OSU PHYSICS DEPARTMENT
COMPREHENSIVE EXAMINATION #96

March 29 and 30, 2004

Comprehensive examination for Spring 2004

PART I, Monday March 29, 9:00am

General Instructions

This Comprehensive Examination for Spring 2004 consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, March 29, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:00 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, March 30.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit - especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Problem 1.

A spinless particle of mass M moving freely in the absence of forces, in one dimension, is described at time $t = 0$ by the wave packet

$$\psi(x,0) = \begin{cases} C[1-\cos(2\pi \frac{x}{L})] & \text{for} \quad 0 \leq x \leq L \\ 0 & \text{otherwise} \end{cases}$$

You need not evaluate any integrals that arise in this problem, except for part b.

a. Find an expression for the constant C.

b. Find an expression for the momentum components $\bar{\psi}(p,0)$. Try to evaluate the integrals.

c. Estimate the uncertainties in the position and momentum, Δx and Δp . Show that your answer satisfies the uncertainty principle.

d. Give an expression for the wave packet at a later time t, $\psi(x,t) =?$

e. What happens to the center of this wave packet as a function of time? What happens to Δx ? What happens to Δp ?

Problem 2.

A skater weighs 650 N . The area of contact between the skates and the ice is $10^{-5}\ m^2$. The temperature of the ice is -4^oC. The latent heat of fusion for ice is 320 $kJ\ kg^{-1}$. The density of ice is 920 $kg\ m^{-3}$. The density of water is 1000 $kg\ m^{-3}$. Determine whether the ice under the skates will melt or not.

Solution to problem 1

a. C is determined by the normalization condition that the total probability is 1:

$$1 = \int_0^L dx \; |\psi(x)|^2 = |C|^2 \int_0^L dx \; [1 - \cos(2\pi x/L)]^2$$

$$C = \frac{1}{\sqrt{\int_0^L dx \; [1 - \cos(2\pi x/L)]^2}} \quad \text{times arbitrary phase factor}$$

b. $$\bar{\psi}(p, 0) = \int_{-\infty}^{\infty} dx \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} \psi(x, 0) = \int_0^L dx \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} C[1 - \cos(2\pi x/L)]$$

$$= \frac{C}{\sqrt{2\pi\hbar}} \frac{\hbar}{i} \left[\frac{e^{-ipx/\hbar}}{-p} - \frac{e^{i(-p+2\pi\hbar/L)x/\hbar}}{2(-p+2\pi\hbar/L)} - \frac{e^{i(-p-2\pi\hbar/L)x/\hbar}}{2(-p-2\pi\hbar/L)} \right]_0^L$$

then $$\bar{\psi}(p, 0) = \frac{C}{\sqrt{2\pi\hbar}} \frac{\hbar}{i} (e^{-ipL/\hbar} - 1) \left(\frac{1}{-p} - \frac{1}{2(-p+2\pi\hbar/L)} - \frac{1}{2(-p - 2\pi\hbar/L)} \right)$$

$$= \frac{C}{\sqrt{2\pi\hbar}} \frac{\hbar}{i} (e^{-ipL/\hbar} - 1) \frac{2(p^2 - (2\pi\hbar/L)^2) - (-2p)(-2p)}{-2p(p^2 - (2\pi\hbar/L)^2)}$$

$$= \frac{C}{\sqrt{2\pi\hbar}} \frac{\hbar}{i} (e^{-ipL/\hbar} - 1) \frac{p^2 + (2\pi\hbar/L)^2}{p(p^2 - (2\pi\hbar/L)^2)}$$

c. from the half-wave-function points, $\Delta x \approx L/2$

from the denominator of $\bar{\psi}(p, 0)$, $\Delta p \approx 2\,(2\pi\hbar/L)$

check uncertainty principle: $\Delta x \, \Delta p \approx 2\pi\hbar > \hbar/2$ √

d. each momentum component propagates freely with a time dependent phase $e^{-iE(p)t/\hbar}$

where $E(p) = \frac{\hbar^2 p^2}{2M}$ so $\psi(x, t) = \int_{-\infty}^{\infty} dp \; \bar{\psi}(p, 0) \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}} e^{-iE(p)t/\hbar}$

e. Wave function remains centered on $x = \frac{L}{2}$. Δp constant, Δx increases, eventually like $\frac{\Delta p}{M} t$

Solution to Problem 2.

