Chapter 2

Waves – a Simple-Minded General Picture

2.1 The Motion of a Simple Oscillator

We will begin with presenting something that is usually referred to as a *Simple Harmonic Oscillator* (SHO), or, more specifically, a *Simple Harmonic Spring Oscillator*. It's a very simple device, it consists of only of a spring and a weight of mass m. On the Web, one can find a whole bunch of pictures and animations showing a spring oscillator – here is one example on YouTube, but you may find many more.

If one stretches a spring, it "protests" – it does not want to be distorted. The "protest" manifests itself as a "pulling-back" force. Suppose that the end of a non-stretched spring is at $x_0 = 0$. Now, you pull it, it elongates, so the end is not at a new position Δx. The spring "protests", and creates a force F that tends to restore its original unstretched length. Therefore, we call this force a *restoring force*. There is a simple relation between the elongation Δx and the restoring force F, known as the Hooke's Law. It states that F is proportional to Δx:

$$F = -k \cdot \Delta x \tag{2.1}$$

where k is a coefficient called the *spring constant*, or the ***spring's stiffness constant***. Its value characterizes a given spring, and it has to be individually determined for each spring – which is not difficult. For example, one may suspend a mass m from the spring and measure how much the spring extends.

The weight pulls down the spring with a force of $m \cdot g$ (where $g = 9.81$ m/s^2 is the acceleration due to gravity), and the spring tends to pull the weight up with a force of the same magnitude. Let's "downwards" be the "positive direction". Hence, the restoring force in the present case is $F = -m \cdot g$, and from the Eq. 2.1 we get:

$$k = -\frac{F}{\Delta x} = -\frac{-m \cdot g}{\Delta x} = \frac{m \cdot g}{\Delta x}. \tag{2.2}$$

The spring extension Δx (we can also call it the *displacement* of the weight m) can be readily measured using a ruler, and after carrying out the multiplication and the division, we get the value of k. From the Eq. 2.1 it follows that the unit of k must be a (unit of force)/(unit of length), i.e., N/m. The unit of the expression in the numerator is (mass uni)·(acceleration unit), i.e., kg·m/s^2 which is the same the Newton, N, and the unit of Δx is a meter, m, so we indeed get k with the right unit.

One more comment: why do we put minus in the Eq. 2.1? Well, the answer is simple, F, the restoring force, always "pulls" in opposite direction relative to the extension, and therefore the minus sign.

Now, lets consider the motion of such an oscillator. In order to make things even simple, let's consider not a weight suspended from a vertical spring, but an oscillator lying horizontally on a table. The other end of the spring is at a fixed position. In such way the only force acting on the mass m is the spring's restoring force, and we don't need to include the gravity – with a single force, the problem indeed gets simpler. You may protest, however: *Simpler? And what about the friction?!*. No, there are nearly frictionless tables – e.g., utilizing an air-cushion – so we can assume that we are using one of them, and we don't need to worry about friction.

So, the only force acting on the weight at a position other than than the equilibrium point x_0 is the spring's restoring force. According to the Second Law of Dynamics, a body of mass m acted upon by a force F moves in a direction parallel to the force vector, with acceleration:

$$a = \frac{F}{m} \tag{2.3}$$

Now, we need to start using some calculus elements.

*** Students allergic to calculus, you may skip the calculations, an important ***
**** thing is only that you understand the final result given by the Eq. 2.6. ****

The acceleration is the second time derivative of the displacement Δx:

$$a = \frac{d^2\Delta x}{dt^2} \tag{2.4}$$

By combining the Eqs. 2.2, 2.3 and 2.4 we obtain:

$$\frac{d^2\Delta x}{dt^2} = -\frac{k}{m}\Delta x \tag{2.5}$$

What we got is the so-called *equation of motion* of the weight m. It's a second-order differential equation (most equations of motion in physics are). The solutions of differential equations are not numbers (like in the case of algebraic equations), but *functions*. We will not explain here how to solve such equations – we will simply show you the solution:

$$\Delta x(t) = A\sin\left(\sqrt{\frac{k}{m}}\cdot t\right) \tag{2.6}$$

We write $\Delta x(t)$ to stress that the displacement is a function of time. Clearly, it is an oscillating function. The A coefficient is called the *amplitude*, and it's the maximum displacement to the left and to the right that may occur in the course of the oscillation process.

