The drag force is immediately large, so $z(t)$ is never quadratic.



(iii)Very similar to (i). The only difference is that the mass first goes upwards, and stops when the velocity decreases to zero. Motion to this point is fairly similar to that with no drag because the velocity magnitude is small. Then the mass falls down, motion initially fairly similar to drag-free motion, but then drag becomes significant and the mass approaches terminal velocity in the same way as in (i).



(iv)The speed decreases, going through "terminal speed", but there is nothing special about this speed on the way up because the net force is <u>never</u> zero on the way up. The mass reaches the top of its trajectory, and then comes down. Near top of trajectory, the motion is similar to the drag-free case, because the velocity is small, but then terminal velocity is approached, and the position changes linearly, in the same way as (i) or (iii).



Consider a system of two identical particles with mass m in a 1D simple harmonic oscillator potential with spring constant k. The Hamiltonian of this system is given by

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega_0^2 x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_0^2 x_2^2 \tag{1}$$

where x_1 is the position of the first particle, and p_1 is the corresponding momentum. Similarly, x_2 and p_2 are the position and momentum of the second particle.

(a) Consider if these two particles are identical spin-zero bosons. What are the two lowest eigenvalues of the Hamiltonian? What are the states (including any possible degeneracies) that have these two lowest energies?

(b) Consider if these two particles are instead identical spin-$\frac{1}{2}$ fermions. In this case, what are the two lowest eigenvalues of the Hamiltonian? What are the states (including any possible degeneracies) that have these two lowest energies?

For the second half of this problem, we will consider only the case of identical spin-zero bosons. Now let us add a small attractive interaction between the particles with potential energy given by:

$$U = \gamma(x_1 - x_2)^4$$

where γ is a small constant. Note that for free particles this cannot be a "small" interaction, since it diverges at large interparticle distances, but for confined particles such as we are considering this can be a small interaction.

(c) What is the dimensionless quantity that must be small in order for us to consider this perturbation to be small?

(d) What is the first-order correction to the energy of the ground state of the two-boson system?

Consider a system of two identical particles with mass m in a 1D simple harmonic oscillator potential with spring constant k. The Hamiltonian of this system is given by

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega_0^2 x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_0^2 x_2^2 \tag{2}$$

where x_1 is the position of the first particle, and p_1 is the corresponding momentum. Similarly, x_2 and p_2 are the position and momentum of the second particle.

(a) Consider if these two particles are identical spin-zero bosons. What are the two lowest eigenvalues of the Hamiltonian? What are the states (including any possible degeneracies) that have these two lowest energies?

Solution:
Bosons will always have a wave function that is symmetric under exchange of particles. Since the spin is zero, there is no need to consider the spin quantum number, which helps keep our life easy and care-free. The two-particle eigenstates are products of single-particle eigenstates because the system is non-interacting (and thus separable). The ground state is simple: both particles are in the ground state:

$$\Psi_0 = |0\,0\rangle \tag{3}$$
$$= \phi_0(x_1)\phi_0(x_2) \tag{4}$$

This is a symmetric state, and it has energy

$$E_0 = \left(\frac{1}{2} + \frac{1}{2}\right)\hbar\omega_0 \tag{5}$$
$$= \hbar\omega_0 \tag{6}$$

The first excited state will have energy $E_1 = 2\hbar\omega_0$, and the eigenstate is given by

$$\Psi_1 = \frac{|1\,0\rangle + |0\,1\rangle}{\sqrt{2}} \tag{7}$$
$$= \frac{\phi_1(x_1)\phi_0(x_2) + \phi_0(x_1)\phi_1(x_2)}{\sqrt{2}} \tag{8}$$

Once again, this eigenstate is symmetric. And it is quite clear that we will not be able to find a lower-energy eigenstate.

(b) Consider if these two particles are instead identical spin-$\frac{1}{2}$ fermions. In this case, what are the two lowest eigenvalues of the Hamiltonian? What are the states (including any possible degeneracies) that have these two lowest energies?

Solution:
Using fermions means our wave function will always be antisymmetric, but now we have to consider the spin component of the wave function as well as the spatial component. Sadly,

only the product of the two need be antisymmetric, so we will have spin singlet eigenstates that are spatially identical to the boson solutions.

$$\Psi_0 = |0\,0\rangle \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \tag{9}$$

$$= \phi_0(x_1)\phi_0(x_2) \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \tag{10}$$

This is an antisymmetric state composed from a symmetric spatial state with an antisymmetric spin state, and just like the similar boson state, it has energy $E_0 = \hbar\omega_0$

The first excited state will have the same energy as the boson system did $E_1 = 2\hbar\omega_0$, but the eigenstates will again be more complicated. We will have a spin singlet state that looks much like the boson first excited state

$$\Psi_{1,\text{singlet}} = \frac{|1\,0\rangle + |0\,1\rangle}{\sqrt{2}} \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \tag{11}$$

$$= \frac{\phi_1(x_1)\phi_0(x_2) + \phi_0(x_1)\phi_1(x_2)}{\sqrt{2}} \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \tag{12}$$

But at this energy we have another option: the spatial state could be antisymmetric while the spin state is symmetric. This is a triplet state, so there are three such states, corresponding to the three ways to construct a symmetric state from two spin-$\frac{1}{2}$ spins:

$$\Psi_{1,\text{triplet},1} = \frac{|1\,0\rangle - |0\,1\rangle}{\sqrt{2}} |\uparrow\uparrow\rangle \tag{13}$$

$$\Psi_{1,\text{triplet},0} = \frac{|1\,0\rangle - |0\,1\rangle}{\sqrt{2}} \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \tag{14}$$

$$\Psi_{1,\text{triplet},-1} = \frac{|1\,0\rangle - |0\,1\rangle}{\sqrt{2}} |\downarrow\downarrow\rangle \tag{15}$$

For the second half of this problem, we will consider only the case of identical spin-zero bosons. Now let us add a small attractive interaction between the particles with potential energy given by:

$$U = \gamma(x_1 - x_2)^4$$

where γ is a small constant. Note that for free particles this cannot be a "small" interaction, since it diverges at large interparticle distances, but for confined particles such as we are considering this can be a small interaction.

(c) What is the dimensionless quantity that must be small in order for us to consider this perturbation to be small?

Solution:
For the perturbation to be small, it must be true that

$$\gamma(x_1 - x_2)^4 \ll \hbar\omega_0. \tag{16}$$

This relies on typical values of $x_1 - x_2$. A reasonable estimate for any distance if we are near the ground state can be found by setting the potential energy to $\hbar\omega_0$:

$$\frac{1}{2}m\omega_0^2\langle x^2\rangle \sim \hbar\omega_0 \tag{17}$$

$$\langle x^2\rangle \sim \frac{\hbar}{m\omega_0} \tag{18}$$

We can thus estimate that for our perturbation to be small it must be true that

$$\gamma\langle x^2\rangle^2 \ll \hbar\omega_0 \tag{19}$$

$$\gamma\left(\frac{\hbar}{m\omega_0}\right)^2 \ll \hbar\omega_0 \tag{20}$$

$$\frac{\gamma\hbar}{m^2\omega_0^3} \ll 1 \tag{21}$$

This is our dimensionless small quantity. Let's just verify the dimensions, as it's a pretty unfamiliar equation.

$$\frac{\frac{\text{energy}}{\text{distance}^4}\text{energy}\cdot\text{time}}{\frac{\text{mass}^2}{\text{time}^3}} = \frac{\text{energy}^2\cdot\text{time}^4}{\text{mass}^2\cdot\text{distance}^4} \tag{22}$$

$$= \text{dimensionless} \tag{23}$$

Yay.

