# OSU PHYSICS DEPARTMENT
# COMPREHENSIVE EXAMINATION #97

September 27 and 28, 2004

Comprehensive examination for Fall 2004

PART I, Monday September 27, 9:00 am

## General Instructions

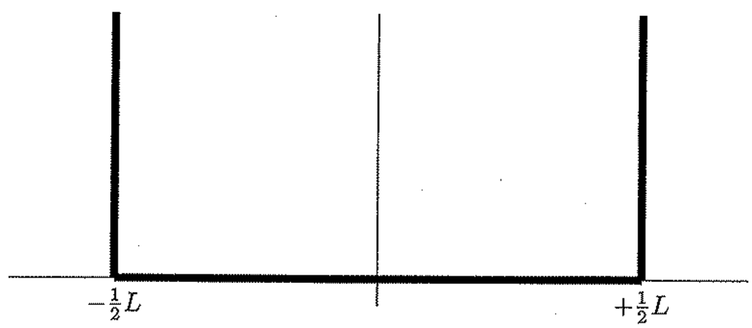
This Comprehensive Examination for Fall 2004 consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, September 27, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:30 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, September 28, at 9:00 am and 1:30 pm.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit - especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

## Problem 1.




A spin-less particle of mass $M$ moving freely in one dimension is confined to a box of length $L$ centered on the origin of a coordinate system. The mean value of the position is observed to vary with time as

$$< x >= A \sin(\omega t + \phi)$$

where

$$\omega = 3\frac{\pi^2 \hbar}{2ML^2}$$

(a.) At time $t = T$ , the energy $E$ of the particle is measured exactly. What values may be found for $E$, and with what probabilities?

(b.) What is the maximum possible value for the amplitude $A$?

HINT:

$$\int dz z \sin(az)\cos(bz) = \frac{1}{2(a+b)^2}[\sin((a+b)z) - (a+b)z\cos((a+b)z)] +$$

$$\frac{1}{2(a-b)^2}[\sin((a-b)z) - (a-b)z\cos((a-b)z)]$$

P1

**Solution to problem UGQM**

**a.** The energy seigenvalues for particle in a box of length $L$ are $E_n = n^2 \frac{\pi^2 \hbar^2}{2ML^2}$ .

Expectation values for a wave function mixture of states $m$ and $n$ vary with the Bohr angular frequency $\omega_{nm} = (E_n - E_m)/\hbar \;\; = (n^2 - m^2)\, \frac{\pi^2 \hbar}{2ML^2}$ .

So $n^2 - m^2 = 3$, and $n = 2$, $m = 1$ . The energies possible are $E_1$ and $E_2$ .

The normalized spatial wave functions,

determined by the boundary condition that they vanish at $x = \pm L/2$,

are $$\phi_1(x) = \sqrt{\frac{2}{L}}\, \cos(\pi x/L), \qquad \phi_2(x) = \sqrt{\frac{2}{L}}\, \sin(2\pi x/L).$$

A linear combination gives the time dependent wave function

$$\psi(x, t) = C_1\, \phi_1(x)\, e^{-iE_1 t/\hbar} + C_2\, \phi_2(x)\, e^{-iE_2 t/\hbar}$$

with $|C_1|^2$ and $|C_2|^2$ the time-independent probabilities of finding energies $E_1$ and $E_2$ respectively.

The mean value is $$\langle x \rangle = \int_{-L/2}^{L/2} dx \;\; x\, \psi(x, t)\;\; \psi^*(x, t)$$

$$= |C_1|^2 \langle x \rangle_1 + |C_2|^2 \langle x \rangle_2 + [C_1 C_2^*\, e^{-i\,\omega_{12} t} + C_1^* C_2\, e^{-i\,\omega_{21} t}]\; \langle x \rangle_{12}$$

where $$\langle x \rangle_i = \int_{-L/2}^{L/2} dx \;\; x\, \phi_i(x)\;\; \phi_i^*(x) \;=\; 0 \text{ by parity asymmetry}$$

and $$\langle x \rangle_{12} = \int_{-L/2}^{L/2} dx \;\; x\, \phi_1(x)\;\; \phi_2^*(x) = \frac{2}{L} \int_{-L/2}^{L/2} dx \;\; x\; \cos(bx)\;\sin(ax)$$

writing $b = \frac{\pi}{L}$ , $a = \frac{2\pi}{L} = 2b$

Using the Hint, $\langle x \rangle_{12} = \frac{2}{L}$.

$$\left( \frac{1}{2(a+b)^2} [\, \sin(a+b)z - (a+b)z\;\; \cos(a+b)z\, ] + \frac{1}{2(a-b)^2} [\sin(a-b)z - (a-b)z \cos(a-b)z\, ] \right)_{-L/2}^{L/2}$$

$$= \frac{2}{bL}\;\; \{[\sin \tfrac{3}{2} Lb - \tfrac{3}{2} Lb\; \cos\, \tfrac{3}{2} Lb\;\; ]/9b \;+\; [\sin Lb/2 - \frac{Lb}{2}\;\; \cos Lb/2\; ]/b\; \}$$

