

OSU PHYSICS DEPARTMENT
COMPREHENSIVE EXAMINATION #115

Monday, January 7 and Tuesday, January 8, 2013

Winter 2013 Comprehensive Examination

PART 1, Monday, January 7, 9:00am

General Instructions

This Winter 2013 Comprehensive Examination consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, January 7, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:00 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, January 8, at 9:00 am and 1:00 pm, respectively. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please return all bluebooks and formula sheets at the end of the exam. Use the last pages of your bluebooks for “scratch” work, separated by at least one empty page from your solutions. “Scratch” work will not be graded.

The Helmholtz free energy of a gas is given by:

$$F = -Nk_B T \left(1 + \ln \left(\frac{(V - Nb)T^{\frac{3}{2}}}{\Phi} \right) \right) - \frac{aN^2}{V} \quad (1)$$

- (a) How much energy is required to raise the temperature of this system to twice its initial temperature, while holding the volume fixed?
- (b) How much does the entropy of this system increase, if it is isothermally expanded to twice its initial volume?
- (c) What is the value of C_p for this gas? C_p is the heat capacity at fixed pressure.

The Helmholtz free energy of a gas is given by:

$$F = -Nk_B T \left(1 + \ln \left(\frac{(V - Nb)T^{\frac{3}{2}}}{\Phi} \right) \right) - \frac{aN^2}{V} \quad (2)$$

- (a) How much energy is required to raise the temperature of this system to twice its initial temperature, while holding the volume fixed?

Solution:

We begin by noting that

$$F = U - TS \quad (3)$$

$$dF = -SdT - pdV \quad (4)$$

$$S = - \left(\frac{\partial F}{\partial T} \right)_V \quad (5)$$

So now we can solve for the entropy. This will be useful, as it will allow us to find out what the internal energy is.

$$S = - \left(\frac{\partial F}{\partial T} \right)_V \quad (6)$$

$$= Nk_B \left(1 + \ln \left(\frac{(V - Nb)T^{\frac{3}{2}}}{\Phi} \right) + \frac{3}{2} \right) \quad (7)$$

The energy needed to raise the temperature at fixed volume is just equal to the change in internal energy, according to the First Law, since the work is zero when the volume is held fixed. Thus we will call our answer ΔU (and call the initial temperature T). But before we do this (as it gets a bit hairy), it's helpful to first just solve for U .

$$U = F + TS \quad (8)$$

$$= -Nk_B T \left(1 + \ln \left(\frac{(V - Nb)T^{\frac{3}{2}}}{\Phi} \right) \right) - \frac{aN^2}{V} + Nk_B T \left(1 + \ln \left(\frac{(V - Nb)T^{\frac{3}{2}}}{\Phi} \right) + \frac{3}{2} \right) \quad (9)$$

$$= \frac{3}{2}Nk_B T - \frac{aN^2}{V} \quad (10)$$

At this point, we should be feeling pretty cozy. If we consider the $a \rightarrow 0$ limit, we will find the comfortable ideal gas internal energy, which is independent of volume. Clearly the a term is just an attractive potential energy.

$$\Delta U = U(2T) - U(T) \quad (11)$$

$$= \frac{3}{2}Nk_B 2T - \frac{aN^2}{V} - \frac{3}{2}Nk_B T + \frac{aN^2}{V} \quad (12)$$

$$= \frac{3}{2}Nk_B T \quad (13)$$

which tells us that this gas (which is just a van der Waals gas) has the same C_V as the ideal gas.

- (b) How much does the entropy of this system increase, if it is isothermally expanded to twice its initial volume?

Solution:

Given the entropy we already found, this is pretty easy.

$$\Delta S = S(2V) - S(V) \tag{14}$$

$$= Nk_B \left(1 + \ln \left(\frac{(2V - Nb)T^{\frac{3}{2}}}{\Phi} \right) + \frac{3}{2} \right) - Nk_B \left(1 + \ln \left(\frac{(V - Nb)T^{\frac{3}{2}}}{\Phi} \right) + \frac{3}{2} \right) \tag{15}$$

$$= Nk_B (\ln(2V - Nb) - \ln(V - Nb)) \tag{16}$$

$$= Nk_B \ln \left(\frac{2V - Nb}{V - Nb} \right) \tag{17}$$

Here we can see that if we set b to zero, we would get the ideal gas solution, which is $Nk_B \ln 2$.

- (c) What is the value of C_p for this gas? C_p is the heat capacity at fixed pressure.

Solution:

The heat capacity at fixed pressure is given by

$$C_p = T \left(\frac{\partial S}{\partial T} \right)_p \tag{18}$$

which would be great if we had S as a function of T and p . But we don't, so we need to think a little further. C_p differs from C_V because of the work that is done by a system as it is heated at fixed pressure, which means you need to add more energy to heat it up (unless its thermal expansion is negative, in which case it is the other way around).

A nice function when working at fixed pressure is the enthalpy, since it automatically subtracts out the work that is done on the surroundings.

$$H = U + pV \tag{19}$$

$$dH = TdS + Vdp \tag{20}$$

From its total differential, you can see that the change in enthalpy is equal to the heat when working at fixed pressure. So we just need to know p ,

and then we can find H from the U that we already computed.

$$p = - \left(\frac{\partial F}{\partial V} \right)_T \quad (21)$$

$$= \frac{Nk_B T}{V - Nb} - \frac{aN^2}{V^2} \quad (22)$$

This is a nice and simple formula!

$$H = U + pV \quad (23)$$

$$= \frac{3}{2}Nk_B T + \frac{Nk_B T V}{V - Nb} - \frac{2aN^2}{V} \quad (24)$$

$$= \frac{3}{2}Nk_B T + \frac{Nk_B T}{1 - \frac{Nb}{V}} - \frac{2aN^2}{V} \quad (25)$$

And now we can find the heat capacity, provided we know how to do a change of variables (or alternatively you could look at it as an application of the chain rule in multiple dimensions).

$$C_p = \left(\frac{\partial H}{\partial T} \right)_p \quad (26)$$

$$= \left(\frac{\partial H}{\partial T} \right)_V + \left(\frac{\partial H}{\partial V} \right)_T \left(\frac{\partial V}{\partial p} \right)_T \quad (27)$$

So now let's evaluate the derivatives we'll need.

$$\left(\frac{\partial H}{\partial V} \right)_T = \left(\frac{\partial U}{\partial V} \right)_T + V \left(\frac{\partial p}{\partial V} \right)_T + p \quad (28)$$

$$\left(\frac{\partial p}{\partial V} \right)_T = - \frac{Nk_B T}{(V - Nb)^2} + \frac{2aN^2}{V^3} \quad (29)$$

$$\left(\frac{\partial U}{\partial V} \right)_T = \frac{aN^2}{V^2} \quad (30)$$

$$\frac{\left(\frac{\partial H}{\partial V} \right)_T}{\left(\frac{\partial p}{\partial V} \right)_T} = \frac{\frac{aN^2}{V^2} + p + V \left(\frac{\partial p}{\partial V} \right)_T}{\left(\frac{\partial p}{\partial V} \right)_T} \quad (31)$$

$$= \frac{\frac{aN^2}{V^2} + p}{\left(\frac{\partial p}{\partial V} \right)_T} + V \quad (32)$$

$$= \frac{\frac{Nk_B T}{V - Nb}}{- \frac{Nk_B T}{(V - Nb)^2} + \frac{2aN^2}{V^3}} + V \quad (33)$$

$$= - \frac{1}{1 - \frac{2aN^2 (V - Nb)^2}{V^3} \frac{1}{Nk_B T}} + V \quad (34)$$

$$\left(\frac{\partial H}{\partial T} \right)_V = \frac{3}{2}Nk_B + \frac{Nk_B}{V - Nb} \quad (35)$$

Putting it all together, we get:

$$C_p = \left(\frac{\partial H}{\partial T} \right)_V + \left(\frac{\partial H}{\partial V} \right)_T \left(\frac{\partial V}{\partial p} \right)_T \quad (36)$$

$$= \frac{3}{2} N k_B + \frac{N k_B}{V - N b} - \frac{1}{1 - \frac{2a N^2 (V - N b)^2}{V^3 N k_B T}} + V \quad (37)$$

Consider a system with one degree of freedom and Hamiltonian $\mathcal{H} = \mathcal{H}(q, p)$.