I (Dr. Tom) dont like to pass information to students by telling them: *It is so!* and then exclaiming: *You have to believe me!*. So, we have to show that the Eq. 2.6 is indeed a solution of the equation of motion. How we do that? It's simple, we plug the solution into the equation and check if the left side (L) indeed is equal to the right side (R). So, at the left side we have the second derivative, which means that we have to differentiate the function twice. We have to use the general differentiation formulas for $\sin(C\cdot t)$ and $\cos(C\cdot t)$ functions, where B and C are arbitrary constants:

$$\frac{d}{dt}B\sin(C\cdot t) = BC\cdot\cos(C\cdot t), \quad \text{and} \quad \frac{d}{dt}B\cos(C\cdot t) = -BC\cdot\sin(C\cdot t)$$

OK, so after the first differentiation of the solution function we get:

$$\frac{d}{dt}A\sin\left(\sqrt{\frac{k}{m}}\cdot t\right) = A\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}\cdot t\right)$$

And after the second:

$$\frac{d}{dt} A\sqrt{\frac{k}{m}} \cos\left(\sqrt{\frac{k}{m}} \cdot t\right) = -A\frac{k}{m} \sin\left(\sqrt{\frac{k}{m}} \cdot t\right)$$

And note that this is exactly the same what we get after inserting the expression for Δx in the Eq. 2.6 into the right side of the Eq. 2.5. So, indeed we get L = R, and Eq. 2.6 is a correct solution of the equation of motion.

** Who decided to skip the calculus-infested part, may continue from here **

Now, we only have to explain what is the physical meaning of the $\sqrt{k/m}$ expression. Suppose that at certain arbitrarily chosen instant of time t' the displacement is $\Delta x(t')$. Suppose that T is the time period needed for making one full oscillation cycle. Therefore, the displacement at t' is the same as at $t' + T$:

$$\Delta x(t') = \Delta x(t' + T)$$

In other words, it must be:

$$A \sin[\sqrt{\frac{k}{m}} \cdot t'] = A \sin[\sqrt{\frac{k}{m}} \cdot (t' + T)] \tag{2.7}$$

And we know that the properties of the $\sin(\alpha)$ function are such for any α:

$$\sin(\alpha) = \sin(\alpha + 2\pi)$$

In view of the above, it means th at the relation between the arguments in the Eq. 2.7 is

$$\sqrt{\frac{k}{m}} \cdot (t' + T) = \sqrt{\frac{k}{m}} \cdot t' + 2\pi$$

From that we get:

$$\sqrt{\frac{k}{m}} \cdot T = 2\pi$$

meaning that ***the oscillation period of the oscillator is:***

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{2.8}$$

Most oscillations we deal with in real life are fast, and very fast. Therefore, the oscillation periods are very small numbers – not too convenient to work with. A much more "user-friendly" parameter is the ***frequency*** f, i.e., the number of oscillation cycles per one second:

$$f = \frac{1}{T} \tag{2.9}$$

By combining the Eqs. 2.8 and 2.9, we get:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{2.10}$$

But physicists are not fully satisfied with the frequency f, they prefer something else they call the ***angular frequency***, which is denoted as ω (lower case Greek omega) and is simply the "ordinary" frequency multiplied by 2π: $\omega = 2\pi f$. Then:

$$\omega = \sqrt{\frac{k}{m}} \tag{2.11}$$

and the Eq. 2.6, describing the time dependence $\Delta x(t)$ of the oscillator's displacement, takes a simple and compact form:

$$\Delta x(t) = A\sin(\omega \cdot t) \tag{2.12}$$

2.2 The Equation of Wave Motion

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We are supposed to talk about waves – so why have we used 5 pages to talk about oscillators, which obviously are ***not*** waves?

The answer is simple. Once mathematical description of simple oscillators is understood, it is easier to understand the equations describing waves!

Let's rewrite the equation for time-dependent displacement in a simple harmonic[1] oscillator: but now we "rename" the oscillation direction to y (x will be reserved for other purposes):

$$\Delta y(t) = A\sin(\omega \cdot t)$$

[1]Not all simple oscillators are harmonic, i.e., such that their displacement vs. time is described by the $\sin(\omega t)$ function. There is a whole variety of oscillators that are simple, but not harmonic: we call them ***anharmonic***. For instance, a simple pendulum is a SHO oscillator as long as the displacement amplitude is small (say, 2-3°, but if it swings as far as 90° from the vertical axis, it becomes strongly anharmonic.

But we will now return to an earlier version in terms of the oscillation ***period*** T:

$$\Delta y(t) = A \sin\left(\frac{2\pi}{T} \cdot t\right) \tag{2.13}$$

About this function, one can tell that it describes a ***displacement periodic in time***. Physicists also like the word ***perturbation***, one of its meanings is a deviation or a displacement from a state of equilibrium or a normal course of action. So, a deviation of an oscillator relative to the equilibrium point can also be called “a perturbation”. Some people even prefer it, because it sounds “more professionally”. Well, perhaps, but we will keep using the “deviation”. Anyway, is you at some moment hear a person talking about a “perturbation” in oscillatory or wave motion, you will know what the person has in mind.

Now, think of a chain of beads strung on a string, as in Fig. 2.1:

Figure 2.1: Beads on a stretched string.