$bL = \pi$ so $\cos Lb/2 = 0 = \cos\frac{3}{2}Lb$ , $\sin Lb/2 = 1$, $\sin\frac{3}{2}Lb = -1$,

$$\langle x\rangle_{12} = \frac{2L}{\pi^2}\left(1-\frac{1}{9}\right) = \frac{16L}{9\pi^2} \quad \text{and } \langle x\rangle = \frac{16L}{9\pi^2}\left[C_1C_2^*\, e^{-i\,\omega_{12}t} + C_1^*C_2\, e^{-i\,\omega_{21}t}\right]$$

$$\langle x\rangle = \frac{16L}{9\pi^2}\left[C_1C_2^*\, e^{-i\,\omega_{12}t} + C_1^*C_2\, e^{-i\,\omega_{21}t}\right]$$

Only relative phase of $C_1$, $C_2$ comes in, say choose real parameters $\alpha$, $\beta$ such that $C_2 = \beta\, e^{i\alpha} C_1$, then $\langle x\rangle = \frac{16L}{9\pi^2}\,|C_1|^2\,\beta\cos(\alpha - \omega_{21}t) = \frac{16L}{9\pi^2}\,|C_1|^2\,\beta\cos(\omega_{21}t - \alpha)$

We see $\alpha = -\phi$. Determine $\beta$, $|C_1|^2$ from $A = \frac{16L}{9\pi^2}\,|C_1|^2\,\beta$ and norm $1 = |C_1|^2(1+\beta^2)$

So $|C_1|^2 = \frac{1}{1+\beta^2}$ and $|C_2|^2 = \frac{\beta^2}{1+\beta^2} = 1 - |C_1|^2$

$$\text{So } A = \frac{16L}{9\pi^2}\,\frac{\beta}{1+\beta^2},\ \beta^2 - \frac{16L}{9\pi^2 A}\,\beta + 1 = 0,\ \beta = \frac{8L}{9\pi^2 A} \pm \sqrt{\left(\frac{8L}{9\pi^2 A}\right)^2 - 1}$$

The two solutions correspond to interchanging the magnitudes of the probabilities of the two states.

**b.** $A$ has its largest value when $\frac{\beta}{1+\beta^2}$ is largest, which is when $\frac{d}{d\beta}\left(\frac{\beta}{1+\beta^2}\right) = 0$

that's when $\frac{1}{1+\beta^2} - 2\beta\,\frac{\beta}{(1+\beta^2)^2} = 0$, $1+\beta^2 - 2\beta^2 = 0$, $\beta^2 = 1$, $\beta = \pm 1$, $A = \frac{8L}{9\pi^2} \approx 0.009L$

## Problem 2.

A particle of mass $m$ is directed upward in a uniform gravitational field. The position and the momentum at time $t$ are given by $z(t)$ and $p(t)$. Suppose that the experiment is repeated in four trials with the following initial conditions:

$$\begin{array}{ll} z_1(0) = z_0 & p_1(0) = p_0 \\ z_2(0) = z_0 + \Delta z_0 & p_2(0) = p_0 \\ z_3(0) = z_0 & p_3(0) = p_0 + \Delta p_0 \\ z_4(0) = z_0 + \Delta z_0 & p_4(0) = p_0 + \Delta p_0 \end{array}$$

Use the solutions to Hamilton's equations of motion to obtain the trajectories of the four trials in phase space and demonstrate the validity of Liouville's Theorem concerning conservation of the phase space spanned by the trajectories.

P2

Solution:

First, set up Hamiltonian $H(p,z)$

$$T=\frac{1}{2}m\dot{z}^2 \qquad U=mgz \qquad \Rightarrow \qquad H=T+U=\frac{1}{2}m\dot{z}^2+mgz$$

$$p=\frac{\partial L}{\partial \dot{z}}=\frac{\partial T}{\partial \dot{z}}=m\dot{z} \qquad \Rightarrow \qquad H=\frac{p^2}{2m}+mgz$$

Hamilton's equations of motion:

$$\dot{z}=\frac{\partial H}{\partial p}=\frac{p}{m} \qquad \dot{p}=-\frac{\partial H}{\partial z}=-mg$$

Integrate: $p(t)=-mgt+p(0)$

Then $\dot{z}=\frac{p}{m}=-gt+\frac{p(0)}{m}$

Integrate: $z(t)=-\frac{1}{2}gt^2+\frac{p(0)}{m}t+z(0)$

Substitute initial conditions:

$$z_1(t)=-\frac{1}{2}gt^2+\frac{p_0}{m}t+z_0 \qquad p_1(t)=-mgt+p_0$$

$$z_2(t)=-\frac{1}{2}gt^2+\frac{p_0}{m}t+z_0+\Delta z_0 \qquad p_2(t)=-mgt+p_0$$

$$z_3(t)=-\frac{1}{2}gt^2+\frac{p_0+\Delta p_0}{m}t+z_0 \qquad p_3(t)=-mgt+p_0+\Delta p_0$$