- (a) A new pair of coordinates Q and P is defined in terms of the coordinates q and p so that

$$q = \sqrt{2P} \sin Q \quad \text{and} \quad p = \sqrt{2P} \cos Q.$$

Prove that if $\partial\mathcal{H}/\partial q = -\dot{p}$ and $\partial\mathcal{H}/\partial p = \dot{q}$, it automatically follows that $\partial\mathcal{H}/\partial Q = -\dot{P}$ and $\partial\mathcal{H}/\partial P = \dot{Q}$.

- (b) Show that the Hamiltonian of a one-dimensional harmonic oscillator with mass $m = 1$ and force constant $k = 1$ is given by $\mathcal{H} = \frac{1}{2} (q^2 + p^2)$.
- (c) Express the Hamiltonian of the one-dimensional harmonic oscillator from part (b) in terms of the coordinates P and Q . Interpret P and Q and your result.
- (d) Solve the Hamiltonian equation for $Q(t)$ and rewrite the result in terms of $q(t)$. Verify that your solution gives the expected behavior.

a) The transformation $q = \sqrt{2F} \sin Q$, (1)
 $p = \sqrt{2F} \cos Q$, implies

$Q = f(q, p)$, $F = g(q, p)$ exists and

$$\begin{aligned} \frac{\partial H}{\partial Q} &= \frac{\partial H}{\partial q} \frac{\partial q}{\partial Q} + \frac{\partial H}{\partial p} \frac{\partial p}{\partial Q} = -\dot{p} \frac{\partial q}{\partial Q} + \dot{q} \frac{\partial p}{\partial Q} \\ \frac{\partial H}{\partial F} &= \dots = -\dot{p} \frac{\partial p}{\partial F} + \dot{q} \frac{\partial p}{\partial F} \end{aligned} \quad (2)$$

$$\begin{aligned} \text{from (1)}: -\dot{p} &= -\frac{1}{\sqrt{2F}} \dot{F} \cos Q + \sqrt{2F} \sin Q \cdot \dot{Q} \\ \dot{q} &= \frac{1}{\sqrt{2F}} \dot{F} \sin Q + \sqrt{2F} \cos Q \cdot \dot{Q} \end{aligned} \quad (3)$$

$$\begin{aligned} \text{from (1)}: \frac{\partial q}{\partial Q} &= \sqrt{2F} \cos Q & \frac{\partial p}{\partial Q} &= -\sqrt{2F} \sin Q \\ \frac{\partial p}{\partial F} &= \frac{1}{\sqrt{2F}} \cos Q & \frac{\partial q}{\partial F} &= \frac{1}{\sqrt{2F}} \sin Q \end{aligned} \quad (4)$$

combine (2)-(4): (2), (4) in (2)

$$\begin{aligned} \frac{\partial H}{\partial Q} &= -\dot{p} \cos^2 Q + \sqrt{2F} \sin Q \cos Q \cdot \dot{Q} \\ &\quad - \dot{p} \sin^2 Q - 2F \sin Q \cos Q \cdot \dot{Q} = -\dot{p} \quad \square \end{aligned}$$

$$\begin{aligned} \frac{\partial H}{\partial F} &= -\frac{1}{2F} \sin Q \cos Q \dot{F} + \dot{Q} \sin^2 Q \\ &\quad + \frac{1}{2F} \sin Q \cos Q \dot{F} + \dot{Q} \cos^2 Q = \dot{Q} \quad \square \end{aligned}$$

$$\begin{aligned} \text{b) } H = T + U &= \frac{1}{2} m \dot{q}^2 + \frac{1}{2} k q^2 \\ &= \frac{p^2}{2m} + \frac{1}{2} k q^2 \quad | m = k = 1 \\ &= \frac{1}{2} (p^2 + q^2) \end{aligned}$$

$$c) \quad H = \frac{1}{2} (2E (\sin^2 Q + \cos^2 Q)) = E$$

E = total energy, is constant of motion

Q : the Hamiltonian does not depend on Q , it is said to be cyclic or ignorable.

\Rightarrow It's corresponding momentum P is a constant of motion, Q is a linear function of time: $Q(t) = at + b$
 (P, Q) are also called action-angle variables.

$$d) \quad \frac{\partial H}{\partial E} = 1 = \dot{Q} \rightarrow Q(t) = t + t_0$$

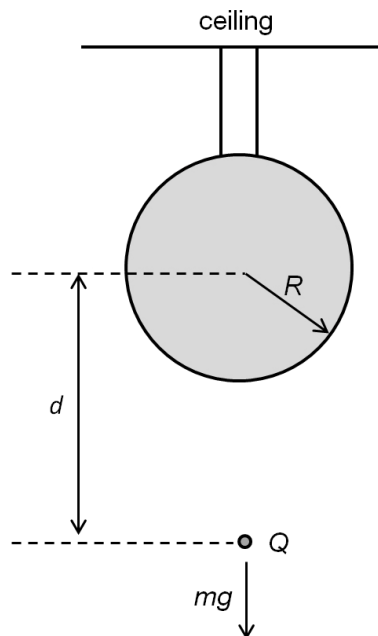
$$\rightarrow q(t) = \sqrt{2E} \sin(t + t_0)$$

= expected oscillatory solution for harmonic oscillator.

A tiny sphere of charge Q and mass m is to be placed at rest at its equilibrium position a distance d directly below the center of an uncharged conducting sphere of radius R . The uncharged conducting sphere is fixed in position by an insulating rod that is attached to the ceiling. The tiny sphere (charge Q , mass m) floats in space despite the downward force of gravity.

Derive an expression for d that is valid when $d \gg R$.

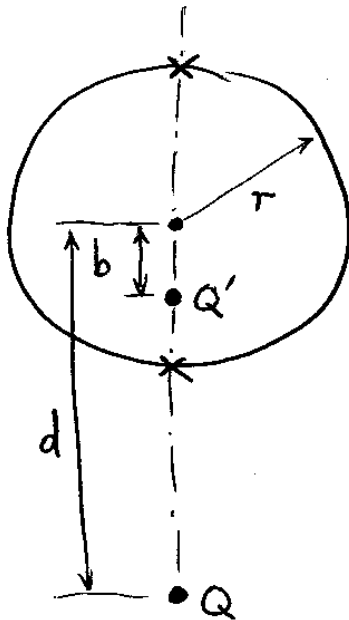
Hint: There exists a spherical zero-potential surface between any two point charges of opposite charge and non-equal magnitude.



p1

Boundary conditions: The surface of the conducting sphere must be an equipotential surface.