(d) What is the first-order correction to the energy of the ground state of the two-boson system?

Solution:
Okay, this requires us to calculate a matrix element:

$$\Delta E = \langle\Psi_0|\gamma(x_1 - x_2)^4|\Psi_0\rangle \tag{24}$$

$$= \gamma\langle 0\,0|(x_1 - x_2)^4|0\,0\rangle \tag{25}$$

Let's begin by expanding the above polynomial. Because $[x_1, x_2] = 0$ we can reorder these operators and combine terms.

$$\Delta E = \gamma\langle 0\,0|x_1^4 - 4x_1^3x_2 + 6x_1^2x_2^2 - 4x_1x_2^2 + x_2^4|0\,0\rangle \tag{26}$$

$$= \gamma\left(\langle 0\,0|x_1^4|0\,0\rangle - 4\langle 0\,0|x_1^3x_2|0\,0\rangle + 6\langle 0\,0|x_1^2x_2^2|0\,0\rangle - 4\langle 0\,0|x_1x_2^2|0\,0\rangle + \langle 0\,0|x_2^4|0\,0\rangle\right) \tag{27}$$

$$= \gamma\left(2\langle 0|x_1^4|0\rangle + 6(\langle 0|x_1^2|0\rangle)^2\right) \tag{28}$$

where in the last step I separated the expectation values into one-particle expectation values, and recognized that $\langle 0|x_1^2|0\rangle = \langle 0|x_2^2|0\rangle$, and eliminated odd terms that integrate to zero.

Now I'll take a brief pause to find x_1 in terms of raising and lowering operators.

$$a^\dagger = \sqrt{\frac{m\omega_0}{2\hbar}}x + i\frac{p}{\sqrt{2m\omega_0\hbar}} \tag{29}$$

$$a = \sqrt{\frac{m\omega_0}{2\hbar}}x - i\frac{p}{\sqrt{2m\omega_0\hbar}} \tag{30}$$

$$a + a^\dagger = \sqrt{\frac{2m\omega_0}{\hbar}}x \tag{31}$$

$$x = \sqrt{\frac{\hbar}{2m\omega_0}}(a + a^\dagger) \tag{32}$$

Thus we can write:

$$\begin{aligned}
\langle 0|x_1^4|0\rangle &= \frac{\hbar^2}{4m^2\omega_0^2}\langle 0|(a+a^\dagger)^4|0\rangle && (33)\\
&= \frac{\hbar^2}{4m^2\omega_0^2}\langle 0|aaa^\dagger a^\dagger + aa^\dagger aa^\dagger|0\rangle && (34)\\
&= \frac{\hbar^2}{4m^2\omega_0^2}(2+1) && (35)\\
&= 3\frac{\hbar^2}{4m^2\omega_0^2} && (36)
\end{aligned}$$

where I have only kept the terms that are non-zero, which eliminates terms that begin with $a^\dagger$ or end with a, etc. Similarly, we find that:

$$\begin{aligned}
\langle 0|x_1^2|0\rangle &= \frac{\hbar}{2m\omega_0}\langle 0|(a+a^\dagger)^2|0\rangle && (37)\\
&= \frac{\hbar}{2m\omega_0}\langle 0|aa^\dagger|0\rangle && (38)\\
&= \frac{\hbar}{2m\omega_0}(1) && (39)\\
&= \frac{\hbar}{2m\omega_0} && (40)
\end{aligned}$$

Putting these into the above equation, we can easily find that the energy shift due to this perturbation is given by

$$\begin{aligned}
\Delta E &= \gamma\left(2\langle 0|x_1^4|0\rangle + 6(\langle 0|x_1^2|0\rangle)^2\right) && (41)\\
&= \frac{\gamma\hbar^2}{4m^2\omega_0^2}(2\cdot 3 + 6\cdot 1) && (42)\\
&= \frac{3\gamma\hbar^2}{m^2\omega_0^2} && (43)
\end{aligned}$$

Consider a solid consisting of N noninteracting nuclei. Each nucleus exhibits energy splitting.

(Case I) The nucleus is spin 1/2: a ground state $E_0 = -\epsilon$ for $m = +1/2$ and an excited state $E_1 = \epsilon$ for $m = -1/2$, when a uniform magnetic field is applied.

(a) Find the internal energy U of the solid as a function of the temperature T.

(b) What is the heat capacity C of the solid? Obtain its approximation in the high temperature limit, $kT \gg \epsilon$.

(Case II) The nucleus is spin 1: a ground state $E_0 = 0$ for $m = 0$ and degenerated excited states $E_1 = \epsilon$ for $m = \pm 1$ due to crystal fields.

(c) Find the internal energy U of the solid as a function of the temperature T.

(d) What is the heat capacity C of the solid? Obtain its approximation in the high temperature limit, $kT \gg \epsilon$.

Consider a solid consisting of N noninteracting nuclei. Each nucleus exhibits energy splitting.

(Case I) The nucleus is spin 1/2: a ground state $E_0 = -\epsilon$ for $m = +1/2$ and an excited state $E_1 = \epsilon$ for $m = -1/2$, when a uniform magnetic field is applied.

(a) Find the internal energy U of the solid as a function of the temperature T.

Solution:
The mean energy of a nucleus is given as

$$\tilde{E} = \frac{\sum_n E_n e^{-\beta E_n}}{\sum_n e^{-\beta E_n}} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = -\frac{\partial \ln Z}{\partial \beta},$$

where $\beta = 1/kT$ and the partition function, $Z = \sum_n e^{-\beta E_n}$. Since the partition function for the spin-1/2 nucleus is

$$Z = e^{-\beta\epsilon} + e^{\beta\epsilon} = 2\cosh(\beta\epsilon),$$

The mean energy is

$$\tilde{E} = -\frac{2\epsilon \sinh(\beta\epsilon)}{2\cosh(\beta\epsilon)} = -\epsilon \tanh\left(\frac{\epsilon}{kT}\right).$$

Then the internal energy of the solid is

$$U = N\tilde{E} = -N\epsilon \tanh\left(\frac{\epsilon}{kT}\right).$$

(b) What is the heat capacity C of the solid? Obtain its approximation in the high temperature limit, $kT \gg \epsilon$.

Solution:
The heat capacity is

$$C = \frac{\partial U}{\partial T} = \frac{N}{kT^2}\epsilon^2 \operatorname{sech}^2\left(\frac{\epsilon}{kT}\right).$$

In the high temperature limit, $\operatorname{sech}\left(\frac{\epsilon}{kT}\right) \cong 1$, thus

$$C \cong Nk\left(\frac{\epsilon}{kT}\right)^2.$$

(Case II) The nucleus is spin 1: a ground state $E_0 = 0$ for $m = 0$ and degenerated excited states $E_1 = \epsilon$ for $m = \pm 1$ due to crystal fields.

