OSU Physics Department
Comprehensive Examination #117

Monday, September 30 and Tuesday, October 1, 2013

Fall 2013 Comprehensive Examination

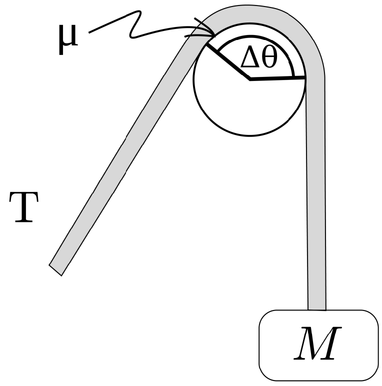
PART 1, Monday, September 30, 9:00am

General Instructions

This Fall 2013 Comprehensive Examination consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, September 30, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:00 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, October 1, at 9:00 am and 1:00 pm, respectively. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please return all bluebooks and formula sheets at the end of the exam. Use the last pages of your bluebooks for "scratch" work, separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Consider a rope wrapped around a cylindrical bar, as shown in the figure below.



(a) Solve for the minimum tension T needed on the left-hand side of the figure in order to prevent the mass M from slipping down. You may assume that the static coefficient of friction μ is the same as the dynamic coefficient of friction.

(b) Estimate how many times you would need to wrap a rope around the bar in order to hold up a car. You may assume a coefficient of friction of 1.0. Make a reasonable estimate for the weight of a car.

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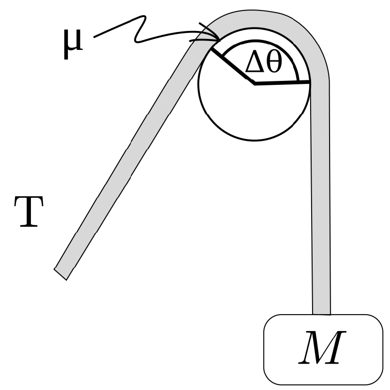
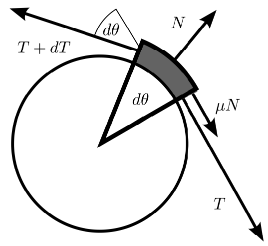


(a) Solve for the minimum tension T needed on the left-hand side of the figure in order to prevent the mass M from slipping down. You may assume that the static coefficient of friction μ is the same as the dynamic coefficient of friction.

Solution:
We know that the frictional force is proportional to the normal force, so we need to begin by finding the normal force, which will naturally be a function of angle, as is the tension. We just need to do some geometry in order to see how they are related.



Note that I have assumed (as instructed) that the friction is at its maximum force in order to find the maximum tension that can be supported. By considering that the net force on this small piece of rope must be zero, we can see that

$$N = T \sin d\theta \tag{1}$$
$$= T d\theta \tag{2}$$

$$T + dT = T - \mu N \tag{3}$$
$$= T - \mu T d\theta \tag{4}$$
$$\frac{dT}{T} = -\mu d\theta \tag{5}$$

Now that we've found a formula for how much the tension changes with angle, we just need to integrate in order to find the tension difference for a finite angle θ.

$$\int_{T(0)}^{T(\theta)} \frac{dT}{T} = -\int_0^\theta \mu d\theta \tag{6}$$

$$\ln\left(\frac{T(\theta)}{T(0)}\right) = -\mu\theta \tag{7}$$

$$T(\theta) = T(0)e^{-\mu\theta} \tag{8}$$

This tells us that the minimum tension exponentially decays with the angle of contact the rope has with the bar:

$$T(\Delta\theta) = Mge^{-\mu\Delta\theta} \tag{9}$$

(b) Estimate how many times you would need to wrap a rope around the bar in order to hold up a car. You may assume a coefficient of friction of 1.0. Make a reasonable estimate for the weight of a car.

Solution:
My car weighs about 3000 lb., and I weight about 150 lb, which is 20 times less. So keeping mind that $\mu \approx 1$ I need to find

$$20 = e^\theta \tag{10}$$

$$\theta = \ln 20 \tag{11}$$

Since $e \approx 2$ (playing it safe), we need $\theta \approx 5$, which would give us a ratio of greater than 32, so I have a double margin. Since $5 < 2\pi$, wrapping the rope around the bar one full time should be plenty to allow me to hold up my car.

Consider a single particle in a 1D simple harmonic potential.

(a) i. Write out the full time-dependent Schrodinger equation with the appropriate potential for this system in the Hamiltonian.

ii. Use separation of variables (e.g. $\Psi(x,t) = \psi(x)\phi(t)$) to write an energy eigenvalue equation. Finally, re-express the resulting time-independent Schrodinger equation in terms of raising and lowering operators.

iii. What are the energies for the ground, 1st and 2nd excited states? (*simply state them, do not solve*)

iv. Why is the ground energy state not zero? Explain qualitatively the origin of this zero-point energy.

(b) **Anharmonic SHO.** Suppose our simple harmonic oscillator is subjected to an anharmoinc perturbation of the form $H_1 = \lambda x^4$. Evaluate the first order energy correction to the n^{th} state. (*show all work*)

(c) **Kicked SHO.** Suppose instead a quantum simple harmonic oscillator is prepared in ground state at time $t = 0$. **Our oscillator now has a weak constant electric perturbation,** $H_1 = -eE_o x$ **that acts only over a time period** $0 < t < T$**.** The applied perturbation is zero otherwise.

i) Use first-order time-dependent perturbation theory to calculate the probability of making a transition to the first excited state as $t \to \infty$. (*show all work*)

ii) Simplify your calculated transition probability $P_{0\to1}$ to show for what values of T is the transition maximal (assume a constant amplitude E_o and fixed frequency ω).

iii) What does first order perturbation theory tell us is the probability of the transition from states $n = 0$ to $n = 2$?