Try placing an image charge (a point charge Q') inside the sphere



The diagram shows two points marked by "X".

The potentials at these points are

$$\frac{Q'}{4\pi\epsilon_0(r-b)} + \frac{Q}{4\pi\epsilon_0(d-r)}$$

and

$$\frac{Q'}{4\pi\epsilon_0(r+b)} + \frac{Q}{4\pi\epsilon_0(d+r)}$$

Can we choose Q' and b such that both potentials are equal and zero?

$$\frac{Q'}{r-b} - \frac{Q}{d-r} = \frac{Q'}{r+b} + \frac{Q}{d+r}$$

$$\Rightarrow Q' \left(\frac{1}{r-b} - \frac{1}{r+b} \right) = Q \left(\frac{1}{d+r} - \frac{1}{d-r} \right)$$

$$\Rightarrow Q' \left(\frac{2b}{r^2-b^2} \right) = Q \left(\frac{-2r}{d^2-r^2} \right)$$

$$\Rightarrow \frac{Q'}{Q} = \left(\frac{-\cancel{2}r}{d^2-r^2} \right) \left(\frac{r^2-b^2}{\cancel{2}b} \right)$$

$$\Rightarrow \boxed{\frac{Q'}{Q} = -\frac{r}{b} \frac{(r^2-b^2)}{(d^2-r^2)}} \quad \text{Eq. 1}$$

We ~~also~~ need a second eqn to solve for Q' & b .

$$\frac{Q'}{r+b} + \frac{Q}{d+r} = 0$$

$$\Rightarrow \boxed{\frac{Q'}{Q} = -\left(\frac{r+b}{r+d} \right)} \quad \text{Eq. 2}$$

Combining Eq. 1 & 2

$$\frac{r}{b} \frac{r^2-b^2}{d^2-r^2} = \frac{r+b}{r+d}$$

$$\frac{r}{b} \frac{r-b}{d-r} = 1$$

p3

$$r(r-b) = b(d-r)$$

$$r^2 - \cancel{rb} = bd - \cancel{rb}$$

$$\boxed{b = \frac{r^2}{d}}$$

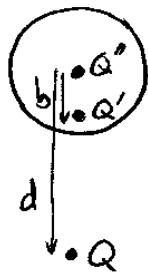
Now using Eq. 2 we find

$$Q' = -Q \left(\frac{r + \frac{r^2}{d}}{r+d} \right)$$

$$= -Q \left(\frac{\frac{r}{d}(d+r)}{\cancel{d+r}} \right)$$

$$\boxed{Q' = -Q \frac{r}{d}}$$

This is not the full story because the conducting sphere is uncharged. We must add a second image charge so that net image charge is zero.



The second image charge, Q'' is placed at the center of the conducting sphere because this will not disrupt the equipotential surface

$$\boxed{Q'' = +Q \frac{r}{d}}$$

p4

Now consider the net force on the small sphere.

$$\sum F = \frac{-|QQ''|}{4\pi\epsilon_0 d^2} + \frac{|QQ'|}{4\pi\epsilon_0 (d-b)^2} - mg = 0$$

$\begin{matrix} \text{(upward)} \\ \text{downward} \end{matrix}$

 $\begin{matrix} \text{down} \\ \text{(upward)} \end{matrix}$

 $\begin{matrix} \text{(downward)} \end{matrix}$

$$\frac{-Q^2 r}{d^2} + \frac{Q^2 r}{(d-b)^2} - 4\pi\epsilon_0 mg = 0$$

$$Q^2 r \left(\frac{1}{(d-b)^2} - \frac{1}{d^2} \right) - 4\pi\epsilon_0 mg = 0$$

$$Q^2 r \frac{1}{d^2} \left(\frac{1}{(1-\frac{b}{d})^2} - 1 \right) - 4\pi\epsilon_0 mg = 0$$

Note that $\frac{b}{d} = \frac{r^2}{d^2} \ll 1$, therefore, use Taylor expansion.

$$\frac{Q^2 r}{d^3} \left(1 + \frac{2r^2}{d^2} - 1 \right) = 4\pi\epsilon_0 mg$$

$$\frac{2Q^2 r^3}{d^5} = 4\pi\epsilon_0 mg$$

$$d = \sqrt[5]{\frac{2Q^2 r^3}{4\pi\epsilon_0 mg}}$$

An electron is subject to a uniform magnetic field $\mathbf{B}_0 = B\mathbf{e}_z$ along z-axis. Measurements of the total spin angular momentum S^2 and the z-component of the spin angular momentum S_z at a time $t (< 0)$ yield $\frac{3}{4}\hbar^2$ and $\frac{\hbar}{2}$, respectively. At $t = 0$, an additional uniform magnetic field $B_1 = \sqrt{3}B\mathbf{e}_y$ along y-axis is suddenly turned on.

- (a) What is the state vector $|\psi(0)\rangle$ at $t = 0$? Express it using the basis vectors $|\uparrow\rangle$ and $|\downarrow\rangle$, where $S_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle$ and $S_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$.
- (b) Describe the temporal evolution of the state vector $|\psi(t)\rangle$ at $t > 0$ in terms of the basis vectors $|\uparrow\rangle$ and $|\downarrow\rangle$.
- (c) Find the probability that a measurement of S_x results in $\frac{\hbar}{2}$ at $t > 0$. What is the maximum probability to get the result? When does it happen?
- (d) Calculate the expectation value of the Spin angular momentum \mathbf{S} at $t > 0$. The spin dynamics is periodic in time. What is the period? Sketch its temporal evolution in the three dimensional real space.

#42.

$$(a) S(S+1)\hbar^2 = \frac{3}{4}\hbar^2, m_s\hbar = \frac{1}{2}\hbar$$

$$\Rightarrow S = \frac{1}{2} \text{ and } m_s = \frac{1}{2}, \text{ thus } |\psi(0)\rangle = |\uparrow\rangle$$

(b) Hamiltonian

$$\hat{H} = -\hat{\mu} \cdot \vec{B} = \frac{e}{m_e c} \hat{S} \cdot \vec{B} = \frac{eB}{m_e c} (\sqrt{3} \hat{S}_y + \hat{S}_z)$$

$$\text{Let } \omega \equiv \frac{eB}{m_e c}, \text{ then } \hat{H} = \frac{\hbar\omega}{2} (\sqrt{3} \hat{O}_y + \hat{O}_z)$$

$$\text{where } \hat{S}_y = \frac{\hbar}{2} \hat{O}_y \text{ and } \hat{S}_z = \frac{\hbar}{2} \hat{O}_z$$

In the matrix formalism

$$\hat{H} = \frac{\hbar\omega}{2} \left\{ \sqrt{3} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} = \frac{\hbar\omega}{2} \begin{pmatrix} 1 & -\sqrt{3}i \\ \sqrt{3}i & -1 \end{pmatrix}, \begin{matrix} \text{basis} \\ |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{matrix}$$