(c) Find the internal energy U of the solid as a function of the temperature T.

Solution:
The partition function for the spin-1 nucleus is

$$Z = 1 + 2e^{-\beta\epsilon}.$$

Thus the mean energy is

$$\tilde{E} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \frac{2\epsilon e^{-\beta\epsilon}}{1 + 2e^{-\beta\epsilon}}.$$

Then the internal energy of the solid is

$$U = N\tilde{E} = \frac{2N\epsilon}{2 + e^{\frac{\epsilon}{kT}}}$$

(d) What is the heat capacity C of the solid? Obtain its approximation in the high temperature limit, $kT \gg \epsilon$.

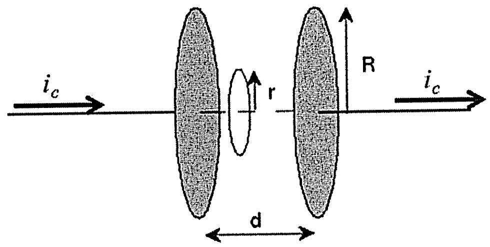
Solution:
The heat capacity is

$$C = \frac{\partial U}{\partial T} = \frac{2N\epsilon^2 e^{\frac{\epsilon}{kT}}}{kT^2(2 + e^{\frac{\epsilon}{kT}})^2}.$$

In the high temperature limit, $e^{\frac{\epsilon}{kT}} \cong 1$

$$C \cong \frac{2N\epsilon^2}{9kT^2} = \frac{2}{9}Nk\left(\frac{\epsilon}{kT}\right)^2.$$

A parallel-plate air-filled capacitor is being charged as shown in the figure below. The circular plates have a radius of R= 5 cm and are spaced 1 mm apart (d = 1 mm). At a particular instant the conductive current (i_c) in the wires in the charging wires is measured to be 0.5 A.



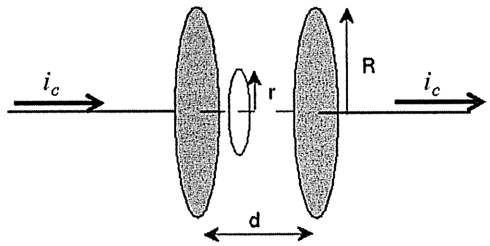
(a) What is the displacement current density in the air space between the plates? (please evaluate numerically)

(b) At what rate does the electric field change between the plates? (express in terms of given parameters, no need to evaluate)

(c) What is the induced magnetic field between the plates when $r < R$? (where r is the distance from the plate center; express in terms of given parameters, no need to evaluate).

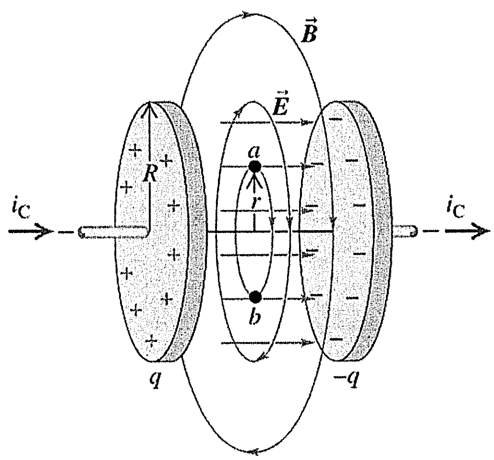
(d) What is the induced magnetic field between the plates when (i) $r \to 0$ and when (ii) $r > R$? (express in terms of given parameters, no need to evaluate).

A parallel-plate air-filled capacitor is being charged as shown in the figure below. The circular plates have a radius of R= 5 cm and are spaced 1 mm apart ($d = 1$ mm). At a particular instant the conductive current (i_c) in the wires in the charging wires is measured to be 0.5 A.



(a) What is the displacement current density in the air space between the plates? (please evaluate numerically)

Solution:



As shown the figure, the conduction current i_C brings charge to the capacitor plates. As this charge builds up, the electric field between the plates increases, as does the electrical flux between the plates. This change in electrical flux is related to a fictitious displacement current i_D defined as

$$i_D = \epsilon_o \frac{d}{dt} \int \mathbf{E} \cdot d\mathbf{A} \tag{44}$$

We will define i_D to be determined by the flux through the whole area between the plates, so that $i_D = i_c$ Since current density is just the current divided by the area, we get

$$j_D = \frac{i_D}{A} = \frac{i_C}{\pi R^2} = \frac{0.5A}{3.14 \times 0.05^2} = 64A/m^2 \tag{45}$$

(b) At what rate does the electric field change between the plates? (express in terms of given parameters, no need to evaluate)

Solution:
We want to know $\frac{dE}{dt}$. The electrical flux can be defined as $\Phi_E = \int \mathbf{E} \cdot d\mathbf{A}$, which gives an alternative definition of i_D

$$j_D = \frac{i_D}{A} = \frac{\epsilon_o}{A}\frac{d}{dt}\int \mathbf{E} \cdot d\mathbf{A} = \frac{\epsilon_o}{A}\frac{d}{dt}(EA) = \epsilon_o \frac{dE}{dt} \tag{46}$$

Solving for $\frac{dE}{dt}$,

$$\frac{dE}{dt} = \frac{j_D}{\epsilon_o} = \frac{i_c}{\epsilon_o \pi R^2} \tag{47}$$

(c) What is the induced magnetic field between the plates when $r < R$? (where r is the distance from the plate center; express in terms of given parameters, no need to evaluate).

Solution:
To find the magnetic field between the plates, we start with the full form of Ampere's law,

$$\int \mathbf{B} \cdot d\mathbf{l} = \mu_o I + \mu_o \epsilon_o \frac{d}{dt}\int \mathbf{E} \cdot d\mathbf{A} \tag{48}$$

We take the Amperian loop to be a circle of radius r, and we take the surface integral over the plane circle bounded by this loop. $I = 0$ in the above equation, since the induced current is effectively zero at a radius, r between the plates. When $r = R$, the second term an above is just $\mu_o i_D$, but when $r < R$ the the magnetic amplitude must scale (linearly) with the distance from the plate center. Specifically, our Amperian loop has area πr^2 and the plate area πR^2; so the magnetic field when $r < R$ gets renormalized by the ratio of the areas.

$$\int \mathbf{B} \cdot d\mathbf{l} = \mu_o i_D \frac{\pi r^2}{\pi R^2} \tag{49}$$

$$2\pi r B = \mu_o i_D \frac{r^2}{R^2} \tag{50}$$

Solving for the magnetic field when $r < R$ and using that the displacement current is numerically the same as the conductive current, we get

$$B = \frac{\mu_o}{2\pi} \frac{r}{R^2} i_c \tag{51}$$

which we could evaluate using the information given to get a magnetic field in Tesla that scale linearly with r.

