1 Physical Constants

$$\text{fine structure constant}: \quad \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$$

$$\text{Rydberg energy}: \quad E_o = \frac{m_e e^4}{2\hbar^2 (4\pi\epsilon_o)^2} = \frac{m_e c^2 \alpha^2}{2}$$

$$\text{Bohr magneton}: \quad \mu_B = \frac{e\hbar}{2m_e}$$

$$\text{Bohr radius}: \quad a_o = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2}$$

2 Math

Vector Calculus

Triple products:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$
$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

Product rules:

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$
$$\nabla \cdot (\phi\mathbf{A}) = \phi\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla\phi$$
$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) + \mathbf{A} \cdot (\nabla \times \mathbf{B})$$
$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A} + + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

Second derivatives:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$
$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

Green's theorem:

$$\int_V \left(\psi\nabla^2\phi - \phi\nabla^2\psi\right) dV = \oint_S (\psi\nabla\phi - \phi\nabla\psi) \cdot \mathbf{dS}$$

Divergence theorem:

$$\int_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot \mathbf{dS}$$

Stoke's theorem:

$$\oint_S (\nabla \times \mathbf{A}) \cdot \mathbf{dS} = \oint_C \mathbf{A} \cdot \mathbf{dl}$$

Spherical coordinates:

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$
$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \theta}$$
$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\hat{\phi}$$
$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Cylindrical coordinates:

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$
$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$
$$\nabla \times \mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\phi} + \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right)\hat{\mathbf{z}}$$
$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Solutions of Laplace's Equation

in cylindrical coordinates (independent of z):

$$\Phi(\rho, \phi) = a_o \log(\rho) + \sum_{n=1}^{\infty}\left(\frac{a_n}{\rho^n} + b_n\rho^n\right)(c_n\cos n\phi + d_n\sin n\phi)$$

in spherical coordinates:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^l + \frac{B_{lm}}{r^{l+1}}\right)Y_{lm}(\theta,\phi)$$

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

(with azimuthal symmetry)

Series and Sums

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta)$$

$$(1+x)^n = \sum_{k=1}^{n}\frac{n!}{k!(n-k)!}x^k$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

$$\frac{1}{|\mathbf{x}-r'\hat{\mathbf{z}}|} = \sum_{l}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)$$

Integrals

$$\int_{-\infty}^{\infty}e^{ixy}dy = 2\pi\delta(x)$$

$$\int_{0}^{\infty}x^n e^{-x}dx = n!\,,\ \text{integer } n$$

$$\int_{-\infty}^{\infty}e^{-ax^2-2bx}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{b^2/a}$$

Combination

The number of possible combinations of r objects from a set of n objects:

$${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Stirling's Approximation

$$\ln(n!) \approx n\log(n) - n \text{ for } n \gg 1$$

Trigonometry Identities

$$\sin(x\pm y) = \sin x\cos y \pm \cos x\sin y$$
$$\cos(x\pm y) = \cos x\cos y \mp \sin x\sin y$$

Special Functions

Spherical Bessel functions:

$$j_0(z) = \frac{\sin z}{z} \qquad n_0(z) = -\frac{\cos z}{z}$$
$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z} \qquad n_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z}$$

Legendre polynomials:

$$P_0(x) = 1 \qquad P_2(x) = \frac{1}{2}\left(3x^2-1\right)$$
$$P_1(x) = x \qquad P_3(x) = \frac{1}{2}\left(5x^3-3x\right)$$

$$P_l(-x) = (-1)^l P_l(x)$$
$$\int_{-1}^{1}P_l(x)P_{l'}(x)dx = \frac{2}{2l+1}\delta_{ll'}$$
$$P_l^m(x) = \left(1-x^2\right)^{m/2}\frac{d^m P_l}{dx^m}$$

Spherical harmonics:

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \qquad Y_{22} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{i2\phi}$$
$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} \qquad Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}$$
$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{20} = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y_{lm}^*(\theta,\phi)$$
$$Y_{l,m}(-\mathbf{x}) = (-1)^l Y_{lm}(\mathbf{x})$$

3 Classical Mechanics

Lagrange's Equations

$L = T - V$ where $T = \frac{1}{2}\sum_{k=1}^{N} m_k v_k^2$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

4 Electromagnetism

Lorentz force: $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

Maxwell's Equations

$$\nabla \cdot \mathbf{D} = \rho \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} + \mathbf{M} = \frac{1}{\mu}\mathbf{B}$$

Scalar and Vector Potentials

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

Electric and Magnetic Dipole Fields

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p}) - \mathbf{p}}{r^3}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m}}{r^3}$$

Energy Density and Poynting Vector

$$U = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

5 Quantum Mechanics

Schrodinger Equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

Ehrenfest's Theorem

$$\frac{d}{dt}\langle A\rangle = \left\langle \frac{i}{\hbar}[H, A] + \left(\frac{\partial A}{\partial t}\right)\right\rangle$$

1D Harmonic oscillator

Creation and annihilation operators: $[a, a^\dagger] = 1$

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{p}{\sqrt{2m\omega\hbar}}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{p}{\sqrt{2m\omega\hbar}}$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

Angular Momentum

Raising and lowering operators:

$$J_\pm|j, m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle$$

Pauli matrices

$$\sigma_x = \frac{2}{\hbar}S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \frac{2}{\hbar}S_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \frac{2}{\hbar}S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Perturbation theory

for nondegenerate states:

$$E_n \approx E_n^o + \langle n| V |n\rangle + \sum_{m\neq n} \frac{|\langle n| V |m\rangle|^2}{E_n - E_m} + \cdots$$

$$|\psi_n\rangle \approx |n\rangle + \sum_{m\neq n} \frac{\langle m| V |n\rangle}{E_n - E_m} |m\rangle$$

6 Statistical Mechanics

Entropy

$$\Delta S = \int \frac{đQ_{\text{quasistatic}}}{T}$$

$$đQ = TdS$$

$$C_\alpha = T\left(\frac{\partial S}{\partial T}\right)_\alpha$$

First Law

$$\Delta U = Q + W$$

$$dU = đQ + đW$$

$$dU = TdS - pdV$$

Probability and Partition Functions

The system is in thermal contact with an environment with temperature T.

$$P_i = \frac{e^{-\beta E_i}}{Z}$$

$$Z = \sum_i^{\text{all states}} e^{-\beta E_i} \qquad \text{where } \beta = \frac{1}{k_B T}$$

$$F = -k_B T \ln Z$$

$$U = \sum_i^{\text{all states}} P_i E_i$$

$$S = -k_B \sum_i^{\text{all states}} P_i \ln P_i$$