A skater weighs 650 N . The area of contact between the skates and the ice is $10^{-5}\ m^2$. The temperature of the ice is -4^oC. The latent heat of fusion for ice is 320 $kJ\ kg^{-1}$. The density of ice is 920 $kg\ m^{-3}$. The density of water is 1000 $kg\ m^{-3}$. Determine whether the ice under the skates will melt or not.

We will need to use the Clapeyron formula. If we follow the Gibbs free energy along the coexistence curve we have (suppressing the constant value of N):

$$G_w(p_{co}(T), T) = G_i(p_{co}(T), T)$$

Take the temperature derivative, using

$$\left(\frac{\partial G}{\partial p}\right)_{T,N} = V$$

$$\left(\frac{\partial G}{\partial T}\right)_{V,N} = -S$$

we get

$$V_w \frac{dp_{co}}{dT}(T) - S_w = V_i \frac{dp_{co}}{dT}(T) - S_i$$

or

$$\frac{dp_{co}}{dT}(T) = \frac{S_w - S_i}{V_w - V_i}$$

This equation or a variation of it is also OK as the starting point for the solution. Derivation is not needed, but can easily be done.
If we take the entropy per kg and the volume per kg and use the relation $L = T\Delta S$ for the latent heat we get

$$\frac{dp_{co}}{dT}(T) = \frac{l_{wi}}{T(v_w - v_i)}$$

Using the numbers from the problem, and taking $T \approx 300K$:

$$\frac{dp_{co}}{dT}(T) = \frac{-320\ kJ\ kg^{-1}}{300\ K(\frac{1}{1000\ kg\ m^{-3}} - \frac{1}{920\ kg\ m^{-3}})}$$

$$\frac{dp_{co}}{dT}(T) \approx \frac{-320}{300\frac{920-1000}{1000\times920}}\ kJ\ m^{-3}\ K^{-1}$$

$$\frac{dp_{co}}{dT}(T) \approx \frac{4\times920}{0.3}\ kJ\ m^{-3}\ K^{-1}$$

$$\frac{dp_{co}}{dT}(T) \approx 12,500\ kJ\ m^{-3}\ K^{-1} = 12.5\times10^6\ N\ m^{-2}\ K^{-1}$$

The extra pressure due to the skater is

$$\Delta p = 650 \times 10^5 \ N \ m^{-2}$$

hence

$$\Delta T = \Delta p \left(\frac{dp_{co}}{dT}(T) \right)^{-1} = \frac{650 \times 10^5}{12.5 \times 10^6} \ K \approx 5 \ K$$

which is sufficient to melt the ice.

Problem 3.

An induction heater consists of a simple set of coils, such as a solenoid, in close proximity to the object to be heated. Such a heater is said to be very efficient, that is, most of the applied electromagnetic power becomes thermal power in the object. Your task is to determine the efficiency of a simple version of an induction heater. The only loss mechanism is assumed to be radiation. The device of length L consists of n loops per unit length with radius R, with $R > L$. A small solid cylinder of aluminum with radius $a << R$ and length $b << L$ is placed at the center of the coil with the axis of the cylinder aligned with the axis of the coil. The conductivity of the object is σ. The frequency at which current $I(t)$ is applied to the coil is ω.

(a) What is the time-averaged power delivered to the cylinder?

(b) What is the time-averaged power dissipated as electromagnetic radiation?

Problem 4.

Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table surface and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, what are the generalized coordinates for the system? Write down the Lagrange equations for the system, and, if possible, discuss the physical significance any of them might have. Reduce the problem to a single second-order differential equation and obtain a first integral of the equation. What is its physical significance? (Consider the motion only so long as neither m_1 nor m_2 passes through the hole.)

PROB. 3

a) Use a simple expression for field in center of $nL = N$ loops

$\vec{B} = B_0 \hat{z}$ $\quad B_0 = \frac{N\mu_0 I}{2R} = \frac{\mu_0}{2\pi R^3}(NI\pi R^2) = \frac{\mu_0}{2\pi R^3} m$

for a flat coil ($R > L$)

or $B_0 = \mu_0 n I = \mu_0 \frac{NI}{L} = \frac{\mu_0}{\pi R^2 L} m$

4

for a solenoid ($R << L$)

Now, determine the induced electric field. Use cylindrical coords

$\vec{B}(\vec{r},t) = \vec{B}(\vec{r}) e^{i\omega t}$ with $\vec{B}(\vec{r}) = \mathcal{B}(\rho)\hat{z}$ with the object.