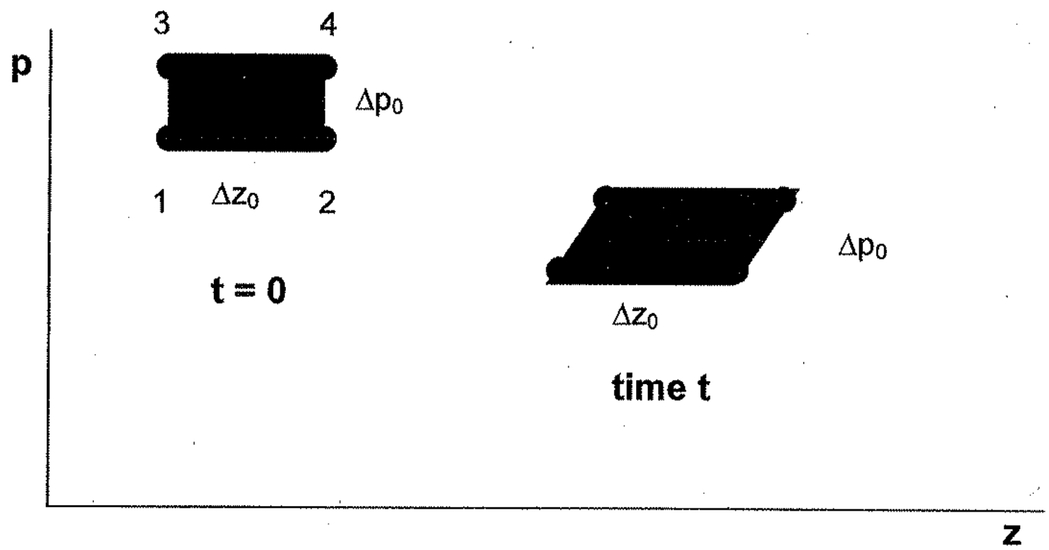
$$z_4(t)=-\frac{1}{2}gt^2+\frac{p_0+\Delta p_0}{m}t+z_0+\Delta z_0 \qquad p_4(t)=-mgt+p_0+\Delta p_0$$

Thus, at any time $t$,

$$p_1(t) = p_2(t) \quad \text{and} \quad p_3(t) = p_4(t) = p_1(t) + \Delta p_0$$

$$z_3(t) = z_1(t) + \frac{\Delta p_0}{m}t \quad \text{and} \quad z_4(t) = z_2(t) + \frac{\Delta p_0}{m}t$$

Phase space plot:




At $t = 0$, the phase space spanned is the area of the rectangle 1-2-3-4 = $\Delta p_0 \, \Delta z$. At any subsequent time, the phase space spanned is the area of the parallelogram, area = base x height = $\Delta p_0 \, \Delta z$. Thus the phase space spanned is conserved.

## Problem 3.

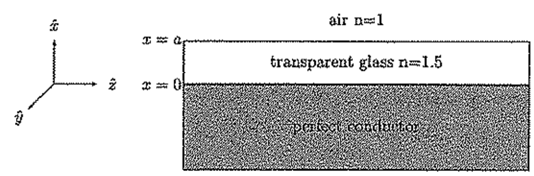
The equation of state of a gas can be written in the form

$$pV = NRT(1 + B_2(T)\frac{N}{V})$$

$B_2$ is the second virial coefficient, and is an increasing function of temperature. Find how the internal energy of the gas depends on volume, i.e. calculate $(\frac{\partial U}{\partial V})_{T,N}$. Is this positive or negative?

## Problem 4.

A *perfect* conductor coated with a layer of transparent dielectric material can act as a planar waveguide at optical frequencies. The surface of the conductor is the $yz$ plane, and the direction of propagation is $\hat{z}$. The dielectric film is of thickness $a$, and above the film there is only air. For the dielectric film, the index of refraction is 1.5 and the permeability is $\mu_o$. Consider the structure to be infinite in the $\hat{y}$ and $\hat{z}$ directions.




(a) For this specific situation, write the appropriate wave equations for $\vec{E}$ and $\vec{B}$

(b) State precisely the boundary conditions on all field components at the two interfaces.

(c) Find the functional form of both $\vec{E}$ and $\vec{B}$ for the first two TE modes.

(d) What is the speed of propagation of the first TE mode?

(e) For a field of vacuum wavelength $\lambda = 500$ nm, what is the minimum film thickness $a$ such that one propagating mode exists?

