OSU Physics Department
Comprehensive Examination #118

Monday, January 6 and Tuesday, January 7, 2014

Winter 2014 Comprehensive Examination

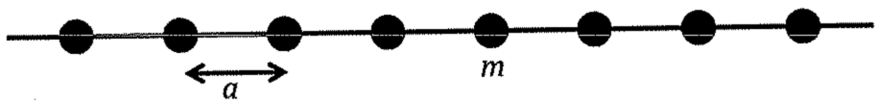
PART 1, Monday, January 6, 9:00am

General Instructions

This Winter 2014 Comprehensive Examination consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, January 6, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:00 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, January 7, at 9:00 am and 1:00 pm, respectively. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please return all bluebooks and formula sheets at the end of the exam. Use the last pages of your bluebooks for "scratch" work, separated by at least one empty page from your solutions. "Scratch" work will not be graded.

A portion of a system with an infinite number of identical coupled masses, m, separated by distance a, is shown below in its equilibrium condition. The masses are coupled by a very light, flexible string under tension T. There is no dissipation in the system, and gravity has no role - imagine that the masses are supported on a frictionless table. Consider small displacements from equilibrium that are transverse to the line of the masses. "Small" in this context means that the flexible string stretches to allow displacement, but not enough to change the tension in the string, and also that the motion is constrained to the direction transverse to the string.



(a) Set up the equation of motion for small transverse oscillations of a generic mass in the nth position that is displaced from its equilibrium position by y_n, showing how it is coupled to the motion of its neighbors. Supply a clear, labeled diagram defining all quantities.

(b) The normal mode solutions are those solutions where all masses oscillate at the same frequency ω, and their displacements at any given time have a sinusoidal envelope with a characteristic wave vector k. Find the normal mode solutions to the coupled equations, and hence the dispersion relation, $\omega(k)$ of this system. Sketch the dispersion relation. Are there maximum and minimum frequencies for which transverse waves can propagate in this system, and if so, what are these frequencies?

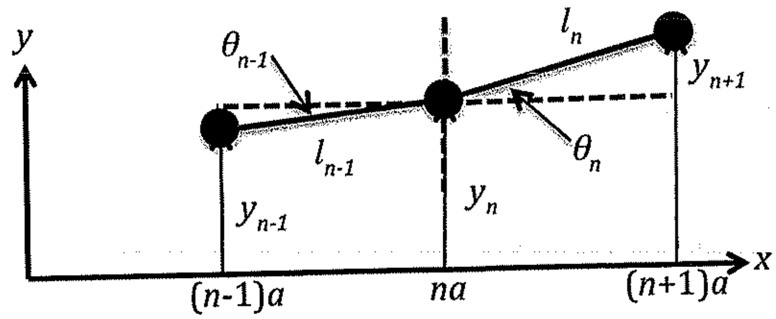
(c) Imagine a piston or other wave generator generating small-amplitude transverse waves at one side of the system, at $x = -\infty$, say. Discuss qualitatively the transverse wave form and energy propagation when the piston generates sinusoidal waves at a frequency (i) within the allowed range (ii) outside the allowed range.

Undergraduate Classical Mechanics: SOLUTION

(a) Define the displaced positions, angles the nth mass:

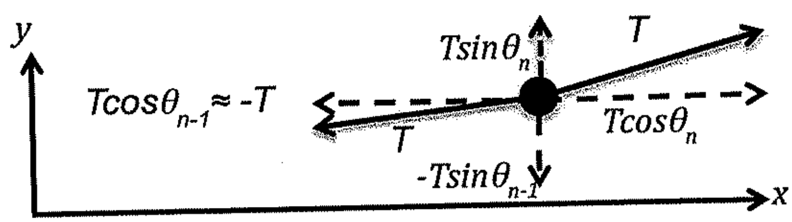
- Let a be the equilibrium separation of the masses (while the string is under tension T but not displaced in the y direction). Then $x_n = na$ labels the position along x of the nth mass.
- Let y_n be the y-displacement of the n^{th} mass at position na. $y = 0$ defines the equilibrium position.
- Let θ_n be the angle the string to its right makes with the x-axis, as shown.
- Let l_n be the length of the segment of the string to the right of the n^{th} mass.

Set up the displacements, angles:



Set up the forces on the nth mass:

- The tension in the string remains T under the condition of small displacements from equilibrium, and the tension force vector points along the string segment. The components of the individual forces on the n^{th} mass are as shown.



- If the angles are small $\cos\theta_n, \cos\theta_{n-1} \approx 1$, and $F_{net,x} = T\cos\theta_n - T\cos\theta_{n-1} \approx 0$
- The restoring force exerted on the nth mass in the y-direction on each mass is the sum of the y-components:

$$F_{net,y} = T\sin\theta_n - T\sin\theta_{n-1}$$

$$\sin\theta_n = \frac{y_{n+1} - y_n}{\ell_n} \approx \frac{y_{n+1} - y_n}{a}$$

The last approximation follows because the displaced length of the string l differs from the equilibrium length a by terms of *second order* in y_{n+1}-y_n, which is negligible if the amplitudes of displacement are small:

$$\ell_n^2 = a^2 + (y_{n+1} - y_n)^2 \Rightarrow \ell_n^2 - a^2 = (y_{n+1} - y_n)^2$$
$$(\ell_n - a)(\ell_n + a) = (y_{n+1} - y_n)^2$$
$$\ell_n - a \approx (y_{n+1} - y_n)^2 / 2a$$
$$\ell_n \approx a$$

The equation of motion is obtained from Newton's law, $F = ma$:

$$m\ddot{y}_n = F_{net,y}$$
$$m\ddot{y}_n \simeq T\left(\frac{y_{n+1} - y_n}{a} - \frac{y_n - y_{n-1}}{a}\right)$$

or

$$\boxed{\ddot{y}_n \simeq \frac{T}{ma}(y_{n+1} - 2y_n + y_{n-1})}$$

This equation is one of n coupled equations linking the displacement, y_n, of mass n to the displacements of its neighbors, y_{n+1} and y_{n-1}. There are no boundary conditions to apply at this time.

The **Euler-Lagrange** method produces the same result, as it should.

$$K = \sum_n \tfrac{1}{2} m\dot{y}_n^2; \quad U = \sum_n T(\ell_n - a)$$

where $(\ell_n - a)$ is the extension of the string segment which is calculated above. Substitute for the extension and form the Lagrangian $L = K - U$

$$L = \sum_n \tfrac{1}{2} m\dot{y}_n^2 - T(\ell_n - a)$$
$$L = \sum_n \tfrac{1}{2} m\dot{y}_n^2 - \frac{T}{2a}(y_{n+1} - y_n)^2$$

The equation of motion of the n^{th} mass is found from

$$\frac{\partial L}{\partial y_n} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_n} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_n} = m\ddot{y}_n \qquad \frac{\partial L}{\partial y_n} = \frac{\partial}{\partial y_n}\left[-\frac{T}{2a}(y_{n+1} - y_n)^2 - \frac{T}{2a}(y_n - y_{n-1})^2\right]$$
$$= \frac{T}{a}(y_{n+1} - y_n) - \frac{T}{a}(y_n - y_{n-1})$$
$$= \frac{T}{a}(y_{n+1} - 2y_n + y_{n-1})$$

and as before:

$$\boxed{m\ddot{y}_n = \frac{T}{a}(y_{n+1} - 2y_n + y_{n-1})}$$

(b) Assume normal mode solutions in which all masses vibrate with the same frequency. There are many such frequencies, which are labeled ω_q. Each set of vibrations is called a mode with mode label q. The displacement of the n^{th} mass in normal mode q is: $y_n^{(q)} = A_n^{(q)} e^{i\omega_q t}$, where $A_n^{(q)}$ is the amplitude. The real part of the displacement is assumed. The above equation becomes

$$-\omega_q^2 y_n = \frac{T}{ma}(y_{n+1} - 2y_n + y_{n-1})$$

Further, each normal mode has a sinusoidal shape, characterized by a wave vector $k_q = \frac{2\pi}{\lambda_q}$.

This means that the amplitudes are $A_n^{(q)} = Ae^{i(k_q na+\delta)}$ where A is a constant, and na labels the equilibrium position of the nth oscillator, and δ is a phase. Thus $y_n^{(q)} = Ae^{i(k_q na+\delta)} e^{i\omega_q t}$, and the real part of the expression describes the displacement.