Allow for a skin depth using $\mathcal{B}(\rho) = B_0 e^{(\rho-a)/\delta}$, $\delta = \sqrt{\frac{2}{\mu\omega\sigma}}$

Use $\vec{\nabla} \times \vec{B} = \mu \vec{J} = \mu\sigma\vec{E}$ in the object,

6 with $\vec{E}(\vec{r},t) = \mathcal{E}(\rho) e^{i\omega t}\hat{\varphi}$

Then $\mathcal{E}(\rho) = -\frac{1}{\delta\mu\sigma} B_0 e^{(\rho-a)/\delta}$

Dissipative power density $= \langle \vec{J}\cdot\vec{E} \rangle_{\text{Time average}}$

$\mathcal{P}(\rho,\varphi,z) = \frac{1}{2}\left(\frac{1}{\delta\mu}\right)^2 \frac{B_0^2}{\sigma} e^{2(\rho-a)/\delta}$

5 Total dissipative power $P = \int_{\text{object}} \mathcal{P}(\rho,\varphi,z)\,dV$

Two possible simplifications: δ is large $\Rightarrow P \simeq \frac{1}{2}\left(\frac{1}{\delta\mu}\right)^2 \frac{B_0^2}{\sigma}\pi a^2 b$ ($\delta >> a$)

δ is small $\Rightarrow P \simeq \frac{1}{2}\left(\frac{1}{\delta\mu}\right)^2 \frac{B_0^2}{\sigma} 2\pi a b \delta$ ($\delta << a$)

*

b) Radiative loss will be based upon magnetic dipole radiation

Magnetic dipole moment is m.

A simple expression is $P_{rad} \propto K^4 \left(\frac{m}{c}\right)^2$. This is the same as

5 the electric dipole expression with the substitution $\frac{m}{c}$ for p.

Radiated Power/Thermal Power $\propto \omega^4/\omega = \omega^3$ using $K = \omega/c$

~~Another approach~~

* Another approach is to calculate $\int \vec{S}\cdot d\vec{a}$ over the surface of the object.

Solution to Problem 4.

Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table surface and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, what are the generalized coordinates for the system? Write down the Lagrange equations for the system, and, if possible, discuss the physical significance any of them might have. Reduce the problem to a single second-order differential equation and obtain a first integral of the equation. What is its physical significance? (Consider the motion only so long as neither m_1 nor m_2 passes through the hole.)

Generalized coordinates: position of mass m_1 on circle: radius r and angle ϕ. If the length of the string is R, we have $r < R$. The height of mass m_2 is $r - R$, which is zero when this mass touches the table.
Kinetic energy of particle 1:

$$K_1 = \frac{1}{2}m_1 v_1^2 = \frac{1}{2}m_1\left((\dot{r})^2 + r^2(\dot{\phi})^2\right)$$

Kinetic energy of particle 2:

$$K_2 = \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_2(\dot{r})^2$$

Particle 1 has no potential energy. Potential energy of particle 2:

$$V_2 = m_2 g(r - R)$$

Lagrangian:

$$\mathcal{L} = \frac{1}{2}m_1\left((\dot{r})^2 + r^2(\dot{\phi})^2\right) + \frac{1}{2}m_2(\dot{r})^2 - m_2 g(r - R)$$

Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \left(\frac{\partial \mathcal{L}}{\partial q}\right)$$

which gives:

$$\frac{d}{dt}(m_1\dot{r} + m_2\dot{r}) = m_1 r(\dot{\phi})^2 - m_2 g$$

$$\frac{d}{dt}\left(m_1 r^2\dot{\phi}\right) = 0$$

The first equation says that the total acceleration of the system along the radial direction is related to the centripetal force and the gravitational force. The second equation is conservation of angular momentum. Denote the value of the angular momentum by L. This gives

$$m_1 r^2 \dot{\phi} = L$$

Use this in the first equation:

$$(m_1 + m_2)\ddot{r} = \frac{L^2}{m_1 r^3} - m_2 g$$

Multiply by $\dot{r}$:

$$(m_1 + m_2)\ddot{r}\dot{r} = \frac{L^2}{m_1 r^3}\dot{r} - m_2 g\dot{r}$$

Integrate:

$$\frac{1}{2}(m_1 + m_2)(\dot{r})^2 = -\frac{L^2}{2m_1 r^2} - m_2 g r + C_0$$

where C_0 turns out to be the total energy!