# Problem 3

The equation of state of a gas can be written in the form

$$pV = NRT(1 + B_2(T)\frac{N}{V})$$

$B_2$ is the second virial coefficient, and is an increasing function of temperature. Find how the internal energy of the gas depends on volume, i.e. calculate $\left(\frac{\partial U}{\partial V}\right)_{T,N}$. Is this positive or negative?

$$\left(\frac{\partial F}{\partial V}\right)_{T,N} = -p(T,V,N) = -\frac{NRT}{V} - B_2(T)\frac{N^2RT}{V^2}$$

Integrate:

$$F(T,V,N) = C(T,N) - NRT\log(V) + B_2(T)\frac{N^2RT}{V}$$

Find the entropy:

$$S(T,V,N) = -\left(\frac{\partial F}{\partial T}\right)_{V,N} =$$

$$-\left(\frac{\partial C}{\partial T}\right)_N + NR\log(V) - B_2(T)\frac{N^2R}{V} - B_2'(T)\frac{N^2RT}{V}$$

$$U(T,V,N) = F + TS = C(T,N) - T\left(\frac{\partial C}{\partial T}\right)_N - TB_2'(T)\frac{N^2RT}{V}$$

Therefore

$$\left(\frac{\partial U}{\partial V}\right)_{T,N} = TB_2'(T)\frac{N^2RT}{V^2}$$

which is positive. If the gas increases its volume, the internal energy goes up, and heat must flow in from the outside to keep the temperature the same. Without such a heat flow the internal energy would be too low and the gas would cool down, as is normally observed.

## Problem 4

(3) a) For this case, $[\nabla^2 + k^2]\{\vec E, \vec B\} \longrightarrow \left[\frac{\partial^2}{\partial x^2} + k^2 - \beta^2\right]\{\vec E, \vec B\} = 0$

Since $\vec E = \vec{\mathcal{E}}(x,y)e^{i\beta z}$, $\vec B = \vec{\mathcal{B}}(x)e^{i\beta z}$ and $k^2 = \beta^2 + \gamma^2$
$k^2 = \omega^2\epsilon\mu$

(3) b) Continuity of tangential E ($E_y$) and tangential B ($B_z$) should suffice
($\mu = \mu_0$ everywhere)

c) From Faraday's law $\vec\nabla \times \vec E = -\dot{\vec B} = +i\omega\vec B$

we find (noting that $E_x = 0$, $B_y = 0$, $E_z = 0$ and all y derivatives $= 0$)

(8) $\frac{\partial E_y}{\partial x} = i\omega B_z$ and $E_y = -\frac{\omega}{\beta} B_x$

From Ampere's law $\vec\nabla \times \vec B = \epsilon\mu\dot{\vec E} = -i\omega\epsilon\mu\vec E$

$$i\beta B_x - \frac{\partial B_z}{\partial x} = -i\omega\epsilon\mu E_y$$

So, substituting for $E_y$ in this expression yields

$$i\beta B_x - \frac{\partial B_z}{\partial x} = \frac{i\omega^2\epsilon\mu}{\beta}B_x \Rightarrow \frac{\partial B_z}{\partial x} = i\beta\left(1 - \frac{\omega^2\epsilon\mu}{\beta^2}\right)B_x$$

For $B_z$, find solution to $\left[\frac{\partial^2}{\partial x^2} + \gamma^2\right]B_z(x) = 0$ for $0 < x < a$

with the boundary condition $\partial B_z/\partial n = 0$ at interfaces

At this point, everybody ignored the fact that the upper region is air, not a conductor. So, the solution to the problem becomes easier.

$B_z(x) = \cos\gamma x$. $\left.\frac{\partial B_z}{\partial x}\right|_{x=a} = 0 \Rightarrow \sin\gamma a = 0 \Rightarrow \gamma = \frac{m\pi}{a}$, $m = 1, 2, 3$

So, $B_x(x) = \frac{-\gamma\sin\gamma x}{i\beta\left(1 - \frac{k^2}{\beta^2}\right)}$ and $E_y(x) = \frac{\omega\gamma}{i(\beta^2 - k^2)}\sin\gamma x = \frac{i\omega\sin\gamma x}{\gamma}$

(3) d) $\beta$ is the projection of $\vec k$ along the z axis, and $\beta = n_{eff}\,\omega/c$ where $1 < n_{eff} < n_{glass}$. So, the apparent phase velocity is $v_p = c/n_{eff} = \omega/\beta$, where $\beta = \sqrt{k^2 - (m\pi/a)^2}$

(3) e) allowed values of $\beta$ are $\beta = \sqrt{k^2 - \gamma^2} = \sqrt{k^2 - (m\pi/a)^2}$

$\beta$ must be real, so $k^2 = \omega^2\epsilon\mu > (m\pi/a)^2$

So for $m = 1$, $a \geq \frac{\pi}{\omega\sqrt{\epsilon\mu}} = \frac{1}{2}\frac{c}{\nu n} = \frac{\lambda}{2n}$, an obvious result.

or $a \geq \frac{500}{3}$ nm

## Problem 5.

Consider a spin $\frac{1}{2}$ particle. Show that in the space of states of a given orbital angular momentum $l$, the operators

$$\frac{l+1+\vec{L}\cdot\vec{\sigma}}{2l+1}$$

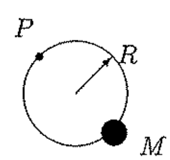
and

$$\frac{l-\vec{L}\cdot\vec{\sigma}}{2l+1}$$

are projectors onto the states of total angular momentum $j = l+\frac{1}{2}$ and $j = l-\frac{1}{2}$ respectively.
Operator notation: $\vec{L} = \vec{r}\times\vec{p}$ is orbital angular momentum, $\vec{S} = \frac{1}{2}\vec{\sigma}$ is spin angular momentum.