Substitute in the equation of motion,

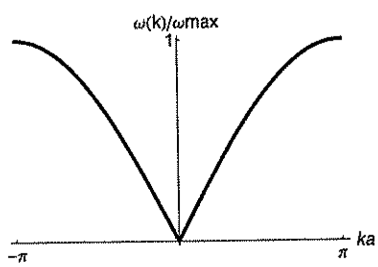
$$-\omega_q^2 Ae^{i(k_q na+\delta)} = \frac{T}{ma}\left(Ae^{i(k_q(n+1)a+\delta)} - 2Ae^{i(k_q na+\delta)} + Ae^{i(k_q(n-1)a+\delta)}\right)$$

and cancel A, $e^{i\delta}$, $e^{ik_n a}$,

$$-\omega_q^2 = \frac{T}{ma}\left(e^{ik_q a} - 2 + e^{-ik_q a}\right)$$

$$\omega_q^2 = \frac{T}{ma}(2 - 2\cos k_q a) = \frac{2T}{ma}(1 - \cos k_q a) = \frac{4T}{ma}\sin^2\frac{k_q a}{2}$$

Identify $\omega_{\max} = \left(\frac{4T}{ma}\right)^{1/2}$ to find the dispersion relation ω(*k*): $\boxed{\omega_q = \omega_{\max}\left|\sin\frac{k_q a}{2}\right|;}$



The dispersion relation is drawn at left – there is indeed a max frequency ω_{max} which occurs for $k = \pi/a$, but no minimum frequency (all frequencies down to ω = 0 are allowed). Modes for wave vectors outside of this range are redundant – the displacements of the masses are the same as for a mode within this range whose wave vector is different by $2\pi/a$.

(c) If the system is driven at frequencies below $\omega_{\max}$, the wave vector $k(\omega)$ is real and there are propagating sinusoidal waves

$$y_n^{(q)} = Ae^{i(k_q na+\delta)} e^{i\omega_q t}$$

the amplitude and phase are determined by the piston motion. The energy put into the system by the piston at, say, $x = 0$ is transferred along the rope and finds its way to the other side.

• If the system is driven at frequencies *above* $\omega_{\max}$, the dispersion relation cannot be satisfied for any real value of k. It can be satisfied for a COMPLEX value of k, however. If k is complex, then the imaginary component means that the waveform is attenuated in space (or exponentially growing, but this is unphysical). In other words, a wave of those frequencies does not propagate in the system, but is exponentially attenuated over a length that depends on how far away the desired frequency is from an allowed frequency. Energy is reflected back to the wave generator.

Consider a system in a (normalized) quantum state $|\psi\rangle$ and define the operator

$$\rho = |\psi\rangle \langle\psi|$$

which is called the density matrix.

(a) Write out the time-dependent Schrodinger equations for both a ket state $|\psi\rangle$ and a bra state $\langle\psi|$. Use the results to prove that equivalent expression for the density matrix can be expressed as,

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho]$$

(b) Show that $\text{Tr}(\rho A)$ is the expectation value of an observable operator A.

(c) Calculate $\text{Tr}(\rho)$.

(d) Now we consider instead the case of spin-1/2 particles. Give the density matrix for the following cases in the $(|\uparrow\rangle, |\downarrow\rangle)$ basis, where $\uparrow$ / $\downarrow$ refers to the sign of the eigenvalue of the spin operator:

Case 1: Suppose an experimentalist prepares atoms with spin in a state $\frac{1}{2}(|\uparrow\rangle + |\downarrow\rangle)$. Express this as a 2×2 density matrix and show your work.

Case 2: Suppose instead the experimentalist prepares atoms with 50% in the $|\uparrow\rangle$ and 50% in the $|\downarrow\rangle$ state. What is ρ now?

Helpful formula: The trace of an operator can be written as $\text{Tr}(A) = \sum_n \langle n| A |n\rangle$, where $|n\rangle$ forms an orthonormal basis.

Consider a system in a (normalized) quantum state $|\psi\rangle$ and define the operator

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(a) Write out the time-dependent Schrodinger equations for both a ket state $|\psi\rangle$ and a bra state $\langle\psi|$. Use the results to prove that equivalent expression for the density matrix can be expressed as,

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho]$$

Solution:
Time-dependent Schrodinger equations for bra and ket states are:

$$\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar}H|\psi\rangle \quad (1)$$
$$\frac{d}{dt}\langle\psi| = \frac{i}{\hbar}\langle\psi| H \quad (2)$$

Likewise using the chain rule to evaluate $\frac{d\rho}{dt}$ we get:

$$\frac{d}{dt}|\psi\rangle\langle\psi| = \left[\frac{d}{dt}|\psi\rangle\right]\langle\psi| + |\psi\rangle\frac{d}{dt}\langle\psi| \quad (3)$$
$$= -\frac{i}{\hbar}H|\psi\rangle\langle\psi| + |\psi\rangle\frac{i}{\hbar}\langle\psi| H \quad (4)$$
$$= -\frac{i}{\hbar}[H,\rho] \quad (5)$$
$$(6)$$

(b) Show that $\mathrm{Tr}(\rho A)$ is the expectation value of an observable operator A.

Solution:

$$Tr(\rho A) = Tr(|\psi\rangle\langle\psi| A) \quad (7)$$
$$= \sum_n < n|\psi > \langle\psi| A |n\rangle \quad (8)$$
$$= \sum_n \langle\psi| A |n\rangle < n|\psi > \quad (9)$$
$$= \langle\psi| A |\psi\rangle \quad (10)$$
$$=< A > \quad (11)$$
$$(12)$$

the last equality holds by the identity that $I = \sum_n |n\rangle \langle n|$. Alternatively,

$$
\begin{aligned}
< A > &= \langle\psi| A |\psi\rangle && (13)\\
&= \sum_{n,m} < \psi |n\rangle \langle n| A |m\rangle < m |\psi\rangle && (14)\\
&= \sum_{n,m} < m |\psi\rangle < \psi |n\rangle \langle n| A |m\rangle && (15)\\
&= \sum_{n,m} \langle m| \rho |n\rangle \langle n| A |m\rangle && (16)\\
&= Tr(\rho A) && (17)\\
& && (18)
\end{aligned}
$$

(c) Calculate $\mathrm{Tr}(\rho)$.

Solution:
$\mathrm{Tr}(\rho) = \sum_n < n|\psi >< \psi|n > = 1$ since it an orthonormal basis set.

(d) Now we consider instead the case of spin-1/2 particles. Give the density matrix for the following cases in the $(|\uparrow\rangle, |\downarrow\rangle)$ basis, where $\uparrow$ / $\downarrow$ refers to the sign of the eigenvalue of the spin operator:

Case 1: Suppose an experimentalist prepares atoms with spin in a state $\frac{1}{2}(|\uparrow\rangle + |\downarrow\rangle)$. Express this as a 2×2 density matrix and show your work.

Case 2: Suppose instead the experimentalist prepares atoms with 50% in the $|\uparrow\rangle$ and 50% in the $|\downarrow\rangle$ state. What is ρ now?

Solution:
Case 1: Multiplying the given state out explicitly.
$\rho = \frac{1}{2}(|\uparrow\rangle \langle\uparrow| + |\downarrow\rangle \langle\downarrow| + |\downarrow\rangle \langle\uparrow| + |\uparrow\rangle \langle\downarrow|)$ OR

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

Case 2: Here instead, the off-diagonal elements are zero. i.e.,
$\rho = \frac{1}{2}(|\uparrow\rangle \langle\uparrow| + |\downarrow\rangle \langle\downarrow|)$ OR

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Helpful formula: The trace of an operator can be written as $\mathrm{Tr}(A) = \sum_n \langle n| A |n\rangle$, where $|n\rangle$ forms an orthonormal basis.

Consider an assembly of N noninteracting atoms at absolute temperature T and placed in an external magnetic field $\mathbf{B}$ pointing along the z direction. The volume of the system is V. The magnetic energy of an atom can be written as $H = -\boldsymbol{\mu}\cdot\mathbf{B}$, where the magnetic moment of the atom is proportional to the spin angular momentum $\mathbf{S}$ of the atom and is written in the form $\boldsymbol{\mu} = g\mu_0\mathbf{S}/\hbar$ with the Bohr magneton μ_0 and the so-called g factor. In a quantum mechanical description S_z has the discrete values given by $S_z = m\hbar$ where $m = -S, -S+1, -S+2, \cdots, S-1, S$.