Problem 5.

The χ mesons are bound states of a heavy "charmed" D quark with a D antiquark, each with spin $\frac{1}{2}\hbar$ and mass $M_D = 1880 MeV/c^2$ (twice the proton's mass). The quark interacts with the antiquark by a central potential $V(r) \approx -\frac{4}{3}\alpha_S \frac{\hbar c}{r}$ where $\alpha_S \approx 1.0$ measures the strength of the strong interaction. Their relative motion is described by a p state (orbital angular momentum $L = 1$).

(a) Give the possible energies, degeneracies, and total angular momenta of the ground state and excited states of the χ meson.

(b) In addition the quark and antiquark interact by a spin-spin interaction $H_{SS} = \vec{S}_1 \cdot \vec{S}_2 \frac{\alpha}{r^3}$ where α is a fairly small positive constant and $\vec{S}_1$ and $\vec{S}_2$ are the spins of the quark and antiquark respectively. Find the degeneracies and total angular momenta of the ground state and excited states of the χ meson, and give approximate expressions for their energies (you need not evaluate radial integrals).

Problem 6.

In a particular solid there are N electrons. Each electron can be in one of two states, either with energy 0 or energy E, with $E > 0$. The distribution of these electrons over the states is governed by the Fermi-Dirac distribution function $\left(e^{\frac{\epsilon-\mu}{k_B T}} + 1\right)^{-1}$.

(a) What is the limit of the chemical potential if the temperature goes to zero?

(b) Calculate the chemical potential for low temperatures, one term beyond the result in (a).

(c) Calculate the low temperature limit of the heat capacity.

(d) How do the results in (a), (b), and (c) change if the high energy state has degeneracy G?

(e) How do the results in (a), (b), and (c) change if there are an infinite number of states available, each singly degenerate, and with energies $E_n = nE$?

Solution to Problem 5

Compare to hydrogen atom, three differences:

i) two equal mass particles $\Rightarrow$ reduced mass is $M_D/2$ instead of $\frac{m_e m_p}{m_e + m_p}$

ii) two spin-1/2 particles, like electron + proton, need to include both spins.

iii) potential is $-\frac{4}{3}\ \alpha_s \frac{\hbar c}{r}$ instead of Coulomb potential

$$V_C = -\frac{e^2}{r} = -\frac{e^2}{\hbar c}\frac{\hbar c}{r} = -\alpha\frac{\hbar c}{r}$$

so need to substitute $\frac{4}{3}\ \alpha_s$ for $\alpha \approx 1/137$

(a) Assume non-relativistic, need to check after computing energy.

Then $E_n = 2M_D c^2 - \frac{Rydberg}{n^2}$, $Rydberg = \frac{1}{2}\left(\frac{4}{3}\alpha_s\right)^2 \frac{M_D}{2}c^2 \approx 830$ MeV $< 2M_D c^2 \approx 3760$ MeV

barely non-relativistic, better for large n .

n is a total quantum number, at least $L+1$, p-state $\Rightarrow L = 1$ so $n = 2$ or greater.

Spin doesn't affect energy, add two spin-1/2 for total spin 0 or 1

Add to spatial angular momentum to get total angular momentum $J = L + S$

Quantum numbers $nLSJM$. For each J, $2J$+1 values of projection M,

	add $L = 1$ to $S = 0$ to get $J = 1$,	$(2J+1) = 3$ states
or	add $L = 1$ to $S = 1$ to get $J = 0$, 1, or 2	$(2\times0+1)$ states $(2\times1+1)$ states $(2\times2+1)$ states
	altogether	$1+3+5 = 9$ states

$$\text{for each } n \geq 2, \begin{pmatrix} 1 \text{ state with } J=0 \\ 2 \infty (2J+1) = 6 \text{ states with } J=1 \\ 2J+1 = 5 \text{ states with } J=2 \end{pmatrix} \text{ altogether 12 states for each } E_n .$$

(b) Degenerate perturbation theory ⇒ diagonalize H_{ss} in subspace of given n .