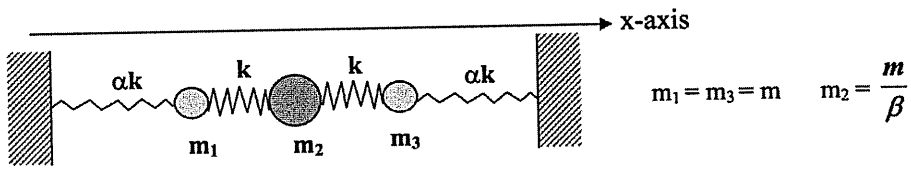
(d) What is the induced magnetic field between the plates when (i) $r \to 0$ and when (ii) $r > R$? (express in terms of given parameters, no need to evaluate).

Solution:
(i) By above the expression the magnetic field goes to zero as $r \to 0$.
(ii) For $r > R$, the surface integral in the generalized Ampere's Law has the same value as for $r = R$, and the magnetic field is the same as for a long wire:

$$B = \frac{\mu_o}{2\pi r} i_c \tag{52}$$

This result is identical to the magnetic field of a wire where the current is uniformly distributed over the cross-sectional area, showing the magnetic field drops off as $1/r$.

Long chain molecules sometimes have two types of bonds–strong bonds within small molecular units, and weak bonds between these molecular units. As a model for the normal mode vibrations in this situation, consider the longitudinal oscillations along the x-axis of the spring and ball system shown below, where the side-walls are rigid. The springs obey Hooke's law with spring constants k and αk, and the masses m_i are related to each other as indicated. α and β are constants with $\alpha < 1$ and β arbitrary.



(a) Give the Lagrangian that describes the vibrations of this system parallel to the x-axis, and use the Lagrangian to obtain the equations of motion.

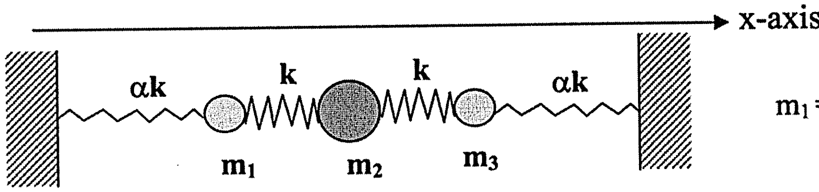
(b) Assume normal mode solutions, i.e. $x_j = a_j e^{i\omega t}$, where x measures the displacement from equilibrium. Find the normal mode frequencies, which you can call ω^0, ω^+, ω^-. The parameterization $\lambda = \omega^2 m/k$ may be useful, in which case find the parameterized versions, λ^0, λ^+, λ^-. One of the frequencies, which you should denote ω_0, depends only on α. One way to solve the problem is to set up a matrix eigenvalue equation.

(c) In the normal mode with frequency ω_0, one atom is stationary. Show which atom is stationary, and find out the motions of the other two atoms (just relative amplitude and phase, not absolute). Sketch and describe this mode qualitatively.

(d) The other 2 normal modes have $a_1 = a_3$ (you don't have to prove this). What does this information imply about the motion of the m_2 atom in these other modes? Your argument can be qualitative, but it must be rigorous. Sketch and describe the other two modes qualitatively. Which has the higher frequency?

Solutions to problem 5

Long chain molecules sometimes have two types of bonds – strong bonds within small molecular units, and weak bonds between these molecular units. As a model for the normal mode vibrations in this situation, consider the longitudinal oscillations along the x-axis of the spring and ball system shown below, where the side-walls are rigid. The springs obey Hooke's law with spring constants k and αk, and the masses m are related to each other as indicated. α and β are constants with $\alpha < 1$ and β arbitrary.



$m_1 = m_3 = m \qquad m_2 = \dfrac{m}{\beta}$

a) Give the Lagrangian that describes the vibrations of this system parallel to the x-axis, and use the Lagrangian to obtain the equations of motion.

Let x_i measure the <u>displacement from each equilibrium position</u> (eqm position of course does not change with time). Kinetic energy T, Potential energy U, Lagrangian L are

$$T = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}\frac{m}{\beta}\dot{x}_2^2 + \tfrac{1}{2}m\dot{x}_3^2$$

$$U = \tfrac{1}{2}\alpha k x_1^2 + \tfrac{1}{2}\alpha k x_3^2 + \tfrac{1}{2}k(x_1 - x_2)^2 + \tfrac{1}{2}k(x_2 - x_3)^2$$

$$\boxed{L = T - U = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}\frac{m}{\beta}\dot{x}_2^2 + \tfrac{1}{2}m\dot{x}_3^2 - \tfrac{1}{2}\alpha k x_1^2 - \tfrac{1}{2}\alpha k x_3^2 - \tfrac{1}{2}k(x_1 - x_2)^2 - \tfrac{1}{2}k(x_2 - x_3)^2}$$

Equations of motion $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}_j} = \dfrac{\partial L}{\partial x_j}$

$$\boxed{\begin{aligned} &j = 1: m\ddot{x}_1 = -\alpha k x_1 - k(x_1 - x_2) \\ &j = 2: \frac{m}{\beta}\ddot{x}_2 = +k(x_1 - x_2) - k(x_2 - x_3) = k(x_1 - 2x_2 + x_3) \\ &j = 3: m\ddot{x}_3 = -\alpha k x_3 + k(x_2 - x_3) \end{aligned}}$$

b) Assume normal mode solutions, i.e. $x_j = a_j e^{i\omega t}$, *where x measures the displacement from equilibrium. Find the normal mode frequencies, which you can call* ω^0, ω^+, ω^-. *The parameterization* $\lambda = \omega^2 \dfrac{m}{k}$ *may be useful, in which case find the parameterized versions,* λ^0, λ^+, λ^-. *One of the frequencies, which you should denote* ω^0, *depends only on* α. *One way to solve the problem is to set up a matrix eigenvalue equation.*

Let $x_j = a_j e^{i\omega t}$ and substitute:

$$j=1: -m\omega^2 a_1 e^{i\omega t} = -\alpha k a_1 e^{i\omega t} - k\left(a_1 - a_2\right)e^{i\omega t}$$
$$j=2: -\frac{m}{\beta}\omega^2 a_2 e^{i\omega t} = k\left(a_1 - 2a_2 + a_3\right)e^{i\omega t}$$
$$j=3: -m\omega^2 a_3 e^{i\omega t} = -\alpha k a_3 + k\left(a_2 - a_3\right)e^{i\omega t}$$

Cancel the exponential, and use the suggested parameterization:

$$\begin{aligned} -\lambda a_1 &= -\alpha a_1 - \left(a_1 - a_2\right) \\ -\frac{\lambda}{\beta}a_2 &= \left(a_1 - 2a_2 + a_3\right) \\ -\lambda a_3 &= -\alpha a_3 + \left(a_2 - a_3\right) \end{aligned} \quad ; \quad \text{rearrange} \Rightarrow \quad \begin{aligned} (\alpha+1)a_1 - a_2 + 0a_3 &= \lambda a_1 \\ -\beta a_1 + 2\beta a_2 - \beta a_3 &= \lambda a_2 \\ 0a_1 - a_2 + (\alpha+1)a_3 &= \lambda a_3 \end{aligned}$$