Note: You can get partial credit if you answer the questions below assuming that the system consists of spin-1/2 particles ($S = 1/2$) instead of an arbitrary spin S.

(a) Find the mean magnetic moment of an atom, $\langle\boldsymbol{\mu}\rangle$.

(b) Find the magnetization $\mathbf{M}$ (mean magnetic moment per unit volume) for $T \to 0$. Justify your answer using physical reasoning.

(c) What is the magnetic susceptibility χ_m (the constant of proportionality, $\mathbf{M} = \chi_m\mathbf{B}$) for $T \to \infty$?

(d) Sketch the magnetization as a function of $1/T$.

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Note: You can get partial credit if you answer the questions below assuming that the system consists of spin-1/2 particles ($S = 1/2$) instead of an arbitrary spin S.

(a) Find the mean magnetic moment of an atom, $\langle\boldsymbol{\mu}\rangle$.

Solution:
The magnetic energy is $E_m = -g\mu_0 Bm$. The probability of a particle in the energy state E_m is

$$P_m = \frac{e^{-\beta E_m}}{\sum_{m=-S}^{S} e^{-\beta E_m}}, \text{ where } \beta = \frac{1}{k_B T}.$$

Since the magnetic field is aligned in the z-axis, $\langle\boldsymbol{\mu}\rangle = \langle\mu_z\rangle\mathbf{e}_z$ with $\mu_z = g\mu_0 S_z/\hbar$:

$$\langle\mu_z\rangle = \frac{\sum_{m=-S}^{S}(g\mu_0 m)e^{\beta g\mu_0 Bm}}{\sum_{m=-S}^{S} e^{\beta g\mu_0 Bm}} = \frac{1}{\beta Z}\frac{\partial Z}{\partial B} = \frac{1}{\beta}\frac{\partial \ln Z}{\partial B},$$

where the partition function

$$\begin{aligned} Z &= \sum_{m=-S}^{S} e^{\beta g\mu_0 Bm} = \sum_{m=-S}^{S} e^{am}, a \equiv \beta g\mu_0 B \\ &= e^{-aS} + e^{-a(S-1)} + e^{-a(S-2)} + \cdots + e^{aS} \\ &= \frac{e^{-aS} - e^{a(S+1)}}{1 - e^{a}} = \frac{e^{a(S+1/2)} - e^{-a(S+1/2)}}{e^{a/2} - e^{-a/2}} \\ &= \frac{\sinh[(S+1/2)a]}{\sinh[a/2]}. \end{aligned}$$

Then,

$$\begin{aligned} \langle\mu_z\rangle &= \frac{1}{\beta}\frac{\partial \ln Z}{\partial a}\frac{\partial a}{\partial B} = g\mu_0\frac{\partial \ln Z}{\partial a} \\ &= g\mu_0\left\{\frac{(S+1/2)\cosh[(S+1/2)a]}{\sinh[(S+1/2)a]} - \frac{1/2\cosh(a/2)}{\sinh(a/2)}\right\} \\ &= \mu_0\left\{\left(S+\frac{1}{2}\right)\coth\left[\left(S+\frac{1}{2}\right)a\right] - \frac{1}{2}\coth\left(\frac{a}{2}\right)\right\} \end{aligned}$$

(b) Find the magnetization $\mathbf{M}$ (mean magnetic moment per unit volume) for $T \to 0$. Justify your answer using physical reasoning.

Solution:
For $T \to 0$, $a = \frac{g\mu_0 B}{k_B T} \to \infty$. Therefore,

$$\coth\left[\left(S+\frac{1}{2}\right)a\right] \cong \frac{e^{(S+1/2)a}}{e^{(S+1/2)a}} = 1 \text{ and } \coth(a/2) \cong \frac{e^{a/2}}{e^{a/2}} = 1.$$

Then the dipole moment is

$$\langle \mu_z \rangle \cong g\mu_0[(S+1/2) - 1/2] = gS\mu_0.$$

The magnetization, $\mathbf{M} = M_z \mathbf{e}_z$ with $M_z = \frac{N}{V}\langle \mu_z \rangle \cong \frac{N}{V} gS\mu_0$.
At $T = 0$, all the particles must be in the ground state ($m = S$), in which the magnetic moments are aligned along the magnetic field, i.e., $\langle \mu_z \rangle = gS\mu_0$.

(c) What is the magnetic susceptibility χ_m (the constant of proportionality, $\mathbf{M} = \chi_m \mathbf{B}$) for $T \to \infty$?

Solution:
For $T \to \infty$, $a = \frac{g\mu_0 B}{k_B T} \to 0$.
Since

$$\begin{aligned}
\coth x &= \frac{1 + \frac{1}{2}x^2 + \cdots}{x + \frac{1}{6}x^3 + \cdots} \cong \frac{1}{x}\left(1 + \frac{1}{2}x^2\right)\left(1 - \frac{1}{6}x^2\right) \\
&\cong \frac{1}{x}\left(1 + \frac{1}{3}x^2\right) = \frac{1}{x} + \frac{1}{3}x \text{ for } x \ll 1,
\end{aligned}$$

$$\coth\left[\left(S+\frac{1}{2}\right)a\right] \cong \frac{1}{(S+1/2)a} + \frac{1}{3}(S+1/2)a \text{ and } \coth(a/2) \cong \frac{2}{a} + \frac{a}{6}.$$

Therefore, the dipole moment is

$$\begin{aligned}
\langle \mu_z \rangle &\cong g\mu_0 \left\{ (S+1/2)\left[\frac{1}{(S+1/2)a} + \frac{1}{3}(S+1/2)a\right] - \frac{1}{2}\left(\frac{2}{a} + \frac{a}{6}\right) \right\} \\
&= g\mu_0 \left[\frac{1}{a} + \frac{1}{3}\left(S + \frac{1}{2}\right)^2 - \frac{1}{a} - \frac{a}{12}\right] = \frac{1}{3} g\mu_0 (S^2 + S)a \\
&= \frac{1}{3} g\mu_0 (S^2+S) \frac{g\mu_0 B}{k_B T} = \frac{g^2 \mu_0^2 S(S+1)}{3k_B T} B
\end{aligned}$$

Then the magnetization is

$$M_z = \frac{V}{N}\langle \mu_z \rangle \cong \frac{N g^2 \mu_0^2 S(S+1)}{3V k_B T} B = \chi_m B,$$

thus

$$\chi_m = \frac{N g^2 \mu_0^2 S(S+1)}{3V k_B T}$$

(d) Sketch the magnetization as a function of $1/T$.

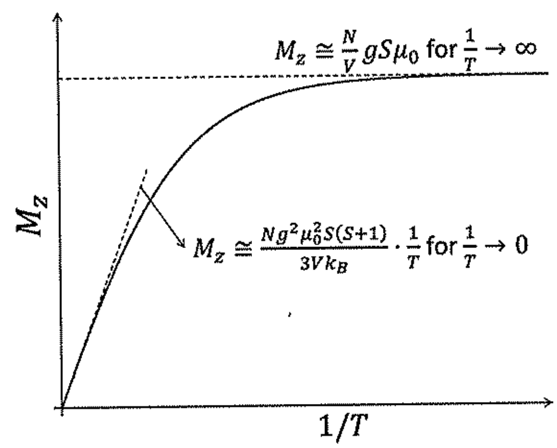
Solution:
From the answers in (b) and (c), we can see that

$$M_z \cong \frac{Ng^2\mu_0^2 S(S+1)B}{3Vk_BT} \propto \frac{1}{T} \text{ for } \frac{1}{T} \to 0$$

and

$$M_z \cong \frac{N}{V} gS\mu_0 = constant \text{ for } \frac{1}{T} \to \infty.$$

Therefore,



Consider a system of four point charges with positive charge Q, arranged in a square with each a distance D from the origin.

(a) Solve for the force on a particle with charge q (which you may assume to be positive) that is near the origin. You may assume that $|\vec{r}| \ll D$. Keep the lowest-order non-zero terms in a power series expansion of the force.

(b) Solve for the motion of the particle as a function of time for a general set of initial conditions. You solution should be valid so long as the particle remains near the origin.

(c) Describe in words the behavior of the particle if its initial position is in the plane of the four charges.

(d) Describe in words the behavior of the particle if its initial position is equidistant from the four charges, but not at the origin.