## Problem 6.

A thin circular hoop of radius $R$ and mass $M$ is suspended from a pivot (P) through one point on the hoop's rim. It is free to oscillate in its own plane. Attached to the hoop is a small bead, also of mass $M$, constrained to slide without friction along the hoop as shown below:




(1) Assume that the oscillations are small and find the frequencies of the normal modes of this system. Hint: the moment of inertia of thin hoop rotating about a point on its rim is $I = 2MR^2$.
(2) Find the eigenvectors and sketch the motions corresponding to the normal modes of this system. Identify the frequencies associated with each normal mode motion.

P5

**Solution to Problem GQM**

To show that they are projection operators, show that their eigenvalues are 1 and zero.

A basis for the state space of orbital angular momentum $l$ and spin 1/2 is the set of simultaneous eigenstates of $\boldsymbol{L}^2$, $\boldsymbol{S}^2$, $\boldsymbol{J}^2$, and $J_z$ with eigenvalues $l(l+1)$, 3/4, $j(j+1)$, and $m_j$ where $l$ is an integer and $j = l \pm 1/2$ and $m$ are half-integers: $|jlsm\rangle$

Check operation of each operator on these basis states.

Note $\boldsymbol{J}^2 = (\boldsymbol{L}+\boldsymbol{S})^2 = \boldsymbol{L}^2 + \boldsymbol{S}^2 + 2\,\boldsymbol{L}\bullet\boldsymbol{S} = \boldsymbol{L}^2 + \boldsymbol{S}^2 + \boldsymbol{L}\bullet\sigma$ so $\boldsymbol{L}\bullet\sigma = \boldsymbol{J}^2 - \boldsymbol{L}^2 - \boldsymbol{S}^2$

Thus the basis states are eigenstates of $\boldsymbol{L}\bullet\sigma$ with eigenvalues $j(j+1) - l(l+1) - 3/4$.

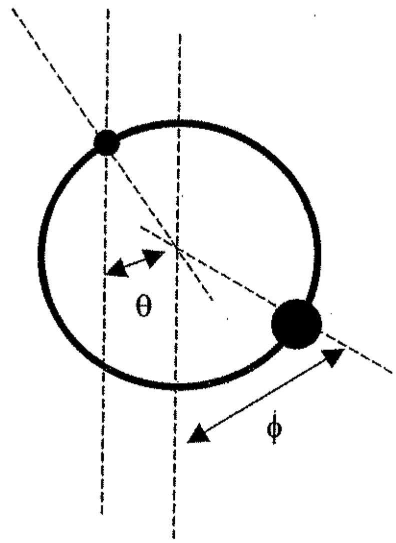
For $j = l + 1/2$, $j(j+1) - l(l+1) - 3/4 = (l+1/2)(l+3/2) - l(l+1) - 3/4 = l$, so the first operator gives 1 and the second gives zero.

For $j = l - 1/2$, $j(j+1) - l(l+1) - 3/4 = (l-1/2)(l+1/2) - l(l+1) - 3/4 = -(l+1)$, so the first operator gives 0 and the second gives 1.

P6

Solution:

(1) <u>Normal mode frequencies</u>




Let $\theta$ denote the angular displacement of the center of the hoop with respect to a vertical line through the pivot and $\phi$ denote the angular displacement of the mass with respect to a vertical line through the center of the hoop.

Hoop center coordinates: $X = R\sin\theta \quad Y = R(1-\cos\theta)$

Mass coordinates: $x = R\sin\theta + R\sin\phi \qquad y = R(1-\cos\theta) + R(1-\cos\phi)$