Then, let someone grabs the left end of the string and starts wiggling it up and down – such an action will create a wave (physicists often say: “will excite a wave”) propagating towards the other end:

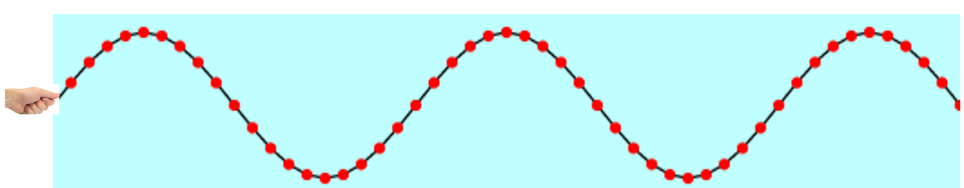

Figure 2.2: Exciting a wave in the bead chain.

It's clear that such an action created a state ***periodic in space*** – along the horizontal direction, call it x, or the ***propagation directio***.

The wave period along this direction – the distance between two consecutive maxima (usually, called "crests"), or two consecutive minima (often called "troughs") is referred to as the ***wavelength***, and conventionally denoted as λ (the lower-case Greek character "lambda"; upper case is Λ, the Roman equivalent is L).

By analogy to the function Eq. 2.13 which describes a periodicity in time, we can write a function describing the periodicity in space – just by replacing the time period T by the period in space λ, and the time variable t in the function's argument by the spatial variable x:

$$\Delta y(x) = A \sin\left(\frac{2\pi}{\lambda} \cdot x\right) \tag{2.14}$$

OK, but we need a function which is periodic ***both in time and space***, right? How we can get it? Well, it's simple: insert into the argument of the function in the Eq. 2.14 a second term corresponding to time periodicity (note that the displacement in the y direction becomes now a function of both x and time, $\Delta y = \Delta y(x, t)$):

$$\Delta y(x, t) = A \sin\left(\frac{2\pi}{\lambda} \cdot x - \frac{2\pi}{T} \cdot t\right). \tag{2.15}$$

This expression is often called a *wave function*, or a *waveform* – the latter term is more often used in electrical engineering, while the former one is more often used by physicists.

Conventionally, in the wave function's argument the "spacial" term with x goes first, and the term with the time variable t goes as the second. But why did I put the minus sign, and not a plus sign in between?

Well, I can put the plus sign, if you wish! Here you go! It's also a perfectly correct wave equation:

$$\Delta y(x, t) = A \sin\left(\frac{2\pi}{\lambda} \cdot x + \frac{2\pi}{T} \cdot t\right). \tag{2.16}$$

Are both correct, indeed? So the sign makes no difference?

Well, it does: the function with the minus sign propagates from the left to the right, and the wave in the Eq. 2.16 with the plus sign travels from the right to the left. Usually, we think about moving from the left to the right as of a motion in the "positive" direction, and motion from the right to the left is a motion in the "negative direction". And we always prefer positive things over negative things, so that if we want just to write down a "generic wave equation", we usually write it with a minus sign.

Hurray!!! Chapeau bas! – as the French say if there is a moment worth celebration (it means: Hats off!). We have derived the simplest wave equation! But even though it is simple, it is very useful, and we will use it several times in this course.

But this is not the end – in fact, we may do several more operations on the equation. However, at this moment it becomes very instructive to "set things in motion", i.e., to start using **animated** versions of Figure 2.2. But, regretfully, your instructor (Dr. Tom, I mean) has not yet learned how to install animated pictures in PDF files. I will figure out, hopefully, but now I have no idea. Well, but it's not a big problem! I know how to put animations into Power Point files! So, please click on this link which will open a Power Point file in which I installed the animations needed, plus a text describing things, and, please, continue reading!

One more important thing before you open the Power Point – a definition. Namely, the argument of the oscillating part of the wave function is called the ***phase*** of the wave. In our notation, what oscillates is the sine function:

$$\sin\left(\frac{2\pi}{\lambda}\cdot x - \frac{2\pi}{T}\cdot t\right),$$

so that the phase in the present case is:

$$\frac{2\pi}{\lambda}\cdot x - \frac{2\pi}{T}\cdot t$$

Often an abbreviation $\Phi(x, t)$ is used, to save space and time needed for typing the whole phase, which may be even more elaborate than in the present case (e.g., in wave functions of two- or three-dimensional waves which we will be talking about in the next chapter).

In addition to the time- and space-dependent terms, in the phase there may be a constant term, which we can call, for instance, Φ_0. Then the whole expression for the phase becomes:

$$\frac{2\pi}{\lambda}\cdot x - \frac{2\pi}{T}\cdot t + \Phi_0 = \Phi(x, t) + \Phi_0$$

The constant term Φ_0 is sometimes called the *phase shift*. Which is not 100% logical if we talk about a **single** wave, because it only makes sense to talk about a shift if it is clearly defined, *relative to what* the shifted object is shifted. But the term makes much sense is we are talking about **two waves**, one of which has a constant term Φ_1 in its phase, and the other has a constant term Φ_2. Then, there obviously is a phase shift *between the two waves*, and its value then is $\Delta\Phi = \Phi_1 - \Phi_2$. If $\Phi_1 > \Phi_2$, we say that Wave One is *leading*, and if $\Phi_1 < \Phi_2$, we say that it is *lagging*.