Rearrange again and cast in eigenvalue form $\mathbf{M}_{3\times3}\mathbf{V}_{1\times3} = \lambda\mathbf{V}_{1\times3}$:

$$\begin{pmatrix} \alpha+1 & -1 & 0 \\ -\beta & 2\beta & -\beta \\ 0 & -1 & \alpha+1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \lambda\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

To find the values of λ and hence ω, solve the secular equation which is

$$\begin{vmatrix} \alpha+1-\lambda & -1 & 0 \\ -\beta & 2\beta-\lambda & -\beta \\ 0 & -1 & \alpha+1-\lambda \end{vmatrix} = 0$$

Expand by 1st row

$$\begin{vmatrix} \alpha+1-\lambda & -1 & 0 \\ -\beta & 2\beta-\lambda & -\beta \\ 0 & -1 & \alpha+1-\lambda \end{vmatrix} = 0$$
$$(\alpha+1-\lambda)\left\{(2\beta-\lambda)(\alpha+1-\lambda)-\beta\right\}-(-1)\left\{-\beta(\alpha+1-\lambda)\right\}=0$$
$$(\alpha+1-\lambda)\left\{(2\beta-\lambda)(\alpha+1-\lambda)-2\beta\right\}=0$$

If the product of two factors is zero, then each factor may be separately zero – the first factor being zero yields one solution for lambda (that we are instructed to label "0"):

$$\alpha + 1 - \lambda^0 = 0 \qquad \text{or} \qquad \lambda^0 = \alpha + 1 \Rightarrow \omega^0 = \sqrt{\frac{k}{m}(\alpha+1)}\ .$$

The second factor being zero yields a quadratic equation in lambda:

$$(2\beta - \lambda)(\alpha + 1 - \lambda) - 2\beta = 0$$
$$2\beta(\alpha+1) - \lambda(\alpha+1) - 2\beta\lambda + \lambda^2 - 2\beta = 0$$
$$\lambda^2 - (2\beta + \alpha + 1)\lambda + 2\alpha\beta = 0$$

that gives two (positive) roots

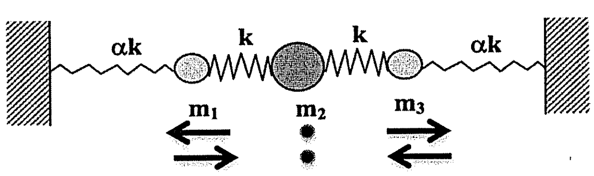
$$\lambda^{\pm} = \frac{(2\beta+\alpha+1) \pm \sqrt{(2\beta+\alpha+1)^2 - 8\alpha\beta}}{2} \Rightarrow \omega^{\pm} = \left[\frac{k}{2m}(2\beta+\alpha+1) \pm \sqrt{(2\beta+\alpha+1)^2 - 8\alpha\beta}\right]^{1/2}$$

c) In the normal mode with frequency ω^0, one atom is stationary. Show which atom is stationary, and find out the motions of the other two atoms (just relative amplitude and phase, not absolute). Sketch and describe this mode qualitatively.

If the normal mode frequency ω^0 is independent of the mass of atom 2, m/β, atom 2 must surely be stationary in that mode. To verify, go back to the equations of motion and substitute $\lambda^0 = \alpha + 1$. The equation for $j = 1$ (also for $j = 3$) yields:
$-(\alpha+1)a_1 = -\alpha a_1 - (a_1 - a_2) \Rightarrow a_2 = 0$.

$a_2 = 0$ means the middle atom of mass m/β is indeed stationary in the ω^0 mode.

The equation for $j = 2$ is $-\frac{\lambda}{\beta}a_2 = (a_1 - 2a_2 + a_3)$ and with $a_2 = 0$ gives
$0 = (a_1 + a_3) \Rightarrow a_1 = -a_3$.
$a_1 = -a_2$ means the two "m" atoms oscillate with equal amplitude and exactly out of phase. Thus the ω^0 (or λ^0) mode is completely antisymmetric.



(Or a plot of Re[$x(t)$] for all 3 masses.)

d) The other 2 normal modes have $a_1 = a_3$ (you don't have to prove this). What does this information imply about the motion of the m_2 atom in these other modes? Your argument can be qualitative, but it must be rigorous. Sketch and describe the other two modes qualitatively. Which has the higher frequency?

The other two modes are the $\omega^{\pm}$ (or $\lambda^{\pm}$) modes. $a_1 = a_3$ means that in these modes, atoms m_1 and m_3 oscillate in phase with each other and with equal amplitude. What about a_2?

The 3 modes are of the form $\begin{pmatrix} a \\ 0 \\ -a \end{pmatrix}$ (λ^0 and $\begin{pmatrix} a' \\ a_2' \\ a' \end{pmatrix}, \begin{pmatrix} a'' \\ a_2'' \\ a'' \end{pmatrix}$.

The orthogonality condition applied to the second two, $a_2'a_2'' = -2a'a''$, says that the m_2 (middle) atom is in phase with m_1 and m_3 in one mode and out of phase with m_1 and m_3 in the other. (Suppose a' and a'' are both positive or both negative, then a_2' and a_2'' have opposite signs. Suppose a' and a'' have different signs, then a_2' and a_2'' have the same sign.) Now – which of λ^+ or λ^- corresponds to m_2 being in phase with the other two? With $a_1 = a_3$, use the equations of motion for $j = 2$:

$$-\frac{\lambda^{\pm}}{\beta}a_2 = (a_1 - 2a_2 + a_3) \Rightarrow -\frac{\lambda^{\pm}}{2\beta}a_2 = (a_1 - a_2) \Rightarrow \frac{a_1}{a_2} = \left(1 - \frac{\lambda^{\pm}}{2\beta}\right)$$

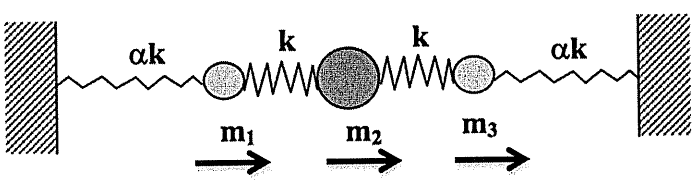
All 3 atoms are in phase if $\frac{a_1}{a_2} > 0 \Rightarrow \frac{\lambda}{2\beta} < 1$. Plug in the values for λ^+ or λ^-

$$\frac{\lambda^{\pm}}{2\beta} = \frac{(2\beta + \alpha + 1) \pm \sqrt{(2\beta + \alpha + 1)^2 - 8\alpha\beta}}{4\beta}$$

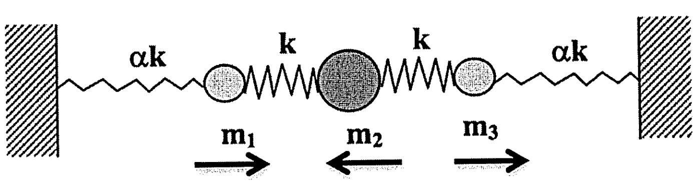
The numerator is of the form $x \pm \sqrt{x^2 - |y|}$ and since $\sqrt{x^2 - |y|} < x$, clearly $\frac{\lambda^-}{2\beta} < 0 < 1$.