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(d) Describe in words the behavior of the particle if its initial position is equidistant from the four charges, but not at the origin.

Solution:

(a) We need the electric field near the origin. We could find this by adding together the electric field of each point charge. However, I'll use the approach of first finding the power series of the electrostatic potential, and then taking its gradient. This has the advantage that we are dealing with a scalar field while doing the annoying portion of the math, and only need deal with the direction at the end.

$$V(\vec{\mathbf{r}}) = \sum_i \frac{kQ}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_i|} \tag{19}$$

$$= kQ\left(\frac{1}{|\vec{\mathbf{r}} - D\hat{\mathbf{x}}|} + \frac{1}{|\vec{\mathbf{r}} + D\hat{\mathbf{x}}|} + \frac{1}{|\vec{\mathbf{r}} - D\hat{\mathbf{y}}|} + \frac{1}{|\vec{\mathbf{r}} + D\hat{\mathbf{y}}|}\right) \tag{20}$$

As you can see, we've got four terms to expand, but they all look basically the same, so I'll start by expanding just one of them (keeping in mind that $r \ll D$.

$$\frac{1}{|\vec{\mathbf{r}} - D\hat{\mathbf{x}}|} = \frac{1}{\sqrt{(x-D)^2 + y^2 + z^2}} \tag{21}$$

$$= \frac{1}{\sqrt{x^2 + y^2 + z^2 - 2xD + D^2}} \tag{22}$$

$$= \frac{1}{\sqrt{r^2 - 2xD + D^2}} \tag{23}$$

$$\frac{1}{|\vec{\mathbf{r}} - D\hat{\mathbf{x}}|} = \frac{1}{\sqrt{r^2 - 2xD + D^2}} \tag{24}$$

$$= \frac{1}{D}\frac{1}{\sqrt{1 - 2\frac{x}{D} + \frac{r^2}{D^2}}} \tag{25}$$

$$= \frac{1}{D}\left(1 - \frac{1}{2}\left(-2\frac{x}{D} + \frac{r^2}{D^2}\right) + \frac{3}{8}\left(-2\frac{x}{D} + \frac{r^2}{D^2}\right)^2 + \cdots\right) \tag{26}$$

$$= \frac{1}{D}\left(1 + \frac{x}{D} + \frac{1}{2}\left(3x^2 - r^2\right)\frac{1}{D^2} + \cdots\right) \tag{27}$$

The $\frac{x}{D}$ term will cancel with an equal and opposite term from the point charge opposite this one (at $-D\hat{\mathbf{x}}$). When we add together the potential due to all four charges, we get:

$$V(\vec{\mathbf{r}}) = \frac{kQ}{D}\left(4 + \frac{1}{2}(6x^2 + 6y^2 - 4r^2)\frac{1}{D^2} + \cdots\right) \tag{28}$$

$$= \frac{kQ}{D}\left(4 + \frac{1}{2}(2(x^2 + y^2) - 4z^2)\frac{1}{D^2} + \cdots\right) \tag{29}$$

Now we just need to compute the gradient of this potential to find the electric field:

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = -\vec{\nabla}V(\vec{\mathbf{r}}) \tag{30}$$

$$= -\frac{kQ}{D^3}\left(2x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} - 4z\hat{\mathbf{z}} + \cdots\right) \tag{31}$$

$$= -\frac{2kQ}{D^3}\left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}} + \cdots\right) \tag{32}$$

where all the remaining terms are higher order in D. Thus the force on a point charge with charge q is:

$$\vec{\mathbf{F}} = -\frac{2kqQ}{D^3}\left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}} + \cdots\right) \tag{33}$$

Alternative approach You could also solve for the force or electric field directly using Coulomb's law, and then take a power series of that. In this case, your solution would look something like this:

$$\vec{\mathbf{F}} = kqQ\left(\frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}_1}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_1|^3} + \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}_2}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_2|^3} + \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}_3}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_3|^3} + \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}_4}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_4|^3}\right) \tag{34}$$

$$= kqQ\left(\frac{\vec{\mathbf{r}} - D\hat{\mathbf{x}}}{|\vec{\mathbf{r}} - D\hat{\mathbf{x}}|^3} + \frac{\vec{\mathbf{r}} + D\hat{\mathbf{x}}}{|\vec{\mathbf{r}} + D\hat{\mathbf{x}}|^3} + \frac{\vec{\mathbf{r}} - D\hat{\mathbf{y}}}{|\vec{\mathbf{r}} - D\hat{\mathbf{y}}|^3} + \frac{\vec{\mathbf{r}} + D\hat{\mathbf{y}}}{|\vec{\mathbf{r}} + D\hat{\mathbf{y}}|^3}\right) \tag{35}$$

As in the previous solution, we will perform the power series expansion on just one of these terms, having recognized that they all look pretty similar.

$$\frac{\vec{\mathbf{r}} - D\hat{\mathbf{x}}}{|\vec{\mathbf{r}} - D\hat{\mathbf{x}}|^3} = \frac{\vec{\mathbf{r}} - D\hat{\mathbf{x}}}{((x-D)^2 + y^2 + z^2)^{\frac{3}{2}}} \tag{36}$$

$$= \frac{\vec{\mathbf{r}} - D\hat{\mathbf{x}}}{(D^2 - 2xD + x^2 + y^2 + z^2)^{\frac{3}{2}}} \tag{37}$$

$$= \frac{\vec{\mathbf{r}} - D\hat{\mathbf{x}}}{(D^2 - 2xD + r^2)^{\frac{3}{2}}} \tag{38}$$

$$= \frac{\vec{\mathbf{r}} - D\hat{\mathbf{x}}}{D^3} \frac{1}{(1 - 2\frac{x}{D} + \frac{r^2}{D^2})^{\frac{3}{2}}} \tag{39}$$

$$= (\vec{\mathbf{r}} - D\hat{\mathbf{x}}) \left(1 - \frac{3}{2}\left(-2\frac{x}{D} + \frac{r^2}{D^2}\right) + \cdots\right) \tag{40}$$

$$= (\vec{\mathbf{r}} - D\hat{\mathbf{x}}) \left(1 + 3\frac{x}{D} + \cdots\right) \tag{41}$$

If we needed to, of course, we could keep another term, but as we shall see, this gives us all that we need. We *must* not keep the $\frac{r^2}{D^2}$ term, unless we also include the $\frac{x^2}{D^2}$ term that arises from the next order (omitted above). The other forces differ only in minus signs and swapping of y for x:

$$\vec{\mathbf{F}} \approx \frac{kqQ}{D^3}\Big[(\vec{\mathbf{r}} - D\hat{\mathbf{x}})\left(1 + 3\frac{x}{D}\right) + (\vec{\mathbf{r}} + D\hat{\mathbf{x}})\left(1 - 3\frac{x}{D}\right)$$
$$+(\vec{\mathbf{r}} - D\hat{\mathbf{y}})\left(1 + 3\frac{y}{D}\right) + (\vec{\mathbf{r}} + D\hat{\mathbf{y}})\left(1 - 3\frac{y}{D}\right)\Big] \tag{42}$$

$$= \frac{kqQ}{D^3}(4\vec{\mathbf{r}} - 6x\hat{\mathbf{x}} - 6y\hat{\mathbf{y}}) \tag{43}$$

$$= \frac{kqQ}{D^3}(-2x\hat{\mathbf{x}} - 2y\hat{\mathbf{y}} + 4z\hat{\mathbf{z}}) \tag{44}$$

(b) To find the motion of the particle, we employ Newton's law

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} \tag{45}$$

Since each component of the force only depends on the corresponding component of the position, we can treat each component separately.

Motion within the plane Let's start with the x component (y will behave similarly):

$$-\frac{2kqQ}{D^3}x = m\ddot{x} \tag{46}$$

$$\ddot{x} = -\frac{2kqQ}{mD^3}x \tag{47}$$

This is a familiar differential equation, which has a solution that is sinusoidal. We can quickly verify that

$$x(t) = x_0 \sin\left(\sqrt{\frac{2kqQ}{mD^3}}t - \phi_x\right) \tag{48}$$

satisfies this differential equation. Of course, we could have used a cos instead, or a sum of sin and cos instead of using an arbitrary phase factor. The y solution is so similar that I won't repeat it here.