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Consider a single particle in a 1D simple harmonic potential.

a) **SHO**
i)
$i\hbar \frac{d\Psi}{dt} = H\Psi,$

separation of variables, $\Psi(x,t) = \psi(x)\varphi(t)$, $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$

$i\hbar\,\psi(x)\frac{d\varphi(t)}{dt} = \varphi(t)H\psi(x) \Rightarrow \frac{i\hbar}{\varphi(t)}\frac{d\varphi(t)}{dt} = \frac{1}{\psi(x)}H\psi(x) \Rightarrow \textit{\textbf{set E}} = \frac{1}{\psi(x)}H\psi(x)$

or $H\psi(x) = E\psi(x)$

i.e. $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi = E\psi$

Expressing in terms of raising and lowering operators,

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right)(a + a_+)^2\psi = E\psi$ (accepted answer #1)

OR $\hbar\omega\left(aa_+ + \frac{1}{2}\right)\psi = E\psi$ (accepted answer #2)

ii) What are energies for the ground, 1st and 2nd excited states? (*simply state them, do not solve*)

$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \Rightarrow \frac{\hbar\omega}{2}, \frac{3\hbar\omega}{2}, \frac{5\hbar\omega}{2}$

iii) Why is the ground energy state not zero? Explain qualitatively the origin of this zero-point energy.

(any reasonable physical discussion involving the uncertatinty principle)

b) **Anharmonic SHO**.

Suppose our simple harmonic oscillator is subjected to an anharmoinc perturbation of the form $H_1 = \lambda x^4$.
Evaluate the first order energy correction to the *n*th state.

In the unperturbed problem, $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$

Using first-order perturbation theory,

$E_o^1 = \langle n|\lambda x^4|\mathrm{n}\rangle = \lambda\left(\sqrt{\frac{\hbar}{2M\omega}}\right)^4 \langle \mathrm{n}|(a_+ + a)^4|\mathrm{n}\rangle$ by othogonality all unequal raising/lowering operations give 0.

So we can simplify the expansion as,

$$E_o^1 = \langle n|\lambda x^4|\mathrm{n}\rangle = \lambda\left(\sqrt{\frac{\hbar}{2M\omega}}\right)^4 < \mathrm{n}\left| a_+a_+aa + aaa_+a_+ + a_+aa_+a + aa_+aa_+ + \mathrm{a}\,\mathrm{a}_+a_+\mathrm{a} + a_+\mathrm{a}\mathrm{a}a_+ \right) \middle| \mathrm{n}\right\rangle$$

$$=\lambda\left(\sqrt{\frac{\hbar}{2M\omega}}\right)^4\left((n+1)(n+2)+(n+1)^2+2n(n+1)+n^2+n(n-1)\right)$$

$$\Rightarrow E_o^1=\frac{3\lambda\hbar^2}{4M^2\omega^2}\left(2n^2+2n+1\right)$$

c) **Kicked SHO**.

i) If this osillator is now perturbed by a weak (transient) electric field, $V(t)=-eE_o x$ what is the probablity of finding it in the first excited state at $t=\infty$?

Use 1st order time-depedent perturbation theory, $P_{o\to1}\approx|c_1(t)|^2$

$$c_1(t)=-\frac{i}{\hbar}\int_{t_o}^{t}e^{i\omega t'}V_{10}(t')\,dt,\quad \text{where } V_{10}(t')=-eE_o<1|x|0>\quad\text{and } \omega=\left(E_1-E_o\right)/\hbar$$

Using ladder operators,

$$\langle1|x|0\rangle=\sqrt{\frac{\hbar}{2m\omega}}<0|a(a+a^+)|0>=\sqrt{\frac{\hbar}{2m\omega}}$$

$$c_1(t)=-\frac{i}{\hbar}eE_o\sqrt{\frac{\hbar}{2m\omega}}\int_0^T e^{i\omega_{10}t'}dt=-\frac{1}{\hbar\omega}eE_o\sqrt{\frac{\hbar}{2m\omega}}\left(\exp(i\omega T)-1\right)$$

$$c_1(t)=-\frac{eE_o}{\hbar\omega}\sqrt{\frac{\hbar}{2m\omega}}\exp\left(\frac{i\omega T}{2}\right)\left(\exp\left(\frac{i\omega T}{2}\right)-\exp\left(-\frac{i\omega T}{2}\right)\right)$$

$$c_1(t)=-\frac{eE_o}{\hbar\omega}\sqrt{\frac{\hbar}{2m\omega}}\exp\left(\frac{i\omega T}{2}\right)2i\left(\sin\left(\frac{\omega T}{2}\right)\right)$$

$$\boldsymbol{P_{o\to1}\approx|c_1(t)|^2=\frac{e^2E_o^2}{m\hbar\omega^3}\sin^2\left(\frac{\omega T}{2}\right)}$$

ii) Inspect the above transition probability $P_{0\to1}$, for value of ω where $P_{0\to1}$ is maximal.

iii) The first order transition probability is zero.

Consider, $E^1=\langle2|\left(a+a_+\right)|0\rangle$. This is always zero by orthogonality. Note, it is not suffient to assume it is zero because it is first order perturbation theory, as one could invent a perturbation where it is non-zero.

A metal has N conduction electrons per unit volume. The mean number of particles in a state of energy $\epsilon = \frac{\hbar^2 k^2}{2m}$ at temperature T is given by the Fermi-Dirac distribution

$$f(\epsilon) = \frac{1}{\exp[(\epsilon - \mu)/k_B T] + 1}$$

(a) Find the "Fermi energy" or the chemical potential μ_0 at the absolute zero temperature, $T = 0$. Note that one electron state exclusively occupies the k-space volume of $(2\pi)^3/V$, where V is the volume of the electron system in the real space.

(b) Find the mean energy E of the electron gas at $T = 0$. What is the electron gas pressure P?

(c) The Fermi energy is only slightly different from μ_0, i.e., $\mu \cong \mu_0 \gg k_B T$, at room temperature. Estimate the heat capacity of the metal per unit volume at T.