It is also clear that $\lambda^+ > \lambda^-$ because $\frac{\lambda^+}{\lambda^-} = \frac{x + \sqrt{x^2 - |y|}}{x - \sqrt{x^2 - |y|}} = \frac{x + \sqrt{|< x^2|}}{x - \sqrt{|< x^2|}}$.

Thus the λ^- **mode has all 3 atoms in phase and it has the lower frequency** (see below) as expected for in-phase motion.

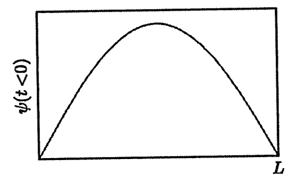


Therefore the λ^+ **mode has the middle atom out-of-phase** with the other two and it has the higher frequency as expected for out-of-phase motion.

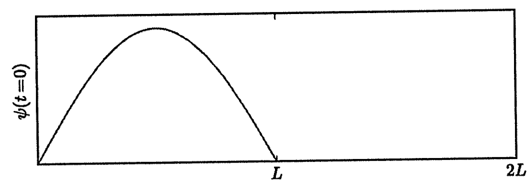


(It wasn't asked, but the λ^0 mode has a frequency intermediate between the other two.)

Consider a particle in the ground state of a one-dimensional box with length L.



At time $t = 0$, the right-hand wall of the box is quickly moved to increase the size of the box to $2L$. This leaves the wave function in the following state immediately after $t = 0$:



(a) What is the expectation value of the energy before and after this change?

(b) What is the probability of finding the particle in the n^{th} energy eigenstate if one measures the energy immediately after expanding the box?

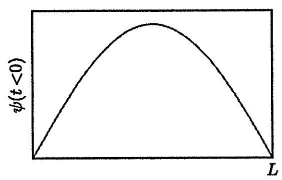
(c) What is the time-dependent wave function for $t > 0$?

(d) At what time for $t > 0$ will there be zero probability of finding the particle on the right-hand side of the box?

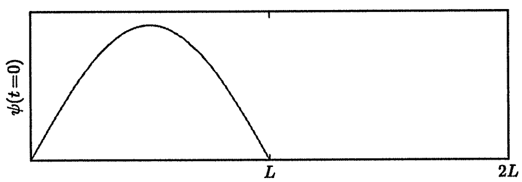
A useful integral:

$$\int_0^{\pi} \sin(x) \sin\left(\frac{2n+1}{2}x\right) dx = \frac{4(-1)^n}{3 - 4n - 4n^2}$$

Consider a particle in the ground state of a one-dimensional box with length L.



At time $t = 0$, the right-hand wall of the box is quickly moved to increase the size of the box to $2L$. This leaves the wave function in the following state immediately after $t = 0$:



(a) What is the expectation value of the energy before and after this change?

Solution:
The wavefunction both before and after the change is identical, so the expectation value of the energy before and after the box is expanded will be identical, and will have the same value as the eigenenergy of the original unexpanded system.

$$\psi(x, t = 0) = \frac{1}{\sqrt{2L}} \sin\left(\frac{\pi x}{L}\right) \tag{53}$$

The energy eigenvalue is easy to find from the momentum (in case you forgot the formula):

$$\langle E \rangle = \frac{\hbar^2 k^2}{2m} \tag{54}$$

$$= \frac{\hbar^2 \pi^2}{2mL^2} \tag{55}$$

which is the same both before and after.

(b) What is the probability of finding the particle in the n^{th} energy eigenstate if one measures the energy immediately after expanding the box?

Solution:
This requires us to find the overlap between the eigenstates and the initial wavefunction.

$$P_n = |\langle n|\psi\rangle|^2 \tag{56}$$

So we just have some unpleasant integrals to perform.

$$\langle n|\psi\rangle = \int_0^L \frac{1}{\sqrt{4L}} \sin\left(\frac{n\pi x}{2L}\right) \frac{1}{\sqrt{2L}} \sin\left(\frac{\pi x}{L}\right) dx \tag{57}$$

$$= \frac{1}{2L\sqrt{2}} \int_0^L \sin\left(\frac{n\pi x}{2L}\right) \sin\left(\frac{\pi x}{L}\right) dx \tag{58}$$

$$u = \frac{\pi x}{L} \tag{59}$$

$$du = \frac{\pi}{L} dx \tag{60}$$

$$\langle n|\psi\rangle = \frac{1}{2\pi\sqrt{2}} \int_0^\pi \sin\left(\frac{nu}{2}\right) \sin(u)\, du \tag{61}$$

At this point, we want to consider separate cases.

If n is equal to 2, we find:

$$\langle 2|\psi\rangle = \frac{1}{2\pi\sqrt{2}} \int_0^\pi \sin^2(u)\, du \tag{62}$$

$$= \frac{1}{4\sqrt{2}} \tag{63}$$

where I used the fact that $\sin^2$ averages to $\frac{1}{2}$.

If n is even and $n > 2$, then our integral is zero for the same reason that Fourier series work.

This leaves odd n, which is handled by the provided integral:

$$\langle 2j+1|\psi\rangle = \frac{1}{2\pi\sqrt{2}} \frac{4(-1)^j}{3-4j-4j^2} \tag{64}$$

$$= \frac{1}{\pi} \frac{\sqrt{2}(-1)^j}{3-4j-4j^2} \tag{65}$$

Putting these together and squaring, we find that the probabilities are:

$$P_1 = \left|\frac{1}{\pi} \frac{\sqrt{2}(-1)^0}{3-4\times 0-4\times 0^2}\right|^2 \tag{66}$$

$$= \left|\frac{\sqrt{2}}{3\pi}\right|^2 \tag{67}$$

$$= \frac{2}{9\pi^2} \tag{68}$$

$$P_2 = \left|\frac{1}{4\sqrt{2}}\right|^2 \tag{69}$$

$$= \frac{1}{32} \tag{70}$$

$$P_{2j+1} = \left| \frac{1}{\pi} \frac{\sqrt{2}(-1)^j}{3 - 4j - 4j^2} \right|^2 \tag{71}$$

$$= \frac{2}{\pi^2} \frac{1}{|4j^2 + 4j - 3|^2} \tag{72}$$

$$P_{n,\text{even}} = 0 \tag{73}$$

And that's it.

(c) What is the time-dependent wave function for $t > 0$?

Solution:
We've already done the grunt work of projecting the initial wave function onto energy eigenstates of the new system. Now we just need to add the time-dependent phase factors.

$$|\psi(t)\rangle = \sum_n e^{-i\frac{E_n t}{\hbar}} |n\rangle \langle n|\psi(t=0)\rangle \tag{74}$$

where I am understanding $|n\rangle$ to not contain time-dependence. Since we've already done the hard part, what remains is to remind ourselves that

$$E_n = \frac{\hbar^2 \pi^2}{8mL^2} \tag{75}$$

where the answer is slightly unfamiliar because our box has size $2L$.