Motion out of the plane Motion in the z direction is not much harder, although it is probably less familiar. In this case we see that

$$+2\frac{2kqQ}{D^3}z = m\ddot{z} \tag{49}$$

$$\ddot{z} = +2\frac{2kqQ}{mD^3}z \tag{50}$$

This differs in sign, meaning that z is accelerated in the direction in which the particle is already located. The solution is in this case a real exponential, where the previous solution could have been written as an exponential of an imaginary quantity. Thus we have

$$z(t) = \left(z_+ e^{2\sqrt{\frac{kqQ}{mD^3}}z} + z_- e^{-2\sqrt{\frac{kqQ}{mD^3}}z}\right) \tag{51}$$

which we could alternatively write as a sum of cosh and sinh.

General motion Putting everything together (and defining a quantity Ω to simplify the formula), we get

$$\vec{\mathbf{r}}(t) = x_0 \sin\left(\sqrt{2}\Omega t - \phi_x\right)\hat{\mathbf{x}} + y_0 \sin\left(\sqrt{2}\Omega t - \phi_y\right)\hat{\mathbf{x}} + \left(z_+ e^{2\Omega z} + z_- e^{-2\Omega z}\right)\hat{\mathbf{z}} \tag{52}$$

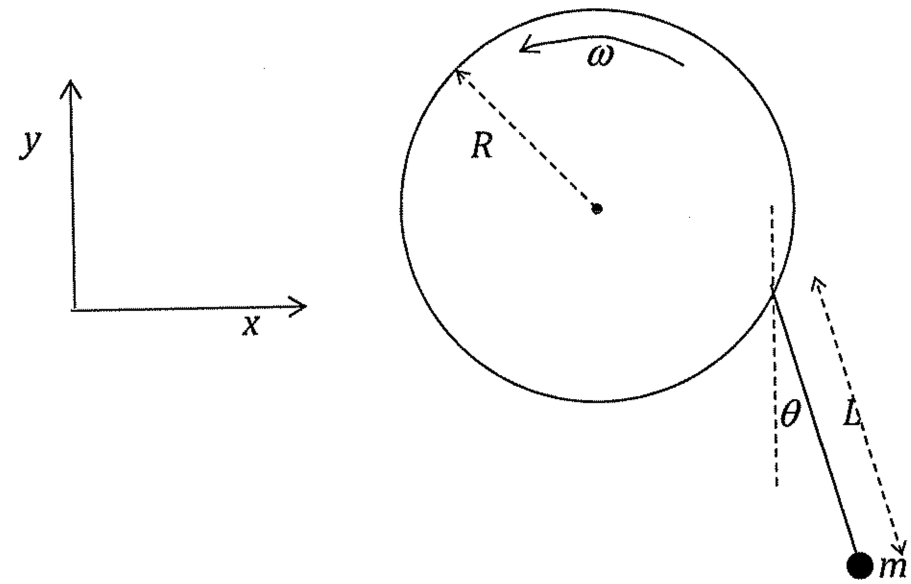
$$\Omega \equiv \sqrt{\frac{kqQ}{mD^3}} \tag{53}$$

which is pretty messy.

(c) If our particle is in the plane of the four charges, $z = 0$ in our coordinate system. Thus there is no force in the $\hat{\mathbf{z}}$ direction, and the particle remains in the plane. That leaves us with simple harmonic motion in the xy plane, since the force is opposite to the displacement of the particle from the origin. The particle will pass through the origin regularly... unless it has an angular momentum, in which case it will have elliptical motion, with the origin at the center of the ellipse.

(d) In this case our particle is on the z axis, and $x = y = 0$. Thus the force is purely in the z direction. z will increase exponentially, with the particle being pushed away from the four charges. Soon our approximation that $r \ll D$ will no longer be valid.

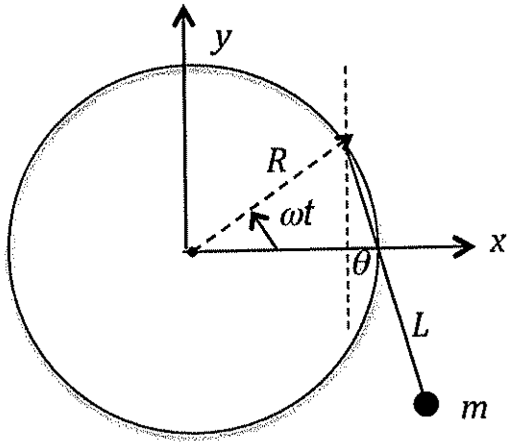
A simple pendulum of length L and mass m is suspended from a (fixed) point on the circumference of a thin massless disk of radius R that rotates with a constant angular velocity ω about an axis through its center and perpendicular to the disk plane, as shown. The pendulum makes an angle θ with the y-axis.



(a) Using Lagrangian formalism, find the equation of motion for the angle $\theta(t)$ of the mass m.

(b) Simplify the equation of motion to show that for small enough frequency $\omega^2 \ll g/R$, the small angle ($\theta \ll 1$) equation of motion becomes the same as that for a simple harmonic oscillator subject to a sinusoidal driving force.

(c) What is the solution to the equation of motion in (b)?



a) Coordinates of position and velocity of the mass m:

$x = R\cos\omega t + L\sin\theta \qquad \dot{x} = -R\omega\sin\omega t + L\dot{\theta}\cos\theta$

$y = R\sin\omega t - L\cos\theta \qquad \dot{y} = R\omega\cos\omega t + L\dot{\theta}\sin\theta$

The Lagrangian (bold **L** just to distinguish from length L) in terms of the kinetic energy

$T = \frac{1}{2}m\left|\dot{\vec{r}}\right|^2 = \frac{1}{2}m\left|\dot{x}\hat{x} + \dot{y}\hat{y}\right|^2$

and the gravitational potential erergy

$U = mgy$:

$\mathbf{L} = T - U = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$

Substitute

$$\mathbf{L} = \tfrac{1}{2}m\left(-R\omega\sin\omega t + L\dot{\theta}\cos\theta\right)^2 + \tfrac{1}{2}m\left(R\omega\cos\omega t + L\dot{\theta}\sin\theta\right)^2 - mg\left(R\sin\omega t - L\cos\theta\right)$$

$$= \tfrac{1}{2}mR^2\omega^2\left(\sin^2\omega t + \cos^2\omega t\right) + \tfrac{1}{2}mL^2\dot{\theta}^2\left(\cos^2\theta + \sin^2\theta\right) + \tfrac{1}{2}mRL\omega\dot{\theta}\left(-2\sin\omega t\cos\theta + 2\cos\omega t\sin\theta\right)$$
$$- mg\left(R\sin\omega t - L\cos\theta\right)$$

$$\mathbf{L} = \tfrac{1}{2}mR^2\omega^2 + \tfrac{1}{2}mL^2\dot{\theta}^2 - mRL\omega\dot{\theta}\sin\left(\omega t - \theta\right) - mg\left(R\sin\omega t - L\cos\theta\right)$$

The equation of motion is

$$\frac{\partial\mathbf{L}}{\partial\theta} - \frac{d}{dt}\frac{\partial\mathbf{L}}{\partial\dot{\theta}} = 0$$

Work out terms:

$$\frac{\partial \mathbf{L}}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\tfrac{1}{2}mR^2\omega^2 + \tfrac{1}{2}mL^2\dot{\theta}^2 - mRL\omega\dot{\theta}\sin(\omega t - \theta) - mg(R\sin\omega t - L\cos\theta)\right)$$

$$= +mRL\omega\dot{\theta}\cos(\omega t - \theta) - mgL\sin\theta$$

$$\frac{d}{dt}\frac{\partial \mathbf{L}}{\partial \dot{\theta}} = \frac{d}{dt}\frac{\partial}{\partial \dot{\theta}}\left(\tfrac{1}{2}mR^2\omega^2 + \tfrac{1}{2}mL^2\dot{\theta}^2 - mRL\omega\dot{\theta}\sin(\omega t - \theta) - mg(R\sin\omega t - L\cos\theta)\right)$$

$$= \frac{d}{dt}\left(mL^2\dot{\theta} - mRL\omega\sin(\omega t - \theta)\right)$$

$$= mL^2\ddot{\theta} - mRL\omega\left(\omega - \dot{\theta}\right)\cos(\omega t - \theta)$$