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Solution:
The chemical potential μ_0 is determined by the particle number conservation. Since the total number of electrons N is the sum over all levels of the mean number in each level,

$$N = \sum_i f(\epsilon_i) = \sum_i \frac{1}{\exp[(\epsilon_i - \mu)/k_B T] + 1}$$

In a gas of free and independent electrons, energies are specified by the wave vector $\mathbf{k}$,

$$\epsilon(\mathbf{k}) = \frac{\hbar^2 k^2}{2m},$$

where each energy level includes the two spin states. Thus, the sum over the energy levels can be transformed to the integration over the wave vector:

$$N = 2 \int \frac{V d^3\mathbf{k}}{(2\pi)^3} f\left(\epsilon(\mathbf{k})\right).$$

Since the integrand depends on $\mathbf{k}$ only through the energy ϵ, we evaluate the integral in spherical coordinates and change variables from k to ϵ:

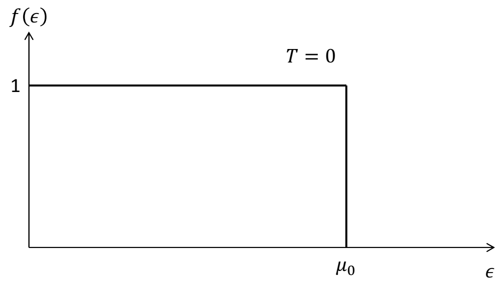
$$N = V \int_0^\infty f\left(\epsilon(\mathbf{k})\right) \frac{k^2 dk}{\pi^2} = V \int_\infty^\infty f(\epsilon) g(\epsilon) d\epsilon,$$

where the density of energy levels per unit volume

$$g(\epsilon) = \begin{cases} \frac{m}{\hbar^2 \pi^2} \sqrt{\frac{2m\epsilon}{\hbar^2}} & \text{for } \epsilon > 0 \\ 0 & \text{for } \epsilon < 0 \end{cases}$$

At $T = 0$, the Fermi-Dirac distribution is

$$f(\epsilon) = \begin{cases} 1 & \text{for } \epsilon \leq \mu_0 \\ 0 & \text{for } \epsilon > \mu_0 \end{cases} \tag{12}$$



Then,

$$N = V \int_0^{\mu_0} \frac{\sqrt{2}m^{3/2}}{\hbar^3\pi^2}\sqrt{\epsilon}d\epsilon = \frac{2\sqrt{2}m^{3/2}}{3\hbar^3\pi^2}\mu_0^{3/2}$$

Thus,

$$\mu_0 = \frac{\hbar^2}{2m}(3\pi^2 N/V)^{2/3} = \frac{\hbar^2}{2m}(3\pi^2 N)^{2/3} \text{ at } V = 1.$$

(b) Find the mean energy E of the electron gas at $T = 0$. What is the electron gas pressure P?

Solution:
The mean energy is

$$E = V \int_{\infty}^{\infty} \epsilon f(\epsilon)g(\epsilon)d\epsilon = V \int_0^{\mu_0} \frac{\sqrt{2}m^{3/2}}{\hbar^3\pi^2}\epsilon^{3/2}d\epsilon = \frac{V}{5\pi^2}\frac{2m^{3/2}}{\hbar^2}\mu_0^{5/2}$$

Therefore,

$$E = \frac{2m^{3/2}}{5\pi^2\hbar^2}\mu_0^{5/2} \text{ at } V = 1.$$

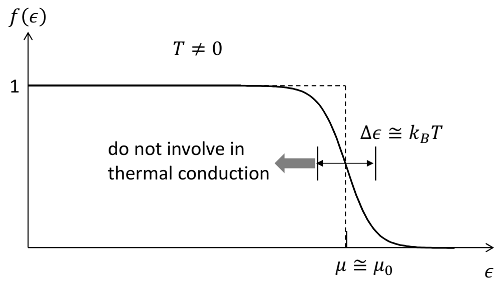
The pressure is

$$P = \left(\frac{\partial E}{\partial V}\right)_T = \frac{2m^{3/2}}{5\pi^2\hbar^2}\mu_0^{5/2}$$

(c) The Fermi energy is only slightly different from μ_0, i.e., $\mu \cong \mu_0 \gg k_B T$, at room temperature. Estimate the heat capacity of the metal per unit volume at T.

Solution:
Since $\mu \cong \mu_0 \gg k_B T$ at room temperature, the Fermi-Dirac distribution is

$$f(\epsilon) \cong \begin{cases} 1 & \text{for } \epsilon \ll \mu_0 \\ 0 & \text{for } \epsilon \gg \mu_0 \end{cases} \tag{13}$$



As shown in the figure, $f(\epsilon)$ varies between 1 and 0 only if $\Delta\epsilon = |\epsilon - \mu_0| < k_B T$. This means that only the electrons within the energy range can be thermally excited and participate in thermal transport. The effective number of electrons involving the heat conduction is estimated as

$$\Delta N \approx g(\mu)\Delta\epsilon \approx \frac{k_B T}{\mu_0} N$$

Since the heat capacitance per a free particle is $\frac{3}{2}k_B$, the heat capacitance for the N-particle system is estimated as

$$C_V \approx \frac{3}{2} k_B \Delta N \approx \frac{3Nk_B^2 T}{2\mu_0}$$

Four identical charges Q are at a distance R from the origin. They are positioned on the corners of a regular tetrahedron, their positions are given by $\vec{R}_i$ for $i = 1, 2, 3, 4$. Remember that a regular tetrahedron is formed by four of the eight corners of a cube. We will make use of the expansion

$$\frac{1}{|\,\vec{r} - \vec{r}\,'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{{r_<}^{l}}{{r_>}^{l+1}} Y_{lm}(\hat{r}) Y^*_{lm}(\hat{r}')$$

You will use this expansion to study the potential $V(\vec{r})$ near the origin, when we have $r \ll R$.

(a) What is the potential at the origin?

(b) For this particular charge distribution, how do the coefficients in the small r expansion relate to the multipole moments one uses in the large r expansion?

(c) What is the electric field at the origin and how does it relate to the expansion you found?

(d) The quadrupole moment is best expressed in Cartesian coordinates via $Q_{\mu,\nu} = \sum_i \left(3(\vec{R}_i)_\mu (\vec{R}_i)_\nu - \delta_{\mu,\nu} |\vec{R}_i|^2\right)$ where $\mu, \nu = x, y, z$ labels the components. Evaluate the quadrupole moment in this form, and use it to find the electric field gradient at the origin.