H_{ss} diagonal when $\boldsymbol{S}^2 = (\boldsymbol{S}_1 + \boldsymbol{S}_2)^2$ is, since

$$\boldsymbol{S_1} \cdot \boldsymbol{S_2} = \frac{1}{2}(S^2 - S_1{}^2 - S_2{}^2) = \frac{1}{2}(S(S+1) - (\frac{1}{2})(\frac{1}{2}+1) - (\frac{1}{2})(\frac{1}{2}+1))$$

$$= \begin{pmatrix} 1/4 \text{ for } S=1 \\ -3/4 \text{ for } S=0 \end{pmatrix}$$

Now, for each $n \geq 2$ there are 3 $S=0$ states and 9 $S=1$ states with JL structure above,

energies shifted by

$$\langle nLSJM \mid H_{ss} \mid nLSJM \rangle = a \langle S \mid \boldsymbol{S_1} \cdot \boldsymbol{S_2} \mid S \rangle \int_0^\infty r^2 dr \; R_{nL}(r)^2 \frac{1}{r^3}$$

where $R_{nL}(r)$ is the radial wave function

The ground state has $n = 2$, and $S = 0$ from the sign of S_{12}, so $J = 1$, degeneracy = 3

The first excited state has $n = 2$, and $S = 1$, so $J = 0,1,2$, degeneracy = 9

This pattern repeats for each higher n.

Solution to Problem 6.

In a particular solid there are N electrons. Each electron can be in one of two states, either with energy 0 or energy E, with $E > 0$. The distribution of these electrons over the states is governed by the Fermi-Dirac distribution function $\left(e^{\frac{\epsilon-\mu}{k_B T}} + 1\right)^{-1}$.

(a) What is the limit of the chemical potential if the temperature goes to zero?

(b) Calculate the chemical potential for low temperatures, one term beyond the result in (a).

(c) Calculate the low temperature limit of the heat capacity.

(d) How do the results in (a), (b), and (c) change if the high energy state has degeneracy G?

(e) How do the results in (a), (b), and (c) change if there are an infinite number of states available, each singly degenerate, and with energies $E_n = nE$?

At zero temperature all electrons are in the state with energy 0. At a very small temperature some will be in the state with the higher energy. Since particles are conserved, we have:

$$1 - \left(e^{\frac{0-\mu}{k_B T}} + 1\right)^{-1} = \left(e^{\frac{E-\mu}{k_B T}} + 1\right)^{-1}$$

$$\frac{e^{\frac{-\mu}{k_B T}}}{e^{\frac{-\mu}{k_B T}} + 1} = \frac{1}{e^{\frac{E-\mu}{k_B T}} + 1}$$

$$\frac{1}{e^{\frac{\mu}{k_B T}} + 1} = \frac{1}{e^{\frac{E-\mu}{k_B T}} + 1}$$

and these are equal if $\mu = \frac{1}{2}E$. This holds for all temperatures! So the answer to (b) is the same as the answer to (a).
The internal energy follows from

$$U = NE\frac{1}{e^{\frac{E-\mu}{k_B T}} + 1} = NE\frac{1}{e^{\frac{E}{2k_B T}} + 1}$$

The heat capacity is equal to

$$C_V = \left(\frac{\partial U}{\partial T}\right)_{V,N} = -NE\frac{-E}{2k_B T^2}\frac{e^{\frac{E}{2k_B T}}}{\left(e^{\frac{E}{2k_B T}} + 1\right)^2}$$

$$C_V = 2Nk_B \frac{E^2}{4k_B^2T^2} \frac{e^{\frac{E}{2k_BT}}}{\left(e^{\frac{E}{2k_BT}} + 1\right)^2}$$