* Eigenvalue and eigen states

$$|\hat{H} - \frac{\hbar\omega}{2}\lambda \hat{I}| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -\sqrt{3}i \\ \sqrt{3}i & -1-\lambda \end{vmatrix} = 0 \Rightarrow -(1-\lambda)(1+\lambda) - 3 = 0$$

$$\Rightarrow \lambda^2 = 4 \Rightarrow \lambda = \pm 2$$

Therefore, eigenvalues are $E_{1,2} = \frac{\hbar\omega}{2}\lambda = \pm\hbar\omega$

$$\text{For } E_1 = \hbar\omega, \hat{H}|\uparrow\rangle = \hbar\omega|\uparrow\rangle, |\uparrow\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\sqrt{3}i \\ \sqrt{3}i & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow a - \sqrt{3}bi = 2a$$

$$\Rightarrow a = -\sqrt{3}ib \text{ with } |a|^2 + |b|^2 = 1 \Rightarrow |\uparrow\rangle = \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ i \end{pmatrix}$$

$$\text{For } E_2 = -\hbar\omega, \hat{H}|\downarrow\rangle = -\hbar\omega|\downarrow\rangle, |\downarrow\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\sqrt{3}i \\ \sqrt{3}i & -1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = -2 \begin{pmatrix} c \\ d \end{pmatrix} \Rightarrow c - \sqrt{3}di = -2c$$

$$\Rightarrow c = \frac{i}{\sqrt{3}}d \text{ with } |c|^2 + |d|^2 = 1 \Rightarrow |\downarrow\rangle = \frac{1}{2} \begin{pmatrix} i \\ \sqrt{3} \end{pmatrix}$$

* Temporal evolution of the state vector

$$|\psi(0)\rangle = |\uparrow\rangle = \alpha|1\rangle + \beta|2\rangle$$

$$\Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\alpha}{2} \begin{pmatrix} \sqrt{3} \\ i \end{pmatrix} + \frac{\beta}{2} \begin{pmatrix} i \\ \sqrt{3} \end{pmatrix} \Rightarrow \begin{cases} 2 = \sqrt{3}\alpha + i\beta \\ 0 = i\alpha + \sqrt{3}\beta \end{cases}$$

$$\Rightarrow \alpha = \frac{\sqrt{3}}{2}, \beta = \frac{1}{2i} \Rightarrow |\psi(0)\rangle = \frac{\sqrt{3}}{2}|1\rangle + \frac{1}{2i}|2\rangle$$

$$\text{Thus, } |\psi(t)\rangle = e^{-i\frac{E}{\hbar}t} |\psi(0)\rangle$$

$$= e^{-i\frac{E}{\hbar}t} \left(\frac{\sqrt{3}}{2}|1\rangle + \frac{1}{2i}|2\rangle \right)$$

$$= \frac{\sqrt{3}}{2} e^{-i\omega t} |1\rangle + \frac{1}{2i} e^{i\omega t} |2\rangle$$

$$= \frac{\sqrt{3}}{2} e^{-i\omega t} \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ i \end{pmatrix} + \frac{1}{2i} e^{i\omega t} \frac{1}{2} \begin{pmatrix} i \\ \sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{4} e^{-i\omega t} + \frac{1}{4} e^{i\omega t} \\ \frac{\sqrt{3}}{4} e^{-i\omega t} + \frac{\sqrt{3}}{4i} e^{i\omega t} \end{pmatrix} = \begin{pmatrix} \cos \omega t - \frac{i}{2} \sin \omega t \\ \frac{\sqrt{3}}{2} \sin \omega t \end{pmatrix}$$

$$\Rightarrow |\psi(t)\rangle = \left(\cos \omega t - \frac{i}{2} \sin \omega t \right) |\uparrow\rangle + \frac{\sqrt{3}}{2} \sin \omega t |\downarrow\rangle$$

$$(c) \quad |S_x = \frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$P(S_x = +\frac{\hbar}{2}) = \left| \langle S_x = \frac{\hbar}{2} | \psi \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} (1 \ 1) \begin{pmatrix} \cos \omega t - \frac{i}{2} \sin \omega t \\ \frac{\sqrt{3}}{2} \sin \omega t \end{pmatrix} \right|^2$$

$$= \frac{1}{2} \left| \cos \omega t - \frac{i}{2} \sin \omega t + \frac{\sqrt{3}}{2} \sin \omega t \right|^2$$

$$= \frac{1}{2} \left(1 + \frac{\sqrt{3}}{2} \sin 2\omega t \right)$$

$$P_{\max} = \frac{1}{2} + \frac{\sqrt{3}}{4} \quad \text{when } 2\omega t = \frac{\pi}{2} + 2n\pi$$

$$\Rightarrow t = \frac{\pi}{2\omega} \left(n + \frac{1}{4} \right), \quad n = 0, 1, 2, \dots$$

$$(d) \hat{S} = \hat{S}_x \hat{e}_x + \hat{S}_y \hat{e}_y + \hat{S}_z \hat{e}_z$$

$$\begin{aligned} \langle \hat{S}_x \rangle &= \frac{\hbar}{2} (\cos \omega t + \frac{i}{2} \sin \omega t, \frac{\sqrt{3}}{2} \sin \omega t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos \omega t - \frac{i}{2} \sin \omega t \\ \frac{\sqrt{3}}{2} \sin \omega t \end{pmatrix} \\ &= \frac{\hbar}{2} (\frac{\sqrt{3}}{2} \sin \omega t, \cos \omega t + \frac{i}{2} \sin \omega t) \begin{pmatrix} \cos \omega t - \frac{i}{2} \sin \omega t \\ \frac{\sqrt{3}}{2} \sin \omega t \end{pmatrix} \\ &= \frac{\sqrt{3}}{4} \hbar \sin 2\omega t \end{aligned}$$

$$\begin{aligned} \langle \hat{S}_y \rangle &= \frac{\hbar}{2} (\cos \omega t + \frac{i}{2} \sin \omega t, \frac{\sqrt{3}}{2} \sin \omega t) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos \omega t - \frac{i}{2} \sin \omega t \\ \frac{\sqrt{3}}{2} \sin \omega t \end{pmatrix} \\ &= \frac{\sqrt{3}}{4} \hbar \sin^2 \omega t = \frac{\sqrt{3}}{8} \hbar (1 - \cos 2\omega t) \end{aligned}$$

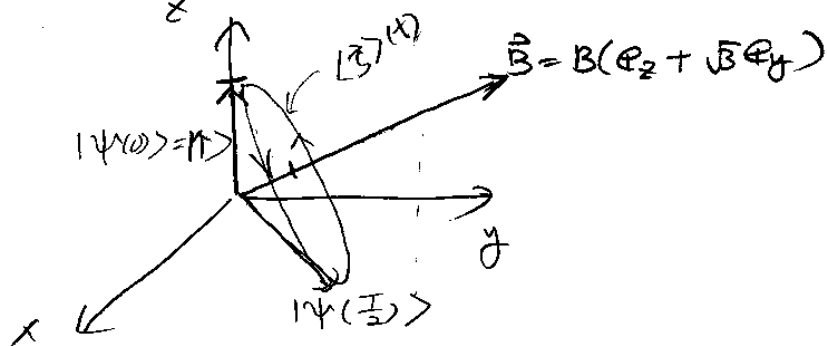
$$\begin{aligned} \langle \hat{S}_z \rangle &= \frac{\hbar}{2} (\cos \omega t + \frac{i}{2} \sin \omega t, \frac{\sqrt{3}}{2} \sin \omega t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos \omega t - \frac{i}{2} \sin \omega t \\ \frac{\sqrt{3}}{2} \sin \omega t \end{pmatrix} \\ &= \frac{\hbar}{2} (1 - \frac{3}{2} \sin^2 \omega t) = \frac{\hbar}{8} (1 + 3 \cos 2\omega t) \end{aligned}$$

$$\Rightarrow \langle \vec{S} \rangle(t) = \langle \hat{S}_x \rangle \hat{e}_x + \langle \hat{S}_y \rangle \hat{e}_y + \langle \hat{S}_z \rangle \hat{e}_z$$