Put together:

$$mRL\omega\dot{\theta}\cos(\omega t - \theta) - mgL\sin\theta - mL^2\ddot{\theta} + mRL\omega\left(\omega - \dot{\theta}\right)\cos(\omega t - \theta) = 0$$

$$mL^2\ddot{\theta} = mRL\omega^2\cos(\omega t - \theta) - mgL\sin\theta$$

$$\ddot{\theta} = \frac{R\omega^2}{L}\cos(\omega t - \theta) - \frac{g}{L}\sin\theta$$

b) Expand the equation of motion:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = \frac{R\omega^2}{L}\cos(\omega t - \theta)$$

$$\ddot{\theta} + \frac{g}{L}\sin\theta = \frac{R\omega^2}{L}\cos\omega t\cos\theta + \frac{R\omega^2}{L}\sin\theta\sin\omega t$$

$$\ddot{\theta} + \left(\frac{g}{L} - \frac{R\omega^2}{L}\sin\omega t\right)\sin\theta = \frac{R\omega^2}{L}\cos\omega t\cos\theta$$

For small enough frequency $R\omega^2 \ll g$, the second term in parentheses can be made negligible, and the equation of motion reduces to

$$\ddot{\theta} + \frac{g}{L}\sin\theta = \frac{R\omega^2}{L}\cos\omega t\cos\theta$$

For small θ, $\sin\theta \approx \theta$, $\cos\theta \approx 1$

$$\ddot{\theta} + \frac{g}{L}\theta = \frac{R\omega^2}{L}\cos\omega t$$

This is the equation of motion of a simple harmonic oscillator driven by a sinusoidal driving force.

(c) The solution to such an equation is the sum of two terms (i) the solution $\theta_c(t)$ to the homogeneous equation with no driving force (also called the complementary solution), and (ii) a particular solution $\theta_p(t)$.

(i) The complementary solution to $\ddot{\theta}+\frac{g}{b}\theta=0$ is $\theta_c(t)=A\cos\left(\sqrt{\frac{g}{L}}t+\varphi\right)$ where A and ϕ are constants determined by initial conditions.

(ii) A particular solution is $\theta_p(t)=B\cos\omega t$ which can be seen by direct substitution, from which we find that

$$-\omega^2 B\cos\omega t+\frac{g}{L}B\cos\omega t=\frac{R\omega^2}{L}\cos\omega t$$

$$\Rightarrow B=\frac{R\omega^2}{\left(g-L\omega^2\right)}$$

Therefore $\theta(t)=A\cos\left(\sqrt{\frac{g}{L}}t+\varphi\right)+\frac{R\omega^2}{\left(g-L\omega^2\right)}\cos\omega t$.

The Schrodinger equation for an electron of mass M in a spherically symmetric potential $V(r)$ can be written as

$$\left[-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)+\frac{1}{2Mr^2}L^2+V(r)\right]\psi(r)=E\psi(r)$$

where L is the angular momentum operator. Throughout we ignore the spin of the electron.

(a) Consider the hydrogen atom with the potential :

$$V(r)=-\frac{e^2}{r}.$$

Assume that ψ is an s-wavefunction such that,

$$\psi(r)=NR(r).$$

for a radial component $R(r)=\exp(-\beta r)$ with N being the normalization constant. Solve for the energy eigenvalue E and evaluate β in terms of $\hbar$, M and e.

(b) An electron is in the ground state of tritium which has a nucleus with one proton, one electron and two neutrons. A nuclear reaction (e.g. a beta particle is ejected) instantaneously changes tritium to a He^3 ion which has two protons, one electron and one neutron. Assume the ejected beta particle does not have time to interact with the existing electron before leaving (i.e. there are no e-e interactions in this problem).

Calculate the numerical probability that the electron is in the ground state of the resulting He^3 ion after this transition.

Possibly useful integrals: $\int_0^\infty r^2\exp(-\gamma r)dr = 2/\gamma^3$ or alternatively $\int_0^\infty x^n\exp(-x)dx = n!$, all information other is given in the problem, final result can be expressed as a *numerical* fraction or decimal probability.

The Schrodinger equation for an electron of mass M in a spherically symmetric potential $V(r)$ can be written as

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Solution:
First the normalization coefficient, N is calculated to be $N=\sqrt{\beta^3/\pi}$, i.e.

$$<\psi|\psi>=4\pi N^2\int r^2e^{-2\beta r}dr=4\pi N^2\frac{2!}{(2\beta)^3}=\frac{\pi N^2}{\beta^3}=1 \tag{54}$$

Next, sub the trial wavefunction suggested into the energy eigenvalue equation, $H|\psi\rangle=E|\psi\rangle$. (Note the wavefunction is purely radial s-orbital, so angular momentum components can be ignored, $l=0$)

$$H|\psi\rangle=\left[-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)\right]N\exp(-\beta r)-\frac{e^2}{r}N\exp(-\beta r) \tag{55}$$

$$=\left[-\frac{\hbar^2}{2M}\left(\beta^2-\frac{2\beta}{r}\right)-\frac{e^2}{r}\right]Ne^{-\beta r} \tag{56}$$

From here there are two options. (1) you can claim that the trial wavefunction given is known to be an exact solution (e.g. the 1s hydrogen atom) to the energy eigenvalue equation, yielding quantized energy levels. Consequently, the ground state energy must be independent of r, which is true if $\beta=\frac{Me^2}{\hbar^2}$. The energy eigenvalue equation may then be solved giving $E=-\frac{\hbar^2}{2M}\beta^2$.

(2) A better solution is obtained using the variational principle, as we do not need to assume (*apriori*) the radial wavefunction gives an exact solution. Plugging into the variational theorem and using the integrals given we get,

$$E \leq \frac{\langle\psi|\, H\, |\psi\rangle}{<\psi|\psi>} \tag{57}$$

$$= 4\pi(\beta^3/\pi) \int r^2 \exp\left(-2\beta r\right) \left[-\frac{\hbar^2}{2M}\left(\beta^2 - \frac{2\beta}{r}\right) - \frac{e^2}{r}\right] dr \tag{58}$$

$$= \frac{\hbar^2\beta^2}{2M} - e^2\beta \tag{59}$$

Minimizing the energy with respect to the variational parameter, β gives,

$$\frac{\partial}{\partial\beta}\left(\frac{\hbar^2\beta^2}{2M} - e^2\beta\right) = \frac{\hbar^2\beta}{M} - e^2 \tag{60}$$

$$= 0 \tag{61}$$

Hence, $\beta = \frac{Me^2}{\hbar^2}$ and $E = -\frac{\hbar^2}{2M}\beta^2$, which is precisely the ground state energy of the hydrogen atom, this worked because the trial wavefunction given was already in the algebraic form of a 1s hydrogenic wavefunction.

(b) An electron is in the ground state of tritium which has a nucleus with one proton, one electron and two neutrons. A nuclear reaction (e.g. a beta particle is ejected) instantaneously changes tritium to a He^3 ion which has two protons, one electron and one neutron. Assume the ejected beta particle does not have time to interact with the existing electron before leaving (i.e. there are no e-e interactions in this problem).

Calculate the numerical probability that the electron is in the ground state of the resulting He^3 ion after this transition.

Solution:
The sudden approximation may be used to find the probability the electron is in ground state of the He^3 ion after beta decay. i.e. the probability coefficient, a_f is approximated by:

$$a_f = \sum_i \langle f|\left(1 - \frac{t_o}{i\hbar}(H_i - H_1)\right)|i\rangle a_i \tag{62}$$

here, t_o is the time for the interaction and we can assume that $t_o \longrightarrow 0$, meaning the total transition probability coefficient simplifies to just the overlap integral between final and initial states, i.e.