For very low temperatures the exponent goes to infinity and we get

$$C_V \approx 2Nk_B \frac{E^2}{4k_B^2T^2} e^{-\frac{E}{2k_BT}}$$

which goes to zero exponentially.
If the higher state has degeneracy G the equation to solve in part (a) is

$$1 - \left(e^{\frac{0-\mu}{k_BT}} + 1\right)^{-1} = G\left(e^{\frac{E-\mu}{k_BT}} + 1\right)^{-1}$$

or

$$\frac{1}{e^{\frac{\mu}{k_BT}} + 1} = \frac{G}{e^{\frac{E-\mu}{k_BT}} + 1}$$

For very low temperatures this is

$$e^{-\frac{\mu}{k_BT}} = Ge^{-\frac{E-\mu}{k_BT}}$$

$$-\frac{\mu}{k_BT} = \log(G) - \frac{E-\mu}{k_BT}$$

$$-\mu = k_BT\log(G) - E + \mu$$

$$\mu = \frac{1}{2}E - \frac{1}{2}k_BT\log(G)$$

and the energy is

$$U = NE\frac{G}{e^{\frac{E-\mu}{k_BT}} + 1} = NE\frac{G}{\sqrt{G}e^{\frac{E}{2k_BT}} + 1}$$

which is for low temperatures:

$$U = NE\sqrt{G}e^{-\frac{E}{2k_BT}}$$

Hence both U and C_V get an extra factor of $\sqrt{G}$.
In the last case we have:

$$1 = \sum_{n=0}^{\infty} \frac{1}{e^{\frac{nE-\mu}{k_BT}} + 1}$$

For very low temperatures we expect again that $\mu \approx \frac{1}{2}E$ since the higher states will have a very small occupation. Therefore, for low temperatures:

$$1 \approx \frac{1}{e^{\frac{-\mu}{k_B T}} + 1} + \sum_{n=1}^{\infty} \frac{1}{e^{\frac{nE-\mu}{k_B T}} + 1}$$

$$1 \approx 1 - e^{\frac{-\mu}{k_B T}} \sum_{n=1}^{\infty} e^{\frac{\mu - nE}{k_B T}}$$

$$e^{\frac{-\mu}{k_B T}} \approx \sum_{n=1}^{\infty} e^{\frac{\mu - nE}{k_B T}} \approx e^{\frac{\mu - E}{k_B T}} (1 + e^{\frac{\mu - E}{k_B T}})$$

$$e^{\frac{E - 2\mu}{k_B T}} \approx (1 + e^{\frac{\mu - E}{k_B T}})$$

$$\frac{E - 2\mu}{k_B T} \approx \log(1 + e^{\frac{\mu - E}{k_B T}}) \approx e^{\frac{\mu - E}{k_B T}}$$

$$\mu \approx \frac{1}{2}(E - k_B T e^{-\frac{E}{2k_B T}})$$

which has an exponentially small temperature correction compared to the answer in (b) before.

Problem 7.

Symmetry in a physical situation often yields surprising simple functional forms for measurable quantities such as fields and potentials. Consider the case of two very thin rings of radius a oriented perpendicularly to the $\hat{z}$ axis and positioned with the centers at $z = \pm b$. The total charge is $+Q$ on the upper ring and $-Q$ on the lower ring, and the charge densities are uniform on both rings. The length R is defined as the distance from the origin to any point on the rings. For the special case $b/R = \sqrt{3/5}$, the electrostatic potential and field have special forms.

(a) Find the charge density $\rho(\vec{r})$.

(b) Determine $\Phi(\vec{r})$ and $\vec{E}(\vec{r})$ for $r > R$ including contributions up to $l = 4$.

(c) Determine $\Phi(\vec{r})$ and $\vec{E}(\vec{r})$ for $r < R$ including contributions up to $l = 4$.

Problem 8.

Consider a very long string of identical masses m connected to nearest neighbors by identical springs with spring constant κ. In equilibrium, all springs have the same length. These masses can only move along the direction of the string. The deviation for mass number n from equilibrium is $x_n(t)$. Mass n is connected to masses $n-1$ and $n+1$ only. The string is very long, so the effects of the end points can be ignored.

(a) Give Newton's equations of motion for this system.