Oscillating at 2ω . \Rightarrow $\boxed{T = \frac{\pi}{\omega}}$: period

$\langle \vec{S} \rangle$ is precessing around the B-field,

$$\vec{B} = B(\hat{e}_x + \sqrt{3}\hat{e}_y)$$

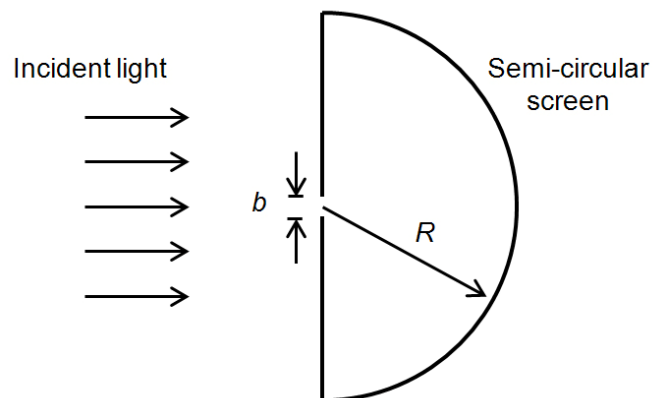


A semi-infinite opaque screen contains a slit which is infinite in length and of width b . The slit is illuminated by a monochromatic plane wave of wavelength λ incident normally upon it.

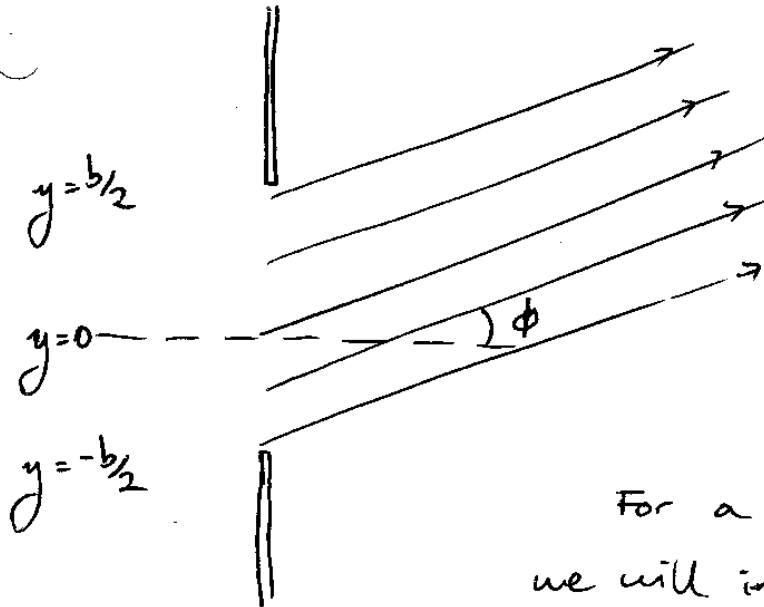
- a) Derive an expression for the light intensity on a semi-circular screen of radius $R \gg b$ that is placed behind the slit (see the figure below).

Hint: Huygen's principle states that this diffraction pattern can be calculated by treating the gap (the slit of width b) as a surface covered by a continuum of point sources.

- b) Sketch this light intensity pattern on a graph for two different cases: $b = \lambda$ and $b = 2\lambda$. In both cases, label the angles at which maxima and minima occur.



①



For a given angle ϕ ,
we will ~~integrate~~ ^{sum} parallel rays
with different starting coordinates
over the range $y = [-b/2, b/2]$.

The phase difference between the $y=0$ ray and
the ~~ray~~ ray from y is

$$\delta = -ky \sin\phi \quad \text{where } k = \frac{2\pi}{\lambda}.$$

$$E_{\text{Tot}} = \int_{-b/2}^{b/2} E_0 e^{-iky \sin\phi} dy$$

$$= \frac{E_0}{-ik \sin\phi} \left[e^{-iky \sin\phi} \right]_{-b/2}^{b/2}$$

(2)

$$= \frac{E_0}{-ik \sin \phi} 2i \left(\sin \left(k \frac{b}{2} \sin \phi \right) \right)$$

$$= \frac{2E_0}{-k \sin \phi} \sin \left(\frac{\pi b}{\lambda} \sin \phi \right)$$

Intensity $\propto |E|^2$

$$I(\phi) \propto \frac{1}{\sin^2 \phi} \sin^2 \left(\frac{\pi b}{\lambda} \sin \phi \right)$$

CASE 1: $\frac{b}{\lambda} = 1 \Rightarrow I \propto \frac{1}{\sin^2 \phi} \sin^2(\pi \sin \phi)$

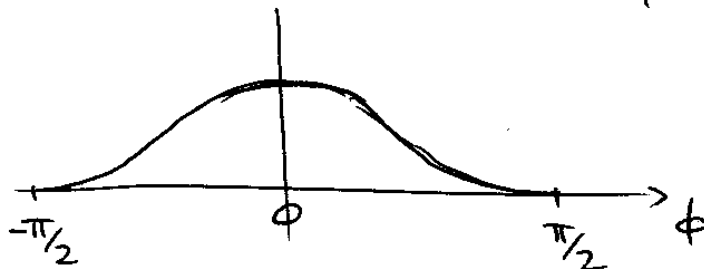
Consider the range $[-\pi/2, \pi/2]$.For small $\phi \sim 0$

$$I \propto \frac{1}{\cancel{\phi^2}} \frac{\pi^2 \cancel{\phi^2}}{\cancel{\phi^2}} \quad \text{local maximum}$$

Zeros occur if $\pi \sin \phi = \pi, 2\pi, 3\pi \dots$

$$\sin \phi = 1, 2, 3 \dots$$

$$\phi = \pm \pi/2$$



(3)

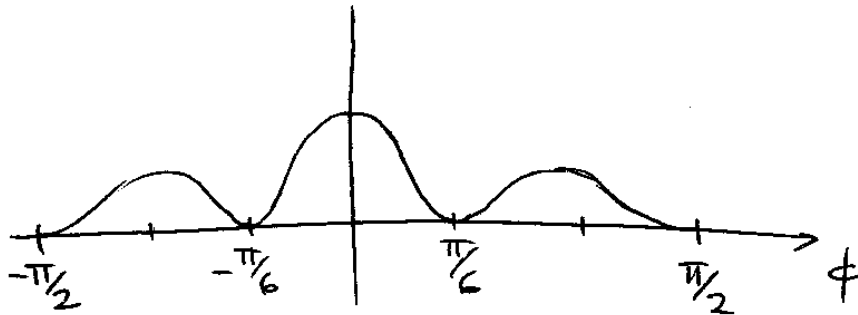
$$\text{CASE 2: } \frac{b}{\lambda} = 2 \Rightarrow I = \frac{1}{\sin^2 \phi} \sin^2(2\pi \sin \phi)$$

For $\phi = 0$, local maximum.