$$a_{|1s\rangle} =< 1s_{He^3}|1s_{H^3} > \tag{63}$$

The total reaction may be written as $H^3 \rightarrow He^3 + e^- + v_e$. Consequently, the initial Hamiltonian is $H_o = -\frac{e^2}{r}$, the intermediate Hamilitionian is $H_i = -\frac{2e^2}{r} + \frac{e^2}{r}$ (i.e. an electron and two protons and an electron which has not yet escaped), and the final Hamiltonian of the He^3 ion, $H_1 = -\frac{2e^2}{r}$. Likewise accounting for the +2 nuclear charges of He^3 (and using the normalization condition found in part a), we have the respective wavefunctions $1s_{H^3} = \sqrt{\beta^3/\pi}\exp(-\beta r)$ and $1s_{He^3} = 2\sqrt{2\beta^3/\pi}\exp(-2\beta r)$. Solving the overlap integral we get,

$$a_{|1s\rangle} =< 1s_{He^3}|1s_{H^3} > \tag{64}$$

$$= (\sqrt{\beta^3/\pi})(2\sqrt{2\beta^3/\pi}) \int \exp(-\beta r)\exp(-2\beta r) r^2 dr d\Omega \tag{65}$$

$$= 4\pi 2\sqrt{2}\frac{\beta^3}{\pi} \int r^2 \exp(-3\beta r) dr \tag{66}$$

$$= 8\sqrt{2}\beta^3 \frac{2!}{(3\beta)^3} \tag{67}$$

$$= \frac{16\sqrt{2}}{27} \tag{68}$$

Lastly, we convert the overlap integral result into a probability, P (numerical fraction need not be evaluated)

$$P = |a_{|1s\rangle}|^2 \tag{69}$$

$$= \frac{2 \times 16^2}{27^2} \tag{70}$$

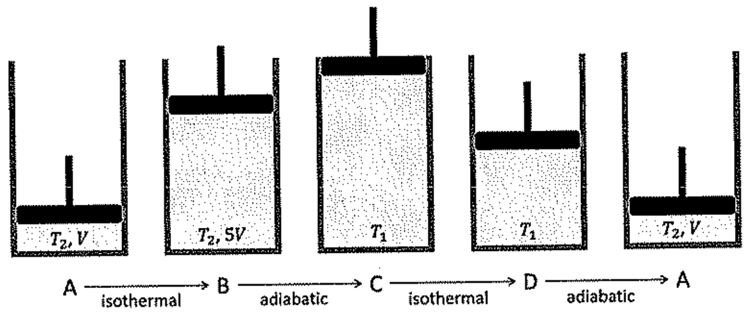
$$= \frac{512}{729} \tag{71}$$

$$=\sim 0.7 \tag{72}$$

Hence, there is a $\sim$ 70% probability our atom is still in it still in the ground state after this nuclear reaction (i.e. a $\sim$ 30% chance you'll find the He^3 ion in an excited state).

Possibly useful integrals: $\int_0^\infty r^2 \exp(-\gamma r) dr = 2/\gamma^3$ or alternatively $\int_0^\infty x^n \exp(-x) dx = n!$, all information other is given in the problem, final result can be expressed as a *numerical* fraction or decimal probability.

One mole of an ideal monoatomic gas performs a Carnot cycle between the temperatures T_2 and T_1 ($T_2 > T_1$). On the upper isothermal transformation ($A \to B$), the initial volume is V and the final volume is $5V$.



(a) Obtain the volume at the states C and D using the first law for an infinitesimal transformation of the system, $dQ = C_v dT + pdV$, where $C_v = \frac{3}{2}R$ is the molar heat capacity of the system.

(b) What are the amounts of heat (Q_1 and Q_2) exchanged with the two sources at T_1 and T_2?

(c) Show that

$$\frac{Q_2}{Q_1} = \frac{T_2}{T_1}$$

(d) What is the work performed during one cycle?

(e) What is the change in entropy during the process: (i) $A \to B$, (ii) $B \to C$, (iii)$C \to D$, and (iv) $D \to A$?

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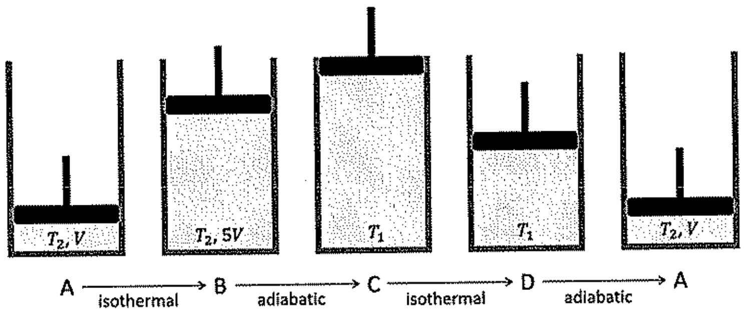

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Solution:
During the adiabatic processes $B \rightarrow C$ and $D \rightarrow A$, $dQ = 0$, thus

$$\begin{aligned} & C_v dT + pdV = \frac{3}{2}RdT + \frac{RT}{V}dV = 0 \\ \Rightarrow \quad & \frac{dT}{T} + \frac{2}{3}\frac{dV}{V} = 0 \Rightarrow \ln T + \frac{2}{3}\ln V = 0 \\ \Rightarrow \quad & \ln(TV^{2/3}) = \textit{constant} \\ \Rightarrow \quad & TV^{2/3} = \textit{constant} \end{aligned}$$

Therefore, at B and C,

$$T_2(5V)^{2/3} = T_1 V_c^{2/3} \Rightarrow V_c^{3/2} = \frac{T_2}{T_1}(5V)^{3/2} \Rightarrow V_C = 5\left(\frac{T_2}{T_1}\right)^{3/2} V$$

and at D and A,

$$T_1 V_D^{2/3} = T_2 V^{2/3} \Rightarrow V_D = \left(\frac{T_2}{T_1}\right)^{3/2} V$$

(b) What are the amounts of heat (Q_1 and Q_2) exchanged with the two sources at T_1 and T_2?

Solution:
For the isothermal process at $T = T_2$, the heat exchange Q_2 must be equal to the work done by the system:

$$Q_2 = W_{AB} = \int_{V_A}^{V_B} pdV' = RT_2 \int_{V}^{5V} \frac{1}{V'}dV' = RT_2 \ln 5$$

For the isothermal process at $T = T_1$,

$$Q_1 = W_{DC} = \int_{V_D}^{V_C} pdV' = RT_1 \int_{V_D}^{V_C} \frac{1}{V'} dV' = RT_1 \ln\left(\frac{V_C}{V_D}\right) = RT_1 \ln 5,$$

where we obtain $\frac{V_C}{V_D} = 5$ from the answers in (a).

(c) Show that

$$\frac{Q_2}{Q_1} = \frac{T_2}{T_1}$$

Solution:

$$\frac{Q_2}{Q_1} = \frac{RT_2 \ln 5}{RT_1 \ln 5} = \frac{T_2}{T_1}$$

(d) What is the work performed during one cycle?

Solution:
The work is $W = Q_2 - Q_1 = R(T_2 - T_1) \ln 5$

(e) What is the change in entropy during the process: (i) $A \to B$, (ii) $B \to C$, (iii)$C \to D$, and (iv) $D \to A$?

Solution:
For the isothermal processes, (i) $A \to B$ and (iii)$C \to D$,

$$\Delta S_2 = \frac{Q_2}{T_2} = R \ln 5 \text{ and } \Delta S_1 = \frac{-Q_1}{T_1} = -R \ln 5$$

For the adiabatic processes (ii) $B \to C$ and (iv) $D \to A$, no heat is exchanged: $\Delta S = 0$.

Consider a standing electromagnetic wave between two parallel plates of (perfect) metal, separated by a distance h, with vacuum separating them. At $t = 0$, the electric field is given by

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}, t = 0) = \begin{cases} E_0 \sin(kx)\hat{\mathbf{z}} & -h/2 < z < h/2 \\ 0 & \text{otherwise} \end{cases}$$

and the magnetic field is zero everywhere (at $t = 0$).

(a) Given the above initial conditions, solve for the electric and magnetic field everywhere in space as a function of both time and position.

(b) Find the surface charge density $\sigma(\vec{\mathbf{r}}, t)$ and surface current density $K(\vec{\mathbf{r}}, t)$ on the metal plates.

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Solution:

(a) *Note: I will be solving this using Gaussian units, which means that magnetic and electric field will have the same dimensions.*

We know from Maxwell's equation that

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{1}{c}\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{73}$$

from which we can conclude that at $t = 0$:

$$\left.\frac{\partial \vec{\mathbf{B}}}{\partial t}\right|_{t=0} = \begin{cases} ckE_0 \cos(kx)\hat{\mathbf{y}} & -h/2 < z < h/2 \\ 0 & \text{otherwise} \end{cases} \tag{74}$$

The tricky part here is getting the sign right, which requires that we remember that $\hat{x} \times \hat{\mathbf{z}} = -\hat{y}$. It's not *that* tricky.