Since we already have all the requisite factors, we will simply write down the answer:

$$|\psi(t)\rangle = \frac{1}{4\sqrt{2}} |2\rangle e^{-i\frac{\hbar\pi^2}{2mL^2}t} + \sum_{j=0}^{\infty} \frac{1}{\pi} \frac{\sqrt{2}(-1)^j}{3 - 4j - 4j^2} |2j+1\rangle e^{-i\frac{\hbar\pi^2(2j+1)^2}{8mL^2}t} \tag{76}$$

(d) At what time for $t > 0$ will there be zero probability of finding the particle on the right-hand side of the box?

Solution:
When we examine the phase factors in the previous equation, we can see that all the frequencies are integer multiples of

$$\omega_1 = \frac{\hbar\pi^2}{8mL^2} \tag{77}$$

This means that when $\omega_1 t = 2\pi n$, all of these phase factors will be equal to 1, and thus the wavefunction will have its initial shape, and will have no probability of being on the right-hand side of the box. Thus the first time at which this will happen will be

$$T = 2\pi \frac{8mL^2}{\hbar\pi^2} \tag{78}$$

$$= \frac{16mL^2}{\hbar\pi} \tag{79}$$

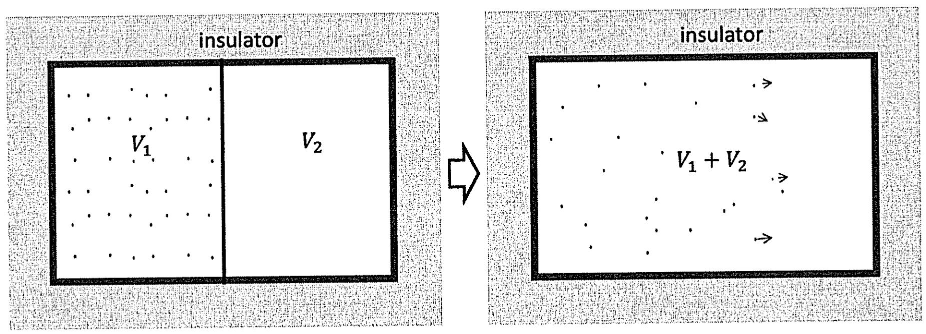
A useful integral:

$$\int_0^\pi \sin(x)\sin\left(\frac{2n+1}{2}x\right)dx = \frac{4(-1)^n}{3-4n-4n^2}$$

One mole ($n = 1$) of a van der Waals gas is confined to a compartment (V_1) separated from an empty compartment (V_2) by a panel, as shown in the figure below. The equation of state is expressed as

$$P = \frac{nRT}{V - nb} - \frac{n^2 a}{V^2} = \frac{RT}{V - b} - \frac{a}{V^2},$$

where $b \ll V$ and $a \ll RVT$.



Now the panel is abruptly removed, and the gas undergoes a free expansion from the initial volume V_1 to the final volume $V_1 + V_2$. You may assume that the heat capacity at constant volume C_v is independent of the volume V and temperature T.

(a) Starting with the fundamental thermodynamic relation, $dU = TdS - PdV$, show that

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$

(b) Show that

$$\left(\frac{\partial U}{\partial V}\right)_T = \frac{a}{V^2}$$

using the fundamental thermodynamic relation, $dU = TdS - PdV$, and the relation you derived in (a),

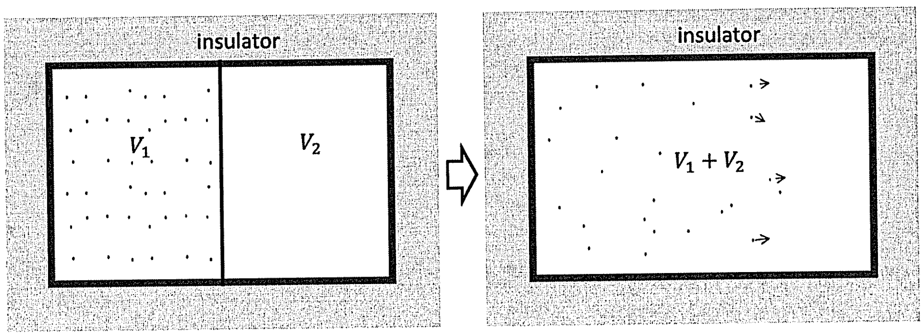
$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V.$$

(c) Find the temperature after the expansion T_2, if the initial temperature is T_1. What is T_2 in the limit of ideal gas, $a, b \to 0$? Explain why T_2 of the van der Waals gas is different from T_2 of an ideal gas.

One mole ($n = 1$) of a van der Waals gas is confined to a compartment (V_1) separated from an empty compartment (V_2) by a panel, as shown in the figure below. The equation of state is expressed as

$$P = \frac{nRT}{V - nb} - \frac{n^2 a}{V^2} = \frac{RT}{V - b} - \frac{a}{V^2},$$

where $b \ll V$ and $a \ll RVT$.



Now the panel is abruptly removed, and the gas undergoes a free expansion from the initial volume V_1 to the final volume $V_1 + V_2$. You may assume that the heat capacity at constant volume C_v is independent of the volume V and temperature T.

(a) Starting with the fundamental thermodynamic relation, $dU = TdS - PdV$, show that

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$

Solution:
Considering T and V as independent variables, we can transform the fundamental thermodynamic relation, $dU = TdS - PdV$, into an expression, $dF = -SdT - PdV$, where we define $F \equiv U - TS$, called Helmholtz free energy. Comparting the expression with

$$dF = \left(\frac{\partial F}{\partial T}\right)_V dT + \left(\frac{\partial F}{\partial V}\right)_T dV,$$

we obtain

$$\left(\frac{\partial F}{\partial T}\right)_V = -S, \left(\frac{\partial F}{\partial V}\right)_T = -P$$

Equality of the cross derivatives,

$$\frac{\partial^2 F}{\partial V \partial T} = \frac{\partial^2 F}{\partial T \partial V},$$

then implies

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$

(b) Show that

$$\left(\frac{\partial U}{\partial V}\right)_T = \frac{a}{V^2}$$

using the fundamental thermodynamic relation, $dU = TdS - PdV$, and the relation you derived in (a),

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V.$$

Solution:
From $dU = TdS - PdV$, we obtain

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - P = T\left(\frac{\partial P}{\partial T}\right)_T - P.$$

The van der Waals equation yields

$$\left(\frac{\partial P}{\partial T}\right)_T = \frac{R}{V-b},$$

and hence,

$$\left(\frac{\partial U}{\partial V}\right)_T = T\frac{R}{V-b} - P = T\frac{R}{V-b} - \left(\frac{RT}{V-b} - \frac{a}{V^2}\right) = \frac{a}{V^2}$$

(c) Find the temperature after the expansion T_2, if the initial temperature is T_1. What is T_2 in the limit of ideal gas, $a, b \to 0$? Explain why T_2 of the van der Waals gas is different from T_2 of an ideal gas.