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Solution:

At the origin the contribution to the potential from all four charges is the same, so we have

$$V(\vec{0}) = 4 \frac{1}{4\pi\epsilon_0} \frac{Q}{R}$$

In general we have

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{|\vec{r} - \vec{r}\,'|} \rho(\vec{r}\,') d^3r'$$

and with our expansion this yields for $r \ll R$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \sum_{lm} \frac{4\pi}{2l+1} \frac{{r_<}^l}{{r_>}^{l+1}} Y_{lm}(\hat{r}) Y^*_{lm}(\hat{r}') \rho(\vec{r}\,') d^3r'$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} Y_{lm}(\hat{r}) \int \frac{r^l}{R^{l+1}} Y^*_{lm}(\hat{r}') \rho(\vec{r}\,') d^3r'$$

With point charges at positions $\vec{R}_i$, all a distance R from the origin, we have

$$V(\vec{r}) = \frac{Q}{\epsilon_0} \sum_{lm} \frac{r^l}{2l+1} Y_{lm}(\hat{r}) \frac{1}{R^{\,l+1}} \sum_i Y^*_{lm}(\hat{R}_i)$$

The expansion at large r looks identical, with the terms r^l and R^{l+1} interchanges. The multipoles in this case are

$$q_{lm} = QR^l \sum_i Y^*_{lm}(\hat{R}_i)$$

and hence we have for small r

$$V(\vec{r}) = \frac{1}{\epsilon_0} \sum_{lm} \frac{r^l}{2l+1} Y_{lm}(\hat{r}) \frac{1}{R^{\,2l+1}} q_{lm}$$

Including factors of 4π and/or $2l+1$ in the definitions of the multipoles is OK. In our case, due to the fact that all charges are at the same distance from the origin, the small r and large r expansion are closely related.

The electric field at the origin is related to the l=1 terms, and hence to the electric dipole moment of the system. In Cartesian coordinates this is

$$\vec{p} = Q \sum_i \vec{R}_i$$

This is zero. One can see this by using the fact that we can take half of the corners of a cube to make a tetrahedron, so we can represent the positions of the charges by

$$(a,a,a)\ ,\ (-a,-a,a)\ ,\ (a,-a,-a)\ ,\ (-a,a,-a)$$

One can also use physics, noting that a 120 degree rotation around a direction towards a charge produces the same tetrahedron. That means that the dipole of the system must point towards that charge. But since that charge is arbitrary, the dipole must point towards all charges, and only the zero vector does that.

Therefore, the electric field at the origin is zero.

The quadrupole moment can be calculated component by component, using the positions given above

$$Q_{x,x} = \sum_i \left(3(\vec{R}_i)^2_x - |\vec{R}_i|^2\right) = 0$$

$$Q_{x,y} = \sum_i 3(\vec{R}_i)_x(\vec{R}_i)_y = 0$$

and it is easy to show that they are all zero. There is no quadrupole moment for this system. Near the origin the second order term in the potential is proportional to

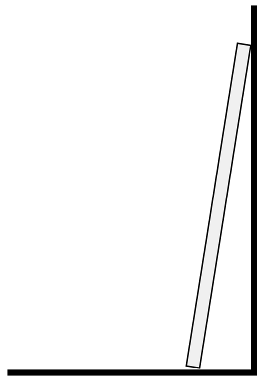
$$\sum_\mu \sum_\nu Q_{\mu,\nu} x_\mu x_\nu$$

this gives for the electric field

$$E_\mu \propto -\sum_\nu Q_{\mu,\nu} x_\nu$$

and hence the components of the field gradient at the origin are proportional to the quadrupole moment. Therefore, they are zero. At the center of our tetrahedron there is no field gradient, the system has too high a symmetry. One can also obtain this result by applying symmetry.

Consider a ladder leaning against a frictionless wall with its foot on a frictionless floor. The ladder is initially almost vertical when it begins to slip. At what height does the ladder lose contact with the wall?



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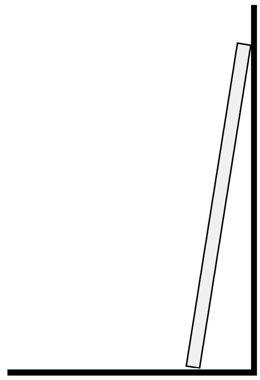


Solution:
This problem will require us to solve the problem in a number of steps. We'll begin by working out the kinetic and potential energies under the assumption that the ladder remains in contact with the wall. Then we can use energy conservation to find out how fast it is falling at each height. From this we can work out the net horizontal force on the ladder. When this force is zero is the point at which the ladder leaves contact with the wall.

Potential energy

$$V = Mg\frac{L}{2}\cos\theta \tag{14}$$

where θ is the angle of the ladder from the vertical, M is the mass of the ladder, and L is its length.

Kinetic energy

$$T_{\text{trans}} = \tfrac{1}{2}M(\dot{x}^2 + \dot{y}^2) \tag{15}$$

where x and y are the center of mass coordinates.

$$x = \tfrac{1}{2}L\sin\theta \tag{16}$$

$$\dot{x} = \tfrac{1}{2}L\cos\theta\dot{\theta} \tag{17}$$

$$y = \tfrac{1}{2}L\cos\theta \tag{18}$$

$$\dot{y} = -\tfrac{1}{2}L\sin\theta\dot{\theta} \tag{19}$$

Putting this together, we get

$$T_{\text{trans}} = \tfrac{1}{8}ML^2(\cos^2\theta\dot{\theta}^2 + \sin^2\theta\dot{\theta}^2) \tag{20}$$

$$= \tfrac{1}{8}ML^2\dot{\theta}^2 \tag{21}$$

The rotational kinetic energy is just a bit more work, since it requires us to get the moment of inertia of the ladder.

$$I = \int r^2 dM \tag{22}$$

$$= \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx \tag{23}$$

$$= \frac{2}{3L} M (L/2)^3 \tag{24}$$

$$= \tfrac{1}{12} M L^2 \tag{25}$$

Once we have the moment of inertial, the rotational kinetic energy is easy.

$$T_{\text{rot}} = \tfrac{1}{2} I \dot{\theta}^2 \tag{26}$$

$$= \tfrac{1}{24} M L^2 \dot{\theta}^2 \tag{27}$$

$$\tag{28}$$

Adding these together we get the total kinetic energy

$$T = \tfrac{1}{8} M L^2 \dot{\theta}^2 + \tfrac{1}{24} M L^2 \dot{\theta}^2 \tag{29}$$

$$= \tfrac{1}{6} M L^2 \dot{\theta}^2 \tag{30}$$

And there we have it.