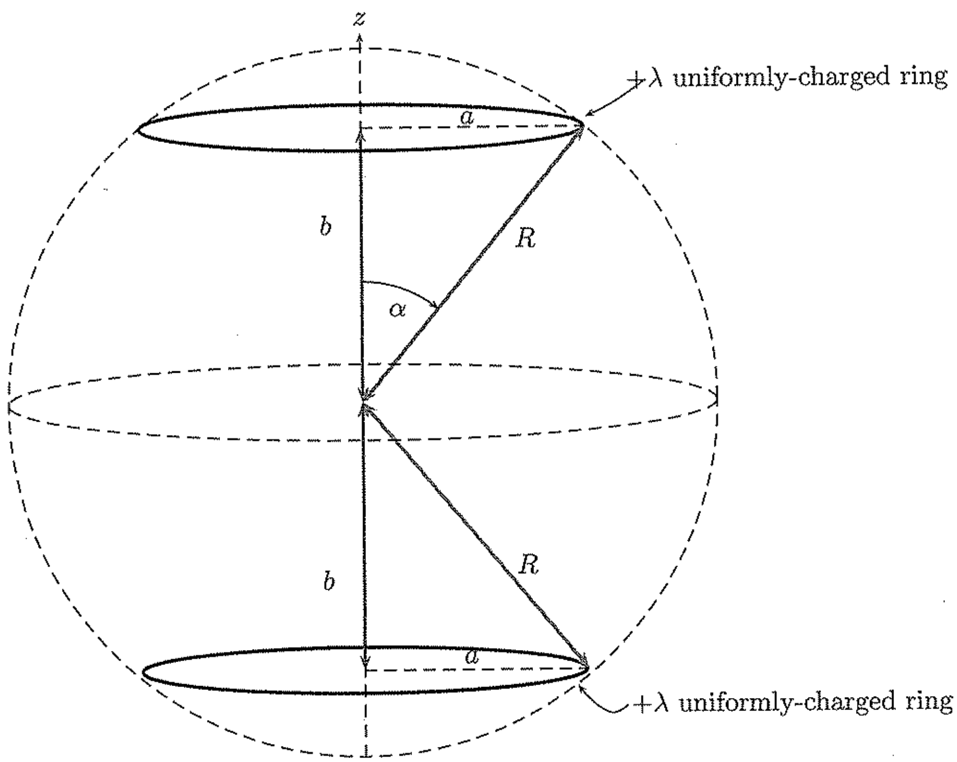
(b) Calculate the normal modes of this system, and calculate the frequency spectrum.

(c) Show that the normal modes are travelling waves, and determine the phase velocity.

(d) The central mass is now replaced by a different mass $m + \Delta$. Consider the following situation. A wave comes in from the left with frequency ω. At the central mass this wave is partly transmitted and partly reflected. Calculated the transmission and reflection coefficients.

Solution

7. (20 points) Symmetry in a physical situation often yields surprising simple functional forms for measurable quantities such as fields and potentials. Consider the case of two very thin rings of radius a oriented perpendicularly to the $\hat{z}$ axis and positioned with the centers at $z = \pm b$. The total charge is $+Q$ on the upper ring and $-Q$ on the lower ring, and the charge densities are uniform on both rings. The length R is defined as the distance from the origin to any point on the rings. For the special case $b/R = \sqrt{3/5}$, the electrostatic potential and field have special forms.



(a) Find the charge density $\rho(\vec{r})$.
Answer (2 points):

$$\rho(\vec{r}) = \rho_{upper}(\vec{r}) + \rho_{lower}(\vec{r}) = \frac{\lambda\delta(r-R)}{r}\left[\delta(\cos\theta - \cos\alpha) - \delta(\cos\theta - \cos(\pi-\alpha))\right] .$$

That $\lambda = Q/2\pi a$ follows from

$$\int \rho_{upper}(\vec{r})dV = 2\pi\lambda\int_0^\infty \frac{\delta(r-R)}{r}r^2dr\int_0^\pi \delta(\cos\theta-\cos\alpha)\sin\theta d\theta = 2\pi\lambda R\sin\alpha = Q$$

and the fact that $R\sin\alpha = a$. This can also be obtained from

$$\int \rho_{lower}(\vec{r})dV = -2\pi\lambda R\sin\alpha = -Q$$

and the fact that $R\sin(\pi-\alpha)=a$.