Zeros occur if $2\pi \sin \phi = \pi, 2\pi, 3\pi \dots$

$$\sin \phi = \frac{1}{2}, 1, \frac{3}{2} \dots$$

$$\phi = \frac{\pi}{6}, \frac{\pi}{2}$$



Consider a liquid composed of N identical molecules, each with a fixed dipole moment D . There are strong interactions between molecules, given by a function of the positions and orientations of all the molecules:

$$E(\mathbf{r}_1, \mathbf{D}_1, \mathbf{r}_2, \mathbf{D}_2, \dots) \quad (38)$$

You may take this function as an input, but may not make any assumptions about its form. This energy E includes all electrostatic interactions between molecules.

Each of your final answers to this problem should use only fundamental constants, the temperature and the energy function above. You may, of course, define and use intermediate results (e.g. your final equation will probably involve β rather than T).

- (a) Find the total polarization of the liquid (i.e. the sum of the dipole moments of all the molecules) at temperature T in the absence of an external electric field. You may assume any symmetries that are present in ordinary liquids (which might not be present in, e.g. liquid crystals).
- (b) Find an expression (involving at most two multidimensional integrals) for the variance of the total polarization $\langle |\mathbf{P}|^2 \rangle$ at temperature T , in the absence of an external electric field.
- (c) Find an expression (involving at most two multidimensional integrals) for the polarizability α of the liquid, defined by

$$\mathbf{P} = \alpha \mathbf{E} \quad (39)$$

where \mathbf{P} is the total polarization of the liquid, and \mathbf{E} is a small electric field. You may assume that the only interaction with the electric field is due to the fixed (in magnitude) dipole moments of the molecules.

- (d) Find an expression with no integrals at all that relates α to $\langle |\mathbf{P}|^2 \rangle$.

Consider a liquid composed of N identical molecules, each with a fixed dipole moment D . There are strong interactions between molecules, given by a function of the positions and orientations of all the molecules:

$$E(\mathbf{r}_1, \mathbf{D}_1, \mathbf{r}_2, \mathbf{D}_2, \dots) \quad (40)$$

You may take this function as an input, but may not make any assumptions about its form. This energy E includes all electrostatic interactions between molecules.

Each of your final answers to this problem should use only fundamental constants, the temperature and the energy function above. You may, of course, define and use intermediate results (e.g. your final equation will probably involve β rather than T).

- (a) Find the total polarization of the liquid (i.e. the sum of the dipole moments of all the molecules) at temperature T in the absence of an external electric field. You may assume any symmetries that are present in ordinary liquids (which might not be present in, e.g. liquid crystals).

Solution:

Ordinary liquids are rotationally symmetric. Therefore, the total polarization must be independent of rotation, and must be zero.

- (b) Find an expression (involving at most two multidimensional integrals) for the variance of the total polarization $\langle |\mathbf{P}|^2 \rangle$ at temperature T , in the absence of an external electric field.

Solution:

We need to average over all possible microstates.

$$\langle |\mathbf{P}|^2 \rangle = \sum_s^{\text{all states}} P_s |\mathbf{P}_s|^2 \quad (41)$$

where P_s is the probability of state s and \mathbf{P}_s is the total polarization of microstate s , which is given by:

$$\mathbf{P}_s = \sum_i^N \mathbf{D}_i^{(s)} \quad (42)$$

where $\mathbf{D}_i^{(s)}$ is the dipole moment of the i th molecule in state s .

We haven't yet defined P_s , so our answer is not yet complete.

$$P_s = \frac{e^{-\beta E(\mathbf{r}_1^{(s)}, \mathbf{D}_1^{(s)}, \dots)}}{Z} \quad (43)$$

$$Z = \sum_s^{\text{all states}} e^{-\beta E(\mathbf{r}_1^{(s)}, \mathbf{D}_1^{(s)}, \dots)} \quad (44)$$

Our answer, however, is still not complete, because we have not yet defined the summation over all states which we have used twice now.

$$Z = \sum_s^{\text{all states}} e^{-\beta E(\mathbf{r}_1^{(s)}, \mathbf{D}_1^{(s)}, \dots)} \quad (45)$$

$$= \int d^3 r_1 d^3 r_2 \dots d^3 r_N \int d^2 D_1 d^2 D_2 \dots d^2 D_N e^{-\beta E(\mathbf{r}_1^{(s)}, \mathbf{D}_1^{(s)}, \dots)} \quad (46)$$

where the $d^2 D_i$ integrals integrate over solid angle describing the direction of each dipole moment. This turns our P_s into a probability density (it is no longer dimensionless), which works out when we write our final answer as:

$$\langle |\mathbf{P}|^2 \rangle = \frac{\int d^3 r_1 d^3 r_2 \dots d^3 r_N \int d^2 D_1 d^2 D_2 \dots d^2 D_N e^{-\beta E(\mathbf{r}_1^{(s)}, \mathbf{D}_1^{(s)}, \dots)} \left(\sum_i^N \mathbf{D}_i \right) \cdot \left(\sum_j^N \mathbf{D}_j \right)}{\int d^3 r_1 d^3 r_2 \dots d^3 r_N \int d^2 D_1 d^2 D_2 \dots d^2 D_N e^{-\beta E(\mathbf{r}_1^{(s)}, \mathbf{D}_1^{(s)}, \dots)}} \quad (47)$$

- (c) Find an expression (involving at most two multidimensional integrals) for the polarizability α of the liquid, defined by

$$\mathbf{P} = \alpha \mathbf{E} \quad (48)$$

where \mathbf{P} is the total polarization of the liquid, and \mathbf{E} is a small electric field. You may assume that the only interaction with the electric field is due to the fixed (in magnitude) dipole moments of the molecules.

Solution:

To begin, we need to recognize how the energy of the system is changed when we apply an electric field. The energy of each molecule is given by

$$U_i = -\mathbf{D}_i \cdot \mathbf{E} \quad (49)$$

and the total energy of interaction with the electric field is thus given by

$$U = -\mathbf{P} \cdot \mathbf{E} \quad (50)$$

At this point, if we hadn't used symmetry in the first part, we could just throw in our extra electric field term, and we'd be golden. Instead, we'll

recognize that the formula for \mathbf{P} is like Equation 47, but with a few minor changes, adding the extra energy contribution and computing an average of \mathbf{D} instead of $|\mathbf{D}|^2$:

$$\mathbf{P}(\mathbf{E}) = \frac{\int d^3r_1 d^3r_2 \cdots d^3r_N \int d^2D_1 d^2D_2 \cdots d^2D_N \left(\sum_i^N \mathbf{D}_i \right) e^{-\beta(E(\mathbf{r}_1^{(s)}, \mathbf{D}_1^{(s)}, \dots) - \sum_i^N \mathbf{D}_i \cdot \mathbf{E})}}{\int d^3r_1 d^3r_2 \cdots d^3r_N \int d^2D_1 d^2D_2 \cdots d^2D_N e^{-\beta(E(\mathbf{r}_1^{(s)}, \mathbf{D}_1^{(s)}, \dots) - \sum_i^N \mathbf{D}_i \cdot \mathbf{E})}} \quad (51)$$