Finding $\dot{\theta}$ Since there is no friction, energy is conserved, and we can conclude that

$$T = V(0) - V(\theta) \tag{31}$$

$$\tfrac{1}{6} M L^2 \dot{\theta}^2 = M g \frac{L}{2} (1 - \cos\theta) \tag{32}$$

$$\dot{\theta}^2 = 3 \frac{g}{L} (1 - \cos\theta) \tag{33}$$

$$\dot{\theta} = \sqrt{3 \frac{g}{L} (1 - \cos\theta)} \tag{34}$$

So there we have $\dot{\theta}$ as a function of θ. Alas, what we actually want is $\ddot{x}$ as a function of θ. Unfortunately (as we'll see below), that requires us to also solve for $\ddot{\theta}$.

Finding $\ddot{\theta}$ Recall that the Lagrangian is

$$L = T - V \tag{35}$$

$$= \tfrac{1}{6} M L^2 \dot{\theta}^2 - M g \frac{L}{2} \cos\theta \tag{36}$$

Finding $\ddot{\theta}$ is precisely what the Euler-Lagrange equation tends to give us.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \tag{37}$$

$$\tfrac{1}{3}ML^2\ddot{\theta} = \tfrac{1}{2}MgL\sin\theta \tag{38}$$

$$\ddot{\theta} = \frac{3}{2}\frac{g}{L}\sin\theta \tag{39}$$

Finding $\ddot{x}$

$$\ddot{x} = \tfrac{1}{2}L(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2) \tag{40}$$

$$= \tfrac{1}{2}L\left(\cos\theta\frac{3}{2}\frac{g}{L}\sin\theta - \sin\theta 3\frac{g}{L}(1-\cos\theta)\right) \tag{41}$$

$$= 0 \tag{42}$$

Finishing it To finish this off, we set $\ddot{x}$ to zero, to see when the normal force on the ladder from the wall switches to being attractive in order to force the constraint that they remain in contact. This is the point at which the ladder leaves the wall.

$$0 = \tfrac{1}{2}L\left(\cos\theta\frac{3}{2}\frac{g}{L}\sin\theta + \sin\theta 3\frac{g}{L}(\cos\theta - 1)\right) \tag{43}$$

$$= \left(\cos\theta\frac{1}{2} + (\cos\theta - 1)\right) \tag{44}$$

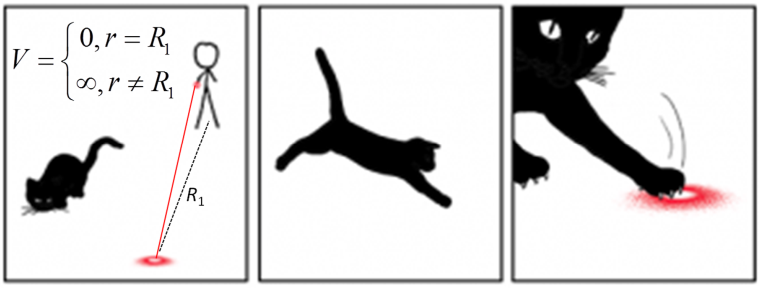
$$\tfrac{3}{2}\cos\theta = 1 \tag{45}$$

$$\cos\theta = \tfrac{2}{3} \tag{46}$$

$$y_{\text{top}} = L\cos\theta \tag{47}$$

$$= \tfrac{2}{3}L \tag{48}$$

So the ladder leaves the wall when its top is one-third of the way down from where it started.



(a) **Cat in a circle** Suppose Heisenberg had a house cat and a laser. One morning, Heisenberg decides to trace the laser spot in a perfect circle on his living room floor at a constant speed. Amazed to observe his cat follows the beam, he calls his friend Schrodinger to tell him about it. Schrodinger immediately inquires about the position and momentum of the cat. Heisenberg responds "you tell me"!

Scrodinger begins by approximating the cat as a point particle of mass, M that is perfectly confined to a circle of radius, R_1. What does Schrodinger tell Heisenberg for:

i) The normalized time-independent wavefunction of the cat (*show all work*).

ii) The allowed energy eigenvalues of the cat.

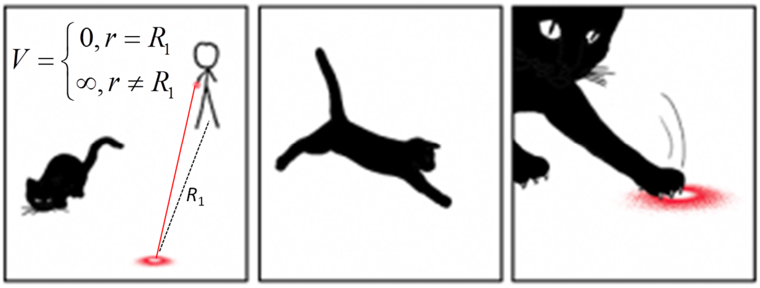
iii) Assuming the cat is confined to a circle, can Schrodinger simultaneously know the cat's radius, angle (θ) and angular momentum (L) with absolute certainty? Evaluate a commutation relation to justify your response.

(b) **Benzene as a circle** Now suppose Heisenberg fires his laser at benzene instead of a cat. Benzene is a ring-like molecule (C_6H_6) with a radius of 1.5×10^{-10} m and six π orbital electrons. Estimate what is the longest wavelength (or lowest energy) laser that Heisenberg needs to resonantly excite benzene (consider the Pauli exclusion principle).

(for any calculations take, $m_e = 1 \times 10^{-30}$ kg and $\hbar = 1 \times 10^{-34}$ Js).