Set up Lagrangian $L = T - U$:

$$T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) \qquad I = 2MR^2$$

$$\dot{x} = R(\cos\theta\dot{\theta} + R\cos\phi\dot{\phi}) \qquad \dot{y} = R(\sin\theta\dot{\theta} + \sin\phi\dot{\phi})$$

$$\dot{x}^2 + \dot{y}^2 = R^2\left[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}(\cos\theta\cos\phi + \sin\theta\sin\phi)\right] \cong R^2\left(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\right)$$

$$T = MR^2\dot{\theta}^2 + \frac{1}{2}MR^2\left(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\right) = \frac{3}{2}MR^2\dot{\theta}^2 + \frac{1}{2}MR^2\dot{\phi}^2 + MR^2\dot{\theta}\dot{\phi}$$

$$U = MgY + Mgy = MgR\left[2(1-\cos\theta) + (1-\cos\phi)\right]$$

Drop constant terms in U:

$$U=-MgR(2\cos\theta+\cos\phi)\cong -MgR\left[2\left(1-\frac{1}{2}\theta^2\right)+\left(1-\frac{1}{2}\phi^2\right)\right]=\frac{1}{2}MgR\left(2\theta^2+\phi^2\right)$$
$$L=T-U=\frac{3}{2}MR^2\dot\theta^2+\frac{1}{2}MR^2\dot\phi^2+MR^2\dot\theta\dot\phi-\frac{1}{2}MgR\left(2\theta^2+\phi^2\right)$$

Lagrange Equation of Motion: $\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial \dot q}=0 \quad q=\theta,\phi$

$q=\theta$ equation: $2MgR\theta+3MR^2\ddot\theta+MR^2\ddot\phi=0$

$q=\phi$ equation: $MgR\phi+MR^2\ddot\phi+MR^2\ddot\theta=0$

Assume harmonic motion: $\theta(t)=a_\theta e^{i\omega t}$ $\phi(t)=a_\phi e^{i\omega t}$ and substitute in equations of motion to get coupled, homgeneous linear equations for $a_\theta$ and $a_\phi$:

$$\left(2MgR-3\omega^2MR^2\right)a_\theta - MR^2\omega^2 a_\phi=0$$
$$-MR^2\omega^2 a_\theta + \left(MgR-\omega^2MR^2\right)a_\phi=0$$

The secular determinant is

$$\begin{vmatrix} 2MgR-3\omega^2MR^2 & -\omega^2MR^2 \\ -\omega^2MR^2 & MgR-\omega^2MR^2 \end{vmatrix}=0$$

Evaluate determinant:

$$2M^2R^4\omega^4-5M^2gR^3\omega^2+2M^2g^2R^2=0$$
$$\omega^4-\frac{5}{2}\left(\frac{g}{R}\right)\omega^2+\left(\frac{g}{R}\right)^2=0$$

Solve quadratic to get the normal mode frequencies:

$$\omega^2=\frac{1}{2}\left[\frac{5}{2}\left(\frac{g}{R}\right)\pm\sqrt{\left(\frac{5}{2}\right)^2\left(\frac{g}{R}\right)^2-4\left(\frac{g}{R}\right)^2}\right]=\frac{1}{2}\left(\frac{g}{R}\right)\left(\frac{5}{2}\pm\frac{3}{2}\right)$$

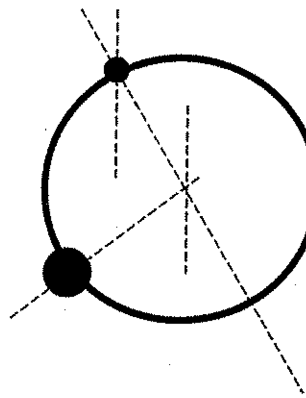
$$\omega_+^2=\frac{2g}{R} \qquad \omega_-^2=\frac{g}{2R}$$

(2) <u>Eigenvectors and mode symmetries</u>

Eigenvector for $\omega_+$:

Substitute $\omega_+^2 = \left(\dfrac{2g}{R}\right)$ in first linear equation

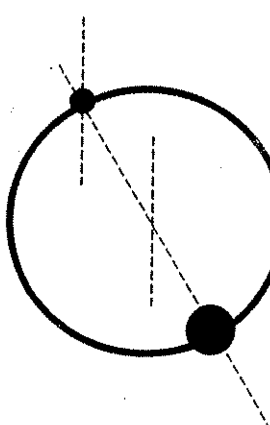
$$\left[2MgR - 3\left(\frac{2g}{R}\right)MR^2\right]a_\theta - MR^2\left(\frac{2g}{R}\right)a_\phi = 0 \quad \Rightarrow \quad a_\phi = -2a_\theta \quad \text{( anti-symmetric mode)}$$



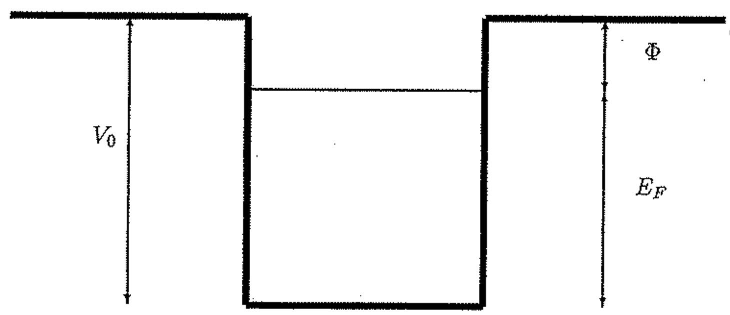
Eigenvector for $\omega_-$:

Substitute $\omega_-^2 = \left(\dfrac{g}{2R}\right)$ in first linear equation

$$\left[2MgR - 3\left(\frac{g}{2R}\right)MR^2\right]a_\theta - MR^2\left(\frac{g}{2R}\right)a_\phi = 0 \quad \Rightarrow \quad a_\phi = a_\theta \quad \text{(symmetric mode)}$$



## Problem 7.




In this problem we model the thermal currents out of a metal in a simple manner. Inside the electrons move freely in a potential which is zero, outside in a potential which is $V_0 > 0$. At $T = 0$ all electrons are inside the metal, and hence the Fermi energy obeys $E_F < V_0$. We consider the electrons inside and outside to be two independent systems, but require that they are in thermal equilibrium with each other. In all practical applications we always have $k_B T \ll E_F$, and hence we can assume that the chemical potential is always equal to the Fermi energy. The Fermi-Dirac distribution functions $f_{FD}$ is given by $f_{FD}(\epsilon; T, \mu) = \left(e^{\frac{\epsilon-\mu}{k_B T}} + 1\right)^{-1}$. At zero temperature the density of the electrons inside the metal is given by

$$n_i = \frac{2}{(2\pi)^3} \int d^3k \Theta(E_F - \frac{\hbar^2 k^2}{2m})$$

where $\Theta$ is the step function.

(a) Give a formula for the density of the electrons outside the metal, in terms of the Fermi-Dirac distribution function.

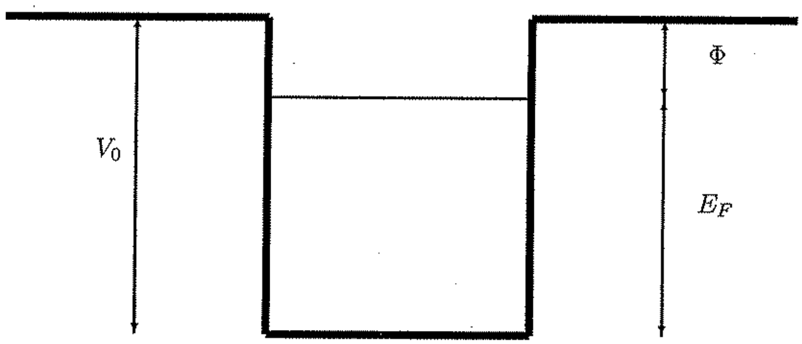
(b) Calculate the flux of electrons from the outside to the inside.

(c) Under the conditions of the problem, what is the flux of electrons from the inside to the outside?

(d) What is the thermal current out of a metal at temperature $T$?

(e) What is the dominant dependence of this current on the work function $\Phi$ and temperature?

# Problem ~~1~~ 7

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$$n_i = \frac{2}{(2\pi)^3} \int d^3k \theta(E_F - \frac{\hbar^2 k^2}{2m})$$

where $\theta$ is the step function.

(a) Give a formula for the density of the electrons outside the metal, in terms of the Fermi-Dirac distribution function.

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(d) What is the thermal current out of a metal at temperature $T$?

(e) What is the dominant dependence of this current on the work function $\Phi$ and temperature?

$$n_o = \frac{2}{(2\pi)^3} \int d^3k \left( e^{\frac{\frac{\hbar^2 k^2}{2m} + V_0 - E_F}{k_B T}} + 1 \right)^{-1}$$

Assume that the interface is at $z = 0$ and that the inside is at positive values of $z$.

$$j_o = \frac{2}{(2\pi)^3}\int_{k_z>0} d^3k \frac{\hbar}{m} k_z \left( e^{\frac{\frac{\hbar^2 k^2}{2m}+V_0-E_F}{k_B T}} + 1 \right)^{-1}$$

Under the conditions stated in the problem we have thermal equilibrium, and hence $j_i = j_o$.

If we allow the outside electrons to disappear, the current from the outside to the inside is now zero, and the thermal current $j_T$ is given by $j_i$.

$$j_T = \frac{2\hbar}{m(2\pi)^3}\int_{k_z>0} d^3k k_z \left( e^{\frac{\frac{\hbar^2 k^2}{2m}+V_0-E_F}{k_B T}} + 1 \right)^{-1}$$

with $k_z = k\cos(\theta)$ this reduces to

$$j_T = \frac{2\hbar}{m(2\pi)^3}\pi \int_0^\infty dk k^3 \left( e^{\frac{\frac{\hbar^2 k^2}{2m}+\Phi}{k_B T}} + 1 \right)^{-1}$$

which is equivalent to

$$j_T = \frac{\hbar}{m(2\pi)^2} e^{-\frac{\Phi}{k_B T}} \int_0^\infty dk k^3 \left( e^{\frac{\hbar^2 k^2}{2m k_B T}} + e^{-\frac{\Phi}{k_B T}} \right)^{-1}$$