To find α , we'll need to take a derivative with respect to \mathbf{E} , and evaluate in the limit as $\mathbf{E} \rightarrow 0$. Since \mathbf{P} and \mathbf{E} are both vectors, we'll need to be a bit careful about how we perform the derivative. We know the polarizability α must be a scalar because of the rotational symmetry of liquids, which tells us that the polarization *must* be in the same direction as the electric field (or opposite). Since we know this, let's just calculate a single component of the derivative of a single component of the polarization:

$$\begin{aligned} \alpha &= \left(\frac{\partial P_x}{\partial E_x} \right)_{E_y, E_z, T} \quad (52) \\ &= \beta \frac{\int d^3r_1 d^3r_2 \cdots d^3r_N \int d^2D_1 d^2D_2 \cdots d^2D_N \left(\sum_i^N D_{ix} \right) \left(\sum_j^N D_{jx} \right) e^{-\beta(E(\mathbf{r}_1^{(s)}, \mathbf{D}_1^{(s)}, \dots) - \sum_i^N \mathbf{D}_i \cdot \mathbf{E})}}{\int d^3r_1 d^3r_2 \cdots d^3r_N \int d^2D_1 d^2D_2 \cdots d^2D_N e^{-\beta(E(\mathbf{r}_1^{(s)}, \mathbf{D}_1^{(s)}, \dots) - \sum_i^N \mathbf{D}_i \cdot \mathbf{E})}} \\ &\quad + \beta \langle P_x \rangle \langle P_x \rangle \quad (53) \end{aligned}$$

where I took a bit of a shorthand (and shortcut on the second term in the derivative, which came about from taking the derivative of the partition function. This term is going to vanish when we consider small \mathbf{E} , since the polarization is zero in that limit. In fact, let's just go ahead and take this limit, setting $\mathbf{E} \rightarrow 0$ so we find the linear response:

$$\alpha = \beta \frac{\int d^3r_1 d^3r_2 \cdots d^3r_N \int d^2D_1 d^2D_2 \cdots d^2D_N \left(\sum_i^N D_{ix} \right) \left(\sum_j^N D_{jx} \right) e^{-\beta E(\mathbf{r}_1^{(s)}, \mathbf{D}_1^{(s)}, \dots)}}{\int d^3r_1 d^3r_2 \cdots d^3r_N \int d^2D_1 d^2D_2 \cdots d^2D_N e^{-\beta E(\mathbf{r}_1^{(s)}, \mathbf{D}_1^{(s)}, \dots)}} \quad (54)$$

- (d) Find an expression with no integrals at all that relates α to $\langle |\mathbf{P}|^2 \rangle$.

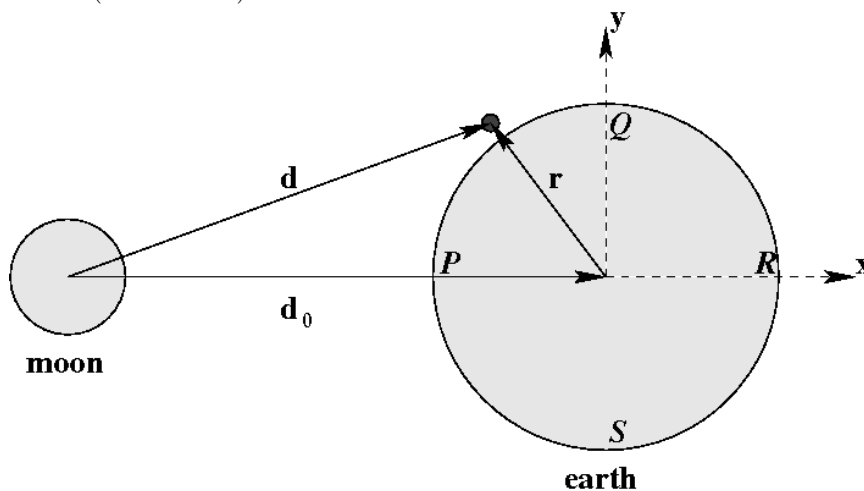
Solution:

With only a very small amount of care, we can now see that Equation 53 is *very* close to our formula for the variance of the polarization. The only difference is that we have just the x components of the dot product that is in Equation 47. Fortunately, but rotational symmetry each of the three components will be identical, so we can just divide by three and write

$$\alpha = \frac{\langle |\mathbf{P}|^2 \rangle}{3k_B T} \quad (55)$$

This is what is known as the fluctuation-dissipation theorem. The unusual-looking factor of 3 comes about because the fluctuation-dissipation theorem in its usual formulation would relate α to $\langle P_x^2 \rangle$.

- (a) Calculate the tidal force on a mass m near the surface of the earth (mass M_e) due to the moon (mass M_m) (see figure). Neglect all other celestial bodies (like the sun).



The tidal force \mathbf{F}_{tid} contains all effects on the mass m due to the moon:

$$m\ddot{\mathbf{r}} = m\mathbf{g} + \mathbf{F}_{\text{tid}}.$$

Here \mathbf{r} is the position in the earth's reference frame, which is not an inertial frame.

- (b) Discuss the relative size and direction of \mathbf{F}_{tid} at the points P, Q, R, S in the figure.
- (c) Calculate the height difference between the tides at points P and Q . Assume that the oceans cover the whole surface of the earth. You may also assume that the surface of the ocean is an equipotential surface. Make appropriate approximations based on the fact that the distance d_0 between the earth and the moon is much larger than the radius R_e of the earth: $R_e/d_0 \ll 1$.

a) The system consisting of the Earth and the Moon is our inertial system S_0 . We are interested in forces as experienced on the Earth (our non inertial reference frame S)

In S_0 , Newton's law holds

$$m \ddot{\vec{r}}_0 = \vec{F}$$

An object on the Earth with position \vec{r} has velocities $\dot{\vec{r}}_0, \dot{\vec{r}}$ in S_0, S connected by the relative velocities of S_0 and S :

$$\dot{\vec{r}}_0 = \dot{\vec{r}} + \vec{V} \quad (\text{non-relativistic})$$

$$\Rightarrow \ddot{\vec{r}}_0 = \ddot{\vec{r}} - \vec{A} \quad | \cdot m$$

$$m \ddot{\vec{r}} = \vec{F} - m \vec{A} \quad (*) \text{ in the non-inertial frame } S.$$

[The additional force $\vec{F}_{\text{inertial}} = -m \vec{A}$ is what you experience in an accelerated ref. frame, for example a plane at take-off]

acceleration of Earth due to moon:

$$\vec{A} = -G M_m \frac{\hat{d}_0}{d_0^2} \quad \text{notation see fig.}$$

in (*)

$$m \ddot{\vec{r}} = \vec{F} - m \vec{A}$$

$$= \left(m \vec{g} - G M_m m \frac{\hat{d}}{d^2} \right) + G M_m m \frac{\hat{d}_0}{d_0^2}$$

$$= m \vec{g} + G M_m m \left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2} \right)$$

$$= \vec{F}_{\text{total}}$$

The tidal force has the simple form of being the difference between the actual force of the moon on mass m and the corresponding force if mass m were located at the center of the earth.

b)

point P: $\vec{d} \parallel \vec{d}_0$, $d < d_0$

$$\Rightarrow \frac{1}{d^2} > \frac{1}{d_0^2} \rightarrow F_{\text{tidal}, P} < 0$$

\rightarrow points towards the moon

point R: $\vec{d} \parallel \vec{d}_0$, $d > d_0$

$$\Rightarrow \frac{1}{d^2} < \frac{1}{d_0^2} \rightarrow F_{\text{tidal}, R} > 0$$

\rightarrow points away from the moon

$F_{\text{tidal}, P}$ and $F_{\text{tidal}, Q}$ are approximately equal in size. They differ on the order of $R_0/d_0 \ll 1$, which we can neglect considering the simplicity of the analysis.

point Q: x components (see fig.) of \vec{d}, \vec{d}_0 almost cancel, but only d has a y component \Rightarrow

$F_{\text{tidal}, Q} < 0$, \rightarrow points towards center of earth

point S: same as for point Q.
 \Rightarrow take home message: 2 tides per day.