(c) The actual absorption resonance for benzene is 260 nm. Can you explain to Schrodinger why his equations describe benzene so well, but the cat so poorly? (Include an estimate how well you could measure the position of a typical house cat using standard household instruments, and the corresponding minimum uncertainty in the velocity of the cat.)

Can you identify 1-2 other reasons why Schrodinger's analysis for both (i.) benzene and (ii.) the cat differ from experiment?



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A. Cat in a circle, condensed solutions.:

i) Chose a coordinate system. In polar coordinates, $\psi(r=R,\theta)=\psi(\theta)$

Consider the time-indendent Schrodinger equation ($H\psi=E\psi$)in polar coordintes at constant R:

$$-\frac{\hbar^2}{2M}\frac{1}{R^2}\frac{d^2\psi}{d\theta^2}=E\psi$$

Guess oscillating solution of form, $\psi(\theta)=Ae^{im\theta}$.

Apply boundary conditions $\psi(\theta=0)=\psi(2\pi)$ or $1=e^{im\theta}$, so m is quantized (...-1,0,1,..).

Find the normalization factor,

$$1=\int_0^{2\pi}|\psi|^2 R d\theta=A^2 2\pi \quad \text{or}$$

$$\Rightarrow \psi(\theta)=\frac{1}{\sqrt{2\pi R}}e^{im\theta}$$

ii) Subbing our solution in the above Schrodinger equation we obtain,

$$-\frac{\hbar^2}{2M}\frac{1}{R^2}(-m^2\psi)=E_{cat}\psi$$

or $E_{cat}=\dfrac{\hbar^2 m^2}{2MR^2}$, (alternatively can use the moment of inertia as $I=MR^2$)

iii) $L\equiv -i\hbar R\times\nabla$

$[\theta,L]\psi=\theta(\hbar mR)\psi-(-i\hbar r\times\nabla)\theta\psi=\theta\hbar mR\psi+i\hbar R\psi-\theta\hbar mR\psi=i\hbar R\psi$

$[\theta,L]=i\hbar R$

Schrodiner can know the absolute radius of the cat, but cannot know the cat's angle and angular momentum simulataneousily because $[\theta,L]\neq 0$.

B. Benzene has six π electrons. Two electrons inhabit the same orbital, so $m_1=$ **0, ±1** are filled. So the lowest trasnsition is from 1 to 2.

$$E=\frac{hc}{\lambda}=\frac{\hbar^2}{2I}\left((m_1+1)^2-m_1^2\right)=\frac{\hbar^2}{2I}(2m_1+1)$$

$$E=\frac{hc}{\lambda}=\frac{3\hbar^2}{2I}\approx\frac{3\cdot 10^{-68}Js}{2\cdot 10^{-30}kg\cdot(1.5\cdot 10^{-10}m)^2} \quad \Rightarrow \lambda=300\ nm$$

C.

Use quantitative uncertainty principle estimate.

$\sigma_L\sigma_\theta\geq\frac{\hbar}{2}$ or $\Delta x\Delta p\geq\frac{\hbar}{2}$

(provide a reasonable estimate of measurement uncertainty for cat, to estimate Δp. *Compare qualitatively with uncertainties for benzene. Enumerate other qualitative reason why the calculation is not exact for (a) the cat (not a point mass, really a many body problem, etc.) and (b) benzene (e.g. pi electrons are not flat, no e-e repulsions term, etc).*

An isolated system consisting of a large number N of weakly interacting particles, each of spin-1/2 and magnetic moment μ_m, is in the presence of an external magnetic field B. The energy of the system is $E = -(n_1 - n_2)\mu_m B$, where n_1 is the number of spins aligned parallel to B and n_2 the number of spins aligned antiparallel to B. We consider the case for large n_1 and n_2 ($\gg 1$).

(a) What is the total number of states $\Omega(E)$ for a given energy E? Write down an expression for $\ln \Omega(E)$ as a function of E and simplify it using the approximation, $\ln n! \cong n \ln n - n$ for large n.

(b) Obtain the parameter, $\beta = \frac{\ln \Omega(E)}{\partial E}$, and express the energy $E(T, B)$ as a function of the absolute temperature T and the magnetic field B. Find the energy E at the extreme temperatures, $T = 0$ and $T \to \infty$, for a finite magnetic field B, and justify your answers using physical intuitions.

(c) Find the total magnetic moment $M(T, B)$ of this system as a function of T and B.

(d) The system in thermal equilibrium has the initial energy E_i. Another isolated system consisting of $N'(\gg 1)$ particles of the initial energy E_i' is then placed in thermal contract with it. Find the energies, E_f and E_f', and the temperature, T_f, of the two systems in the final thermal equilibrium.

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Solution:
The particle number is $N = n_1 + n_2$, and hence the energy can be written as

$$E = (N - 2n_1)\mu_m B \text{ for } n_1 = 1, 2, \cdots, N$$

The total number of possible states for n_1 is

$$\Omega(n_1) = \frac{N!}{n!(N - n_1)!}$$

where $n_1 = N/2 - E/2\mu_m B$. Therefore,

$$\Omega(E) = \frac{N!}{\left(\frac{N}{2} - \frac{E}{2\mu_m B}\right)! \left(\frac{N}{2} + \frac{E}{2\mu_m B}\right)!}$$

Taking the logarithm of Ω and using the approximation $\ln n! \cong n \ln n - n$, we obtain

$$\begin{aligned}
\ln \Omega(E) &= \ln N! - \ln n_1! - \ln n_2! \\
&\cong (N \ln N - N) - (n_1 \ln n_1 - n_1) - (n_2 \ln n_2 - n_2) \\
&= N \ln N - n_1 \ln n_1 - n_2 \ln n_2 \\
&= N \ln N - \left(\frac{N}{2} - \frac{E}{2\mu_m B}\right) \ln \left(\frac{N}{2} - \frac{E}{2\mu_m B}\right) \\
&\quad - \left(\frac{N}{2} + \frac{E}{2\mu_m B}\right) \ln \left(\frac{N}{2} + \frac{E}{2\mu_m B}\right)
\end{aligned}$$

(b) Obtain the parameter, $\beta = \frac{\ln \Omega(E)}{\partial E}$, and express the energy $E(T, B)$ as a function of the absolute temperature T and the magnetic field B. Find the energy E at the extreme temperatures, $T = 0$ and $T \to \infty$, for a finite magnetic field B, and justify your answers using physical intuitions.