Now we do the same to find the time derivative of the electric field at $t = 0$, ignoring for the moment any surface currents:

$$\vec{\nabla} \times \vec{\mathbf{B}} = \frac{1}{c}\frac{\partial \vec{\mathbf{E}}}{\partial t} \tag{75}$$

The curl of the magnetic field is zero (as is the magnetic field), so this tells us that at $t = 0$, the electric field is not changing. This looks like a standing wave. A quick glance at the Poynting vector shows us that no energy is propogating at $t = 0$, which rules out a traveling wave. We expect sinusoidal time dependence for both electric and magnetic field, and the time derivatives above tell us that the electric field will vary as $\cos(\omega t)$, and the magnetic field will vary as $\sin(\omega t)$. Thus we guess that:

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}, t) = \begin{cases} E_0 \sin(kx)\cos(\omega t)\hat{\mathbf{z}} & -h/2 < z < h/2 \\ 0 & \text{otherwise} \end{cases} \tag{76}$$

where we will need to solve for ω. For the magnetic field, we guess

$$\vec{\mathbf{B}}(\vec{\mathbf{r}}, t) = \begin{cases} B_0 \cos(kx)\sin(\omega t)\hat{\mathbf{z}} & -h/2 < z < h/2 \\ 0 & \text{otherwise} \end{cases} \tag{77}$$

where we will need to solve for B_0. At this stage, a quick look at Eq. 74 tells us that

$$ckE_0 = \omega B_0 \tag{78}$$

Now we can evaluate Maxwell's equations at an arbitrary time, and solve for both B_0 and ω (which hopefully you can already guess), and at the same time prove to ourselves that we have found the solution to Maxwell's equations with the specified initial conditions. Applying the curl $\vec{\mathbf{E}}$ equation and taking a direct derivative, we find that

$$\frac{\partial\vec{\mathbf{B}}}{\partial t} = \begin{cases} ckE_0\cos(kx)\cos(\omega t)\hat{\mathbf{y}} & -h/2 < z < h/2 \\ 0 & \text{otherwise} \end{cases} \tag{79}$$

$$= \begin{cases} \omega B_0\cos(kx)\cos(\omega t)\hat{\mathbf{y}} & -h/2 < z < h/2 \\ 0 & \text{otherwise} \end{cases} \tag{80}$$

This is promising, but doesn't help us solve for ω or B_0. This is not surprising, since we already used this equation effectively. Let's now look at the $\vec{\nabla}\times\vec{\mathbf{B}}$ equation, which gives us the time derivative of $\vec{\mathbf{B}}$ (which was trivial at $t = 0$) and connect that with a directly-computed derivative:

$$\frac{\partial\vec{\mathbf{E}}}{\partial t} = \begin{cases} -ckB_0\sin(kx)\sin(\omega t)\hat{\mathbf{z}} & -h/2 < z < h/2 \\ 0 & \text{otherwise} \end{cases} \tag{81}$$

$$= \begin{cases} -\omega E_0\sin(kx)\sin(\omega t)\hat{\mathbf{z}} & -h/2 < z < h/2 \\ 0 & \text{otherwise} \end{cases} \tag{82}$$

When taking this curl, there were three minus signs different from $\vec{\nabla}\times\vec{\mathbf{E}}$: one from the fact that we're taking an x derivative of $\cos(kx)$ rather than $\sin(kx)$, one from the different directions ($\hat{\mathbf{y}}$ rather than $\hat{\mathbf{z}}$), and one from the fact that the $\vec{\nabla}\times\vec{\mathbf{E}}$ Maxwell equation differs from the $\vec{\nabla}\times\vec{\mathbf{B}}$ equation by a sign. We can now see that

$$ckB_0 = \omega E_0 \tag{83}$$

$$ckE_0 = \omega B_0 \tag{84}$$

$$\omega = ck \tag{85}$$

$$B_0 = E_0 \tag{86}$$

which we learn is true for an electromagnetic wave in a vacuum.

(b) We now seek the surface charge density $\sigma(\vec{\mathbf{r}},t)$ and the surface current density $\vec{\mathbf{K}}(\vec{\mathbf{r}},t)$. We will do this by making use of Gauss's law and Ampere's law (which is to say, the integral versions of the $\vec{\nabla}\cdot\vec{\mathbf{E}}$ and $\vec{\nabla}\times\vec{\mathbf{B}}$ Maxwell equations).

To solve for the charge density, we use

$$\vec{\nabla}\cdot\vec{\mathbf{E}} = 4\pi\rho \tag{87}$$

and consider a tiny pill box around the surface. The charge enclosed is σdA, and the flux out of the surface is $E_z dA$ if the pill box is on the lower metal surface, and it is $-E_z dA$ if the pill box is on the upper metal surface. Thus we have

$$4\pi\sigma dA = \begin{cases} E_0 \sin(kx)\cos(\omega t) dA & z = -h/2 \\ -E_0 \sin(kx)\cos(\omega t) dA & z = h/2 \end{cases} \tag{88}$$

$$\sigma(\vec{\mathbf{r}}, t) = \begin{cases} \frac{E_0}{4\pi} \sin(kx)\cos(\omega t) & z = -h/2 \\ -\frac{E_0}{4\pi} \sin(kx)\cos(\omega t) & z = h/2 \end{cases} \tag{89}$$

The opposite signs make sense, since our metal plates are behaving like a capacitor. It also makes sense from the perspective of field likes beginning and ending at positive and negative charges.

The current density The current density is only slightly more exciting (because it is a vector quantity). The Maxwell equation in question is

$$\vec{\nabla} \times \vec{\mathbf{B}} = \frac{1}{c}\frac{\partial \vec{\mathbf{E}}}{\partial t} + \frac{4\pi}{c}\vec{\mathbf{J}} \tag{90}$$

We've already established that this equation is satisfied in the vacuum. Now we draw a very small rectangular loop that goes along the surface in the yz plane. The magnitude of the integral of $\vec{\mathbf{B}}$ around the loop is given by $B_y dy$. The sign of this integral, of course, is affected by which way we perform the integral, and we'll discuss that momentarily. The $\frac{\partial \vec{\mathbf{E}}}{\partial t}$ term does not contribute if we make $dz \ll dy$, since the magnitude of the current through the loop is given by $I = K dy$. Putting this together, we find that

$$B_y dy = \pm \frac{4\pi}{c} K dy \tag{91}$$

$$\vec{\mathbf{K}} = \pm \frac{c}{4\pi} E_0 \cos(kx)\sin(\omega t)\hat{\mathbf{x}} \tag{92}$$

where the $\pm$ reflects the relative sign of the two terms, and is opposite on the upper and lower metal surfaces (since $\vec{\mathbf{B}}$ is nonzero on opposite sides of the loop in each case).

Right hand rule To get the sign right, we apply the right hand rule to relate the direction around our loop to the normal of the surface integral used in the current flux. For convenience, let us define the flux to be positive if $\vec{\mathbf{K}}$ is in the $+\hat{\mathbf{x}}$ direction. In this case, when we are examining the upper metal surface (so the $\vec{\mathbf{B}}$ field is nonzero on the lower edge of the loop), when $\vec{\mathbf{B}}$ is in the $\hat{\mathbf{y}}$ direction we have a positive loop integral. So we find that for $z = h/2$

$$B_y dy = +\frac{4\pi}{c} K_x dy \tag{93}$$

$$\vec{\mathbf{K}} = +\frac{c}{4\pi} E_0 \cos(kx)\sin(\omega t) \tag{94}$$

Recognizing that on the bottom the loop integral changes sign while the flux integral does not, we find that

$$\vec{\mathbf{K}}(\vec{\mathbf{r}}, t) = \begin{cases} \frac{cE_0}{4\pi} \cos(kx)\sin(\omega t) & z = -h/2 \\ -\frac{cE_0}{4\pi} \cos(kx)\sin(\omega t) & z = h/2 \end{cases} \tag{95}$$

Checking charge conservation We can check the consistency of our solution by verifying that it satisfies charge conservation. In 3D terms, this means that

$$\vec{\nabla} \cdot \vec{\mathbf{J}} = -\frac{\partial \rho}{\partial t} \tag{96}$$

This constraint tells us for the bottom surface that

$$\vec{\nabla} \cdot \vec{\mathbf{K}} = -\frac{\partial \sigma}{\partial t} \tag{97}$$

$$-k\frac{cE_0}{4\pi}\cos(kx)\cos(\omega t) = -\omega\frac{E_0}{4\pi}\cos(kx)\cos(\omega t) \tag{98}$$

$$-kc = -\omega \tag{99}$$

which is true (given our formula for ω), indicating that we are indeed conserving charge. If we had gotten a sign wrong on one of our calculations above, this would have come out wrong.