- c) In the absence of dynamic effect the surface of the ocean must be an equipotential surface, because water cannot exert shear forces and all forces on a fluid element are normal:

$$U(P) = U(Q)$$

$$mgh_P + U_{tid}(P) = mgh_Q + U_{tid}(Q)$$

$$mgh = U_{tid}(Q) - U_{tid}(P)$$

the tidal force from part a) can be written as

$$\vec{F}_{tidal} = -\nabla U_{tidal}$$

$$= -GM_m m \left(\frac{L}{d} + \frac{x}{d^3} \right)$$

for point Q: with $d = \sqrt{d_0^2 + r^2}$, $r \approx R_e$
and $x = 0$

$$U_{tid}(Q) = -GM_m m \left(\frac{1}{d_0 \sqrt{1 + \left(\frac{R_e}{d_0}\right)^2}} \right) \left| \sqrt{1+\epsilon} \approx 1 - \frac{1}{2}\epsilon \right.$$

$$\approx -\frac{GM_m m}{d_0} \left(1 - \frac{R_e^2}{2d_0^2} \right)$$

for point P: with $d = d_0 - R_e$, $x = -R_e$

$$U_{tid}(P) = -\frac{GM_m m}{d_0} \left(1 - \frac{R_e}{d_0} - \frac{R_e}{d_0^2} \right) \approx -\frac{GM_m m}{d_0} \left(1 + \frac{R_e^2}{d_0^2} \right)$$

$$\text{and } h = \frac{3}{2} \frac{M_m R_e^4}{M_e d_0^3} \quad \text{using } \frac{1}{1-\epsilon} \approx 1 + \epsilon + \epsilon^2 \quad g = \frac{GM_e}{R_e^2}$$

In an ionic crystal, a spinless ion of charge $-e$ and mass m is placed in a cubic lattice which gives rise to a spherically symmetric harmonic potential

$$V(\mathbf{r}) = \frac{1}{2}m\omega^2r^2 = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2)$$

where r is the distance from the equilibrium position of the ion.

- (a) Find the energy eigenstates and eigenvalues of the ion in the three-dimensional harmonic potential. If any energy level is degenerated, find its degree of degeneracy.
- (b) When a weak, uniform magnetic field is applied along the z axis, magnetic dipole interaction gives rise to the perturbation Hamiltonian,

$$H_1 = -\boldsymbol{\mu}_L \cdot \mathbf{B} = \frac{eB}{2mc}L_z = \frac{eB}{2mc}(xp_y - yp_x)$$

where B is the magnetic field amplitude, c is the speed of light, and L_z is the z component of the orbital angular momentum \mathbf{L} . Calculate the energy shifts of the ground state and the 1st excited states up to the 1st order of the perturbation theory.

8.

$$(a) \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{r}^2 = \hat{H}_x + \hat{H}_y + \hat{H}_z$$

sum of three 1D SHOs

$$\begin{cases} \hat{H}_x |n_x\rangle = (n_x + \frac{1}{2}) \hbar \omega |n_x\rangle \\ \hat{H}_y |n_y\rangle = (n_y + \frac{1}{2}) \hbar \omega |n_y\rangle \\ \hat{H}_z |n_z\rangle = (n_z + \frac{1}{2}) \hbar \omega |n_z\rangle \end{cases}, \text{ where } n_x, n_y, n_z = 0, 1, 2, \dots$$

$$\Rightarrow \hat{H} |n_x n_y n_z\rangle = (n_x + n_y + n_z + \frac{3}{2}) \hbar \omega |n_x n_y n_z\rangle$$

$$= (n + \frac{3}{2}) \hbar \omega |n_x n_y n_z\rangle$$

where $|n_x n_y n_z\rangle = |n_x\rangle \otimes |n_y\rangle \otimes |n_z\rangle$

and $n = n_x + n_y + n_z$

For a given n ,

n_x	n_y	n_z	# of states
0	$\begin{cases} n \\ n-1 \\ \vdots \\ 1 \end{cases}$	$\begin{cases} 0 \\ 1 \\ \vdots \\ n \end{cases}$	$\begin{cases} n+1 \\ n+1 \\ \vdots \\ n+1 \end{cases}$
1	$\begin{cases} n-1 \\ n-2 \\ \vdots \\ 0 \end{cases}$	$\begin{cases} 0 \\ 1 \\ \vdots \\ n-1 \end{cases}$	$\begin{cases} n \\ n \\ \vdots \\ n \end{cases}$
\vdots	\vdots	\vdots	\vdots
n	0	0	1

$$\text{degeneracy } g_n = \sum_{i=1}^{n+1} i = \frac{1}{2} (n+1)(n+2)$$

$$(b) \hat{H}_1 = \frac{eB}{2mC} \hat{L}_z = \frac{eB}{2mC} [2\hat{P}_y - \hat{P}_x]$$

$$\begin{aligned} \hat{L}_z |n_x n_y n_z\rangle &= \left[\sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_x + \hat{a}_x^\dagger) i \sqrt{\frac{m\hbar\omega}{2}} (\hat{a}_y^\dagger - \hat{a}_y) \right. \\ &\quad \left. - \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_y + \hat{a}_y^\dagger) i \sqrt{\frac{m\hbar\omega}{2}} (\hat{a}_x^\dagger - \hat{a}_x) \right] |n_x n_y n_z\rangle \\ &= i\hbar (\hat{a}_x \hat{a}_y^\dagger - \hat{a}_x^\dagger \hat{a}_y) |n_x n_y n_z\rangle \end{aligned}$$

$$\Rightarrow \hat{L}_z |n_x n_y n_z\rangle = i\hbar (\sqrt{n_x} \sqrt{n_y+1} |n_x-1 n_y+1 n_z\rangle - \sqrt{n_x+1} \sqrt{n_y} |n_x+1 n_y-1 n_z\rangle)$$

(i) For the ground state, $n=0$, $|n_x n_y n_z\rangle = |000\rangle$

$$\hat{L}_z |000\rangle = 0 \Rightarrow \boxed{E^{(1)} = \frac{eB}{2mC} \langle 000 | \hat{L}_z | 000 \rangle = 0}$$

(ii) For the 1st excited state, $n=1$ (3 states are degenerate)

$$|n_x n_y n_z\rangle = |100\rangle, |010\rangle, |001\rangle$$

$$\hat{L}_z |100\rangle = i\hbar |010\rangle, \hat{L}_z |010\rangle = -i\hbar |100\rangle, \hat{L}_z |001\rangle = 0$$

In the matrix formalism

$$\hat{H}_1 = \frac{eB\hbar}{2mC} \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{with the basis vectors} \quad \begin{aligned} |100\rangle &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ |010\rangle &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ |001\rangle &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

$$\Rightarrow |\hat{H}_1 - \lambda \mathbb{I}| = 0 \Rightarrow \lambda = 0, \pm \frac{eB\hbar}{2mC}$$

Applying the degenerate perturbation theory

$$\boxed{E^{(1)} = \pm \frac{eB\hbar}{2mC}, 0}$$