Solution:
The parameter $\beta = \frac{1}{k_B T}$, where k_B is the Boltzmann constant.

$$\begin{aligned}
\beta &= \frac{\partial}{\partial E} \ln \Omega(E) \\
&= \frac{1}{2\mu_m B} \ln\left(\frac{N}{2} - \frac{E}{2\mu_m B}\right) + \frac{1}{2\mu_m B} \\
&\quad - \frac{1}{2\mu_m B} \ln\left(\frac{N}{2} + \frac{E}{2\mu_m B}\right) - \frac{1}{2\mu_m B} \\
&= \frac{1}{2\mu_m B} \ln\left(\frac{\frac{N}{2} - \frac{E}{2\mu_m B}}{\frac{N}{2} + \frac{E}{2\mu_m B}}\right)
\end{aligned}$$

Taking exponential on both sides, we get

$$\begin{aligned}
e^{2\beta\mu_m B} &= \frac{\frac{N}{2} - \frac{E}{2\mu_m B}}{\frac{N}{2} + \frac{E}{2\beta\mu_m B}} \\
\Rightarrow E &= N\mu_m B\left(\frac{1 - e^{2\beta\mu_m B}}{1 + e^{2\beta\mu_m B}}\right) \\
&= -N\mu_m B \tanh(\beta\mu_m B)
\end{aligned}$$

Therefore,

$$E = -N\mu_m B \tanh\left(\frac{\mu_m B}{k_B T}\right)$$

(i) At $T = 0$, $\frac{1}{T} \to \infty$
Using the limiting behavior, $\tanh(x) \to 1$ for $x \to \infty$, we can get

$$E(T = 0) = -N\mu_m B$$

At the absolute zero temperature, all the particles must be in the ground state, i.e., $n_1 = N$ and $n_2 = 0$. Therefore, $E = -N\mu_m B$ at $T = 0$.
(ii) For $T \to \infty$, $\frac{1}{T} \to 0$

$$E(T \to \infty) = -N\mu_m B \tanh(0) = 0.$$

When the temperature is very high, the magnetic momenta of the particles are randomly oriented, i.e., $n_1 \cong n_2 \cong N/2$. Therefore, $E(T \to \infty) = -(n_1 - n_2)\mu_m B \cong 0$.

(c) Find the total magnetic moment $M(T, B)$ of this system as a function of T and B.

Solution:
The total magnetic moment is the sum of the spin-up ($n_1\mu_m$) and spin-down ($-n_2\mu_m$) moments:

$$M = n_1\mu_m + n_2(-\mu_m) = (n_1 - n_2)\mu_m$$

From the energy relation,

$$E = -(n_1 - n_2)\mu_m B = -MB \Rightarrow M = -\frac{E}{B}$$

Therefore,

$$M = N\mu_m \tanh\left(\frac{\mu_m B}{k_B T}\right)$$

(d) The system in thermal equilibrium has the initial energy E_i. Another isolated system consisting of $N'(\gg 1)$ particles of the initial energy E'_i is then placed in thermal contract with it. Find the energies, E_f and E'_f, and the temperature, T_f, of the two systems in the final thermal equilibrium.

Solution:
The total number of states of the combined system is

$$\Omega^T(E) = \Omega(E)\Omega'(E_0 - E)$$

where the total energy, $E_0 = E_i + E'_i$
At the thermal equilibrium, $\beta_f = \beta'_f$:

$$\begin{aligned}
\frac{\partial}{\partial E}\ln\Omega(E)\bigg|_{E=E_f} &= \frac{\partial}{\partial E'}\ln\Omega(E')\bigg|_{E'=E'_f} \\
\Rightarrow -\frac{E_f}{\mu_m^2 B^2 N} &= -\frac{E'_f}{\mu_m^2 B^2 N'} \\
\Rightarrow E'_f &= \frac{N'}{N}E_f
\end{aligned}$$

The energy conservation, $E'_f = E_0 - E_f$, leads to

$$\begin{aligned}
E_0 - E_f &= \frac{N'}{N}E_f \\
\Rightarrow &\left\{\begin{array}{l} E_f = \frac{E_0}{1+\frac{N'}{N}} \\ E'_f = \frac{E_0}{1+\frac{N}{N'}} \end{array}\right.
\end{aligned}$$

The temperature at the final equilibrium state is

$$\begin{aligned}
T_f &= \frac{1}{k_b\beta_f} = -\frac{\mu_m^2 B^2 N}{k_B E_f} = -\frac{\mu_m^2 B^2 N}{k_B}\frac{1+\frac{N'}{N}}{E_0} \\
&= -\frac{\mu_m^2 B^2}{k_B}\frac{N+N'}{E_i + E'_i}
\end{aligned}$$

Space is divided into two parts. We have vacuum for $z < 0$ and a medium with conductivity σ, and dielectric constant $\epsilon_r\epsilon_0$ and magnetic permeability $\mu_r\mu_0$ where the vacuum values are ϵ_0 and μ_0. The conductivity σ is a real number. In the vacuum region we have an incoming plane wave (magnitude of the wave vector k_0 and angular frequency ω) traveling in the positive z direction, with electric field in the x-direction, and a reflected wave in the opposite direction. In the medium we have a wave traveling in the positive z direction and the magnitude of the wave vector is k. There are no net charges anywhere in space.

(a) Find k as a function of ω inside the medium.

(b) Show that when the conductivity is large we have

$$k \approx \pm\sqrt{\frac{\omega\sigma\mu_r}{\epsilon_0 c^2}}e^{\imath\frac{\pi}{4}}$$

(c) What does it mean that k is complex when ω is real?

(d) What does it mean that ω is complex when k is real?

Suppose the electric field of the incoming wave is given by $\vec{E}(\vec{r},t) = E_0\hat{x}e^{\imath(k_0 z-\omega t)}$

(e) Find the electric field component of the wave inside the medium.