(b) Determine $\Phi(\vec{r})$ and $\vec{E}(\vec{r})$ for $r>R$ including contributions up to $l=4$.
Answer (9 points):
For $r>R$,

$$\Phi=\sum_l \Phi_l \text{ and } \Phi_l=\frac{4\pi}{2l+1}\frac{1}{r^{l+1}}\sum_{m=-l}^{l} Y_{lm}(\Omega)q_{lm}\,.$$

Symmetry dictates that only $m=0$ can contribute, so

$$q_{l0}=\int\rho(\vec{r})r^l Y_{l0}(\Omega)r^2 dr d\Omega$$

$$=\lambda\sqrt{\frac{2l+1}{4\pi}}\int_0^{2\pi}d\phi\int_0^{\infty}\delta(r-R)r^{l+1}dr\int_0^{\pi}P_l(\cos\theta)\left[\delta(\cos\theta-\cos\alpha)-\delta(\cos\theta-\cos(\pi-\alpha))\right]\sin\theta d\theta$$

$$=2\pi\lambda\sqrt{\frac{2l+1}{4\pi}}R^{l+1}\sin\alpha\left[P_l(\cos\alpha)-P_l(\cos(\pi-\alpha))\right]\,.$$

Using $\cos(\pi-\alpha)=-\cos\alpha$ and $P_l(-x)=(-1)^l P_l(x)$,

$$q_{l0}=2\lambda\sqrt{\frac{2l+1}{4\pi}}R^{l+1}\sin\alpha P_l(\cos\alpha)\left[1-(-1)^l\right]\,.$$

Hence, $q_{l0}=0$ if l is even. The first non-zero term is

$$q_{10}=2\pi\lambda\sqrt{\frac{3}{4\pi}}R^2\sin\alpha\cos\alpha\,[2]=2Q\sqrt{\frac{3}{4\pi}}R\cos\alpha\,.$$

The $l=3$ term is zero by symmetry

$$q_{30}=2\pi\lambda\sqrt{\frac{5}{4\pi}}R^4\sin\alpha\frac{1}{2}(5\cos^2\alpha-3\cos\alpha)\,[2]=0\,,$$

since $\cos^2\alpha=3/5$. The final result for $l\leq 4$ is

$$\Phi(r>R,\theta)=4\pi\lambda R^2\sin\alpha\cos\alpha\frac{\cos\theta}{r^2}=2Qb\frac{\cos\theta}{r^2}\,,$$

since $b=R\cos\alpha$. Note that the dipole moment is $\vec{p}=2Qb\hat{z}$. Finally,

$$\vec{E}=-\vec{\nabla}\Phi=\frac{p}{r^3}(2\cos\theta\hat{r}+\sin\theta\hat{\theta})\,,$$

.

(c) Determine $\Phi(\vec{r})$ and $\vec{E}(\vec{r})$ for $r<R$ including contributions up to $l=4$.
Answer (9 points):
For $r<R$,

$$\Phi_l=\frac{4\pi}{2l+1}r^l\sum_{m=-l}^{l}Y_{lm}(\Omega)I_{lm}\,,$$

where

$$I_{lm}=\int\frac{\rho(\vec{r})}{r^{l+1}}Y^*_{lm}(\Omega)dV\,.$$

Again, only $m = 0$ contributes. Since the ϕ and θ integrations are the same as in part (b), l must be odd, and $I_{30} = 0$. The radial integration is

$$\int_0^\infty \frac{\delta(r-R)}{r} \frac{1}{r^{l+1}} r^2 dr = \frac{1}{R^l}\,.$$

Hence, only I_{10} contributes,

$$I_{10} = 2\pi\lambda \sqrt{\frac{3}{4\pi}} \frac{1}{R} 2 \sin\alpha \cos\alpha\,.$$

So,

$$\Phi(r < R, \theta) = 4\pi\lambda \frac{r}{R} \sin\alpha \cos\alpha \cos\theta = \frac{2Qb}{R^3} r \cos\theta = \frac{2Qb}{R^3} z\,.$$

Finally,

$$\vec{E} = -\vec{\nabla}\Phi = -\frac{\partial}{\partial z}\Phi\hat{z} = -\frac{2Qb}{R^3}\hat{z}\,,$$

for $r < R$. This properly becomes 0 as $\alpha \to 0$ and as $\alpha \to \pi/2$.

Solutions for Problem 8.

Consider a very long string of identical masses m connected to nearest neighbors by identical springs with spring constant κ. In equilibrium, all springs have the same length. These masses can only move along the direction of the string. The deviation for mass number n from equilibrium is $x_n(t)$. Mass n is connected to masses $n-