Solution:
In the free expansion, there is no heat exchange with the outside world, and no work is done. Hence $dU = 0$. Considering V and T as independent variables,

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV = C_v dT + \frac{a}{V^2}dV = 0.$$

Therefore,

$$dT = -\frac{a}{C_v V^2},$$

and hence

$$T_2 = T_1 + \left(\frac{1}{V_1 + V_2} - \frac{1}{V_1}\right)\frac{a}{C_v}.$$

In the ideal gas limit, $a = 0$, $T_2 = T_1$, i.e., there is no change in temperature during expansion.

From a qualitative microscopic point of view, the term a/V^2 represents the additional positive pressure due to long-range attractive forces between molecules. An expansion increases spacing between the molecules, and hence the attractive potential energy (< 0) increases (note that the potential energy reaches 0 for $V \to \infty$). Since the internal energy is conserved, the kinetic energy must decrease to compensate the change in the potential energy. Since the temperature of a gas is proportional to its kinetic energy, the final temperature after an expansion is lower than the initial temperature in the van der Waals gas. On the other hand, the expansion of an ideal gas does not change its kinetic energy, and hence there is no change in temperature.

A plane electromagnetic wave of angular frequency, ω is **normally** incident on a planar slab of copper metal.

(a) Derive an expression for the fraction of the power reflected ($R = \left|\frac{\mathbf{E}_{reflected}}{\mathbf{E}_{incident}}\right|^2$) of the wave in terms of ω and the conductivity of copper, σ.
You may safely assume in copper that $1 >> \epsilon_o\omega/\sigma$. Please show all of your work.

(b) If the EM wave has $\omega = 10^{10}s^{-1}$ and copper has a conductivity of $\sigma = 10^7\Omega^{-1}m^{-1}$, what does this tell you about what happens (qualitatively, but justify from above equations) to the EM wave at the copper interface?

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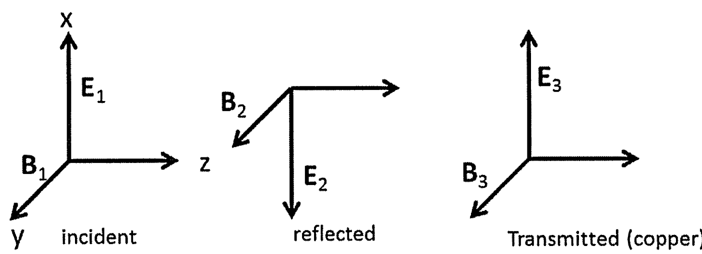
Solution:
We first need to define a coordinate system for the incident(1), reflected(2) and transmitted(3) EM fields.



Boundary conditions at the interface further tell us that: (1) $\mathbf{E}_1 - \mathbf{E}_2 = \mathbf{E}_3$ and (1) $\mathbf{B}_1 + \mathbf{B}_2 = \mathbf{B}_3$

We begin by solving Maxwell's equations (note, $\epsilon_o << \sigma/\omega$) for the transmitted wave in the metal (also making use of Ohm's Law $\mathbf{J} = \sigma\mathbf{E}$):

$$\nabla \times \mathbf{B}_3 = \mu_o \mathbf{J}_3 = \mu_o\sigma\mathbf{E}_3 \quad (80)$$

$$\nabla \times \nabla \times \mathbf{B}_3 = \mu_o\sigma\nabla \times \mathbf{E}_3 = -\mu_o\sigma\frac{\partial \mathbf{B}_3}{\partial t} \quad (81)$$

$$-\nabla^2\mathbf{B}_3 = -\mu_o\sigma\frac{\partial \mathbf{B}_3}{\partial t} \quad (82)$$

The solution to the differential goes as, $B_3\hat{z} \sim e^{ik_3 z - i\omega t}\hat{z}$, where $k_3^2 = i\omega\sigma/c^2 = i\omega\mu_o\epsilon_o\sigma$ or $k_3 = \sqrt{\frac{\sigma\mu_o\epsilon_o\omega}{2}}(1+i)$.

We can further evaluate the curl of the B-field in the copper (making reference to the diagram):

$$\nabla \times \mathbf{B}_3 = \sigma\mu_o\mathbf{E}_3 \quad (83)$$

$$ik_3 B_3\hat{z} \times \hat{y} = \sigma\mu_o E_3\hat{x} \quad (84)$$

This gives us, $B_3 = \frac{-\sigma\mu_o}{ik_3}E_3$. Using our boundary conditions $\mathbf{B}_1 + \mathbf{B}_2 = \mathbf{B}_3$ and $\mathbf{E}_1 - \mathbf{E}_2 = \mathbf{E}_3$, and in the vacuum region we can set the magnitudes, $E_1 = B_1$ and $E_2 = B_2$, we combine with our expression for B_3 to get:

$$E_1 + E_2 = \frac{-\sigma\mu_o}{ik_3}(E_1 - E_2) \tag{85}$$

$$E_1\left(1 + \frac{-\sigma\mu_o}{ik_3}\right) = E_2\left(1 - \frac{\sigma\mu_o}{ik_3}\right) \tag{86}$$

$$\frac{E_2}{E_1} = \frac{\left(1 - \frac{\sigma\mu_o}{ik_3}\right)}{\left(1 + \frac{-\sigma\mu_o}{ik_3}\right)} \tag{87}$$

$$\frac{E_2}{E_1} = \frac{ik_3/\sigma\mu_o + 1}{1 - ik_3/\sigma\mu_o} \tag{88}$$

But we can further simplify as $\epsilon_o\omega/\sigma << 1$, hence $k_3/\sigma << 1$ so we get

$$\left|\frac{E_2}{E1}\right| \approx \left|\frac{ik_3}{\sigma\mu_o} + 1\right| \tag{89}$$

$$= \left|\frac{i\sqrt{\frac{\sigma\mu_o\epsilon_o\omega}{2}}(1+i)}{\sigma\mu_o} + 1\right| \tag{90}$$

$$\approx 1 - \sqrt{2\omega\epsilon_o/\sigma} \tag{91}$$

Our resulting reflectivity is just:

$$\left|\frac{E_2}{E_1}\right|^2 \approx (1 - \sqrt{2\epsilon_o\omega/\sigma})^2 \tag{92}$$

$$= 1 - 2\sqrt{2\epsilon_o\omega/\sigma} + 2\epsilon_o\omega/\sigma \tag{93}$$

$$\approx 1 - \sqrt{8\omega\epsilon_o/\sigma} \tag{94}$$

(b) If the EM wave has $\omega = 10^{10}s^{-1}$ and copper has a conductivity of $\sigma = 10^7\Omega^{-1}m^{-1}$, what does this tell you about what happens (qualitatively, but justify from above equations) to the EM wave at the copper interface?

Solution:

$$R \approx 1 - \sqrt{8\omega\epsilon_o/\sigma} = 1 - \sqrt{8 \cdot 10^{10} \cdot 10^{-11}/10^7} \approx 1 - 3 \times 10^{-4} \tag{95}$$

Clearly almost all of the incident EM field gets reflected in the normal incident geometry, for most any planar, highly conducting solid. This is why metals make good coatings for mirrors.