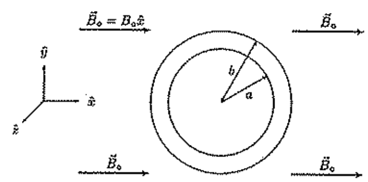
In this expression we can approximate the integral because the first exponent is always larger than one, and we have

$$j_T \approx \frac{\hbar}{m(2\pi)^2} e^{-\frac{\Phi}{k_B T}} \int_0^\infty dk k^3 e^{-\frac{\hbar^2 k^2}{2m k_B T}}$$

and hence the dominant dependence is $T^2 e^{-\frac{\Phi}{k_B T}}$

**Problem 8.**

Hollow permeable objects can serve as magnetic shields for sensitive instruments. Consider a very long, hollow, cylinder of circular cross-section oriented along the $\hat{z}$ axis. A uniform external magnetic field $\vec{B}_\circ = B_\circ\hat{x}$ is applied in the $\hat{x}$ direction perpendicular to the axis of the cylinder. The inner and outer radii of the cylinder are $a$ and $b$, respectively. The permeable tube then has a wall thickness of $b - a$ and permeability $\mu >> \mu_\circ$.




1. One way to find the magnetic field inside the cylinder is to define an appropriate potential and apply boundary conditions. How would you define such a potential, and why can you do so?
2. Write the general form for your potential in a three regions (outside the cylinder, within the permeable material and inside the cylinder), keeping only those terms which are relevant given the symmetry of the situation.
3. Apply the boundary conditions, and find the magnetic fields in all three regions.
4. Show that as $\mu \to \infty$ the field inside the cylinder $\to 0$.

Prob. 8

(3) a) $\vec{H} = -\vec{\nabla}\Phi$ since there are no currents in the region of interest. ($\vec{\nabla}\wedge\vec{H} = 0$)

b) Use the 2-dimensional potential with only $\cos\varphi$ dependence.

$$\Phi(\rho,\varphi) = A_0 + B_1 \ln\rho + \sum_{n=1}^{\infty} \rho^n [C_n \cos n\varphi + S_n \sin n\varphi] + \sum_{n=1}^{\infty} \rho^{-n} [c_n \cos n\varphi + s_n \sin n\varphi]$$

(7) outside: $\Phi^> = -H_0 \rho \cos\varphi + \frac{\alpha}{\rho}\cos\varphi$

in material $\Phi = \beta \ln\rho + \rho\gamma\cos\varphi + \delta/\rho \cos\varphi$

inside shield $\Phi^< = \rho\epsilon\cos\varphi$

c) Normal $\vec{B}$ and tangential $\vec{H}$ are continuous

(7)

$$\text{(I)}\ -\mu_0 \frac{\partial}{\partial\rho}\Phi^>\Big|_{\rho=b} = -\mu \frac{\partial\Phi}{\partial\rho}\Big|_{\rho=b} \quad \text{and} \quad \text{(II)}\ -\mu\frac{\partial\Phi}{\partial\rho}\Big|_{\rho=a} = -\mu_0 \frac{\partial}{\partial\rho}\Phi^<\Big|_{\rho=a}$$

$$\text{(III)}\ -\frac{1}{\rho}\frac{\partial\Phi^>}{\partial\varphi}\Big|_{\rho=b} = -\frac{1}{\rho}\frac{\partial\Phi}{\partial\varphi}\Big|_{\rho=b} \quad \text{and} \quad \text{(IV)}\ -\frac{1}{\rho}\frac{\partial\Phi}{\partial\varphi}\Big|_{\rho=a} = -\frac{1}{\rho}\frac{\partial\Phi^<}{\partial\varphi}\Big|_{\rho=a}$$

(I) $\Rightarrow \left[-H_0 - \frac{\alpha}{b^2}\right]\cos\varphi = \frac{\mu\beta}{b} + \mu\left[\gamma - \frac{\delta}{b^2}\right]\cos\varphi \qquad \left(\mu \equiv \frac{\mu}{\mu_0}\right)$

(II) $\Rightarrow \mu\left[\gamma - \frac{\delta}{b^2}\right] = \epsilon$ $\qquad$ conclude $\beta = 0$

(III) $\Rightarrow \frac{\alpha}{b^2} - H_0 = \gamma + \frac{\delta}{b^2}$ $\qquad$ (IV) $\Rightarrow \epsilon = \gamma + \delta/a^2$

Important quantity is $\epsilon$, but need $\gamma$ and $\delta$ and $\alpha$

$$\gamma = \left(\frac{1}{1-\mu}\right)\left[\frac{2\alpha - \delta(1+\mu)}{b^2}\right]$$

(3) d) But in the limit $\mu \to \infty$, $\gamma \to -\frac{1}{\mu}\left(-\frac{\delta\mu}{b^2}\right) = \frac{\delta}{b^2}$

so $\epsilon \to \mu\left[\frac{\delta}{b^2} - \frac{\delta}{b^2}\right] = 0$