(f) Find the current density in the medium.

(g) If the conductivity becomes infinitely large, will the rate of the Joule heat produced in the material be zero, infinity, or a finite value?

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Solution:
The Maxwell equations inside the medium are

$$\vec{\nabla} \cdot \vec{D} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

The constituent relations are

$$\vec{D} = \epsilon_r \epsilon_0 \vec{E}$$

$$\vec{B} = \mu_r \mu_0 \vec{H}$$

and the current follows from

$$\vec{J} = \sigma \vec{E}$$

Inside the medium we have in general

$$\vec{E}(\vec{r}, t) = E_m e^{\imath(\vec{k}\cdot\vec{r} - \omega t)}$$

and similarly for the other fields. The Maxwell equations simplify to

$$\imath\vec{k} \cdot \vec{D} = 0$$

$$\imath\vec{k} \cdot \vec{B} = 0$$

$$\imath\vec{k} \times \vec{E} = \imath\omega\vec{B}$$

$$\imath\vec{k} \times \vec{H} = -\imath\omega\vec{D} + \vec{J}$$

Now we write everything in terms of E and H fields:

$$\vec{k} \cdot \vec{E} = 0$$

$$\vec{k} \cdot \vec{H} = 0$$

$$\imath\vec{k} \times \vec{E} = \imath\omega\mu_r\mu_0\vec{H}$$

$$\imath\vec{k} \times \vec{H} = -\imath\omega\epsilon_r\epsilon_0\vec{E} + \sigma\vec{E}$$

We now eliminate the H field

$$\vec{k} \times (\vec{k} \times \vec{E}) = \omega\mu_r\mu_0\vec{k} \times \vec{H} = -\omega^2\epsilon_r\epsilon_0\mu_r\mu_0\vec{E} - \imath\omega\mu_r\mu_0\sigma\vec{E}$$

but we also have

$$\vec{k} \times (\vec{k} \times \vec{E}) = (\vec{k} \cdot \vec{E})\vec{k} - k^2\vec{E}$$

and because $\vec{k} \cdot \vec{E} = 0$ this gives

$$-k^2\vec{E} = -\omega^2\epsilon_r\epsilon_0\mu_r\mu_0\vec{E} - \imath\omega\mu_r\mu_0\sigma\vec{E}$$

This gives us the relation we need

$$k^2 = \omega^2\epsilon_r\mu_r\epsilon_0\mu_0(1 + \frac{\imath\sigma}{\epsilon_r\epsilon_0\omega})$$

If the conductivity is large we have approximately

$$k^2 \approx \omega^2\epsilon_r\mu_r\epsilon_0\mu_0(\frac{\imath\sigma}{\epsilon_r\epsilon_0\omega})$$

$$k^2 \approx \imath\omega\sigma\mu_r\mu_0$$

The speed of light in vacuum is

$$c^2 = \frac{1}{\epsilon_0\mu_0}$$

and hence we get

$$k^2 \approx \imath\frac{\omega\sigma\mu_r}{\epsilon_0 c^2}$$

or

$$k \approx \pm\sqrt{\frac{\omega\sigma\mu_r}{\epsilon_0 c^2}}e^{\imath\frac{\pi}{4}}$$

If ω is real, k becomes complex, and the plane wave has an exponential part in space. This means that the intensity of the wave is decreasing when we go away from the interface. The waves are shielded in the medium.

If k is real, ω is complex, and the plane wave has an exponential part in time. The wave is decaying everywhere because of Joule heating.

We have three waves, all the same real frequency,

$$\vec{E}_{in}(\vec{r},t) = E_0\hat{x}e^{\imath(k_0 z-\omega t)}$$

$$\vec{E}_{refl}(\vec{r},t) = E_1\hat{x}e^{\imath(-k_0 z-\omega t)}$$

$$\vec{E}_{trans}(\vec{r},t) = E_2\hat{x}e^{\imath(k_0-\omega t)}$$

The corresponding H fields follow from

$$\vec{k}_0 \times \vec{E}_{in} = \omega\mu_0\vec{H}_{in}$$

$$\vec{k}_0 \times \vec{E}_{refl} = \omega\mu_0\vec{H}_{refl}$$

$$\vec{k} \times \vec{E}_{trans} = \omega\mu_r\mu_0\vec{H}_{trans}$$

which gives using the appropriate directions of the k vectors

$$\vec{H}_{in} = \frac{k_0}{\omega\mu_0}E_0\hat{y}$$

$$\vec{H}_{refl} = -\frac{k_0}{\omega\mu_0}E_1\hat{y}$$

$$\vec{H}_{trans} = \frac{k}{\omega\mu_r\mu_0}E_2\hat{y}$$

and hence we have at the interface for continuity of these parallel components:

$$E_0 + E_1 = E_2$$

$$\frac{k_0}{\omega\mu_0}E_0 - \frac{k_0}{\omega\mu_0}E_1 = \frac{k}{\omega\mu_r\mu_0}E_2$$

or

$$E_0 - E_1 = \frac{k}{k_0\mu_r}E_2$$

which gives

$$E_2 = \frac{2}{1 + \frac{k}{k_0\mu_r}}E_0$$

Using

$$k = \sqrt{\frac{\omega\sigma\mu_r}{\epsilon_0 c^2}}e^{\imath\frac{\pi}{4}}$$

$$k_0 = \frac{\omega}{c}$$

this gives

$$E_2 = \frac{2}{1 + \sqrt{\frac{\sigma}{\epsilon_0\omega\mu_r}}e^{\imath\frac{\pi}{4}}}E_0$$

and with large conductivity we have

$$E_2 \approx 2\sqrt{\frac{\epsilon_0\omega\mu_r}{\sigma}}e^{-\imath\frac{\pi}{4}}E_0$$

The current density is therefore

$$\vec{J} = \sigma E_2\hat{x} \approx \sqrt{\epsilon_0\omega\mu_r\sigma}e^{-\imath\frac{\pi}{4}}E_0$$

The rate of the Joule heat produced in the medium is the product of J and E_2, which is

$$P_{Joule} = 2\omega\epsilon_0\mu_r E_0^2$$

and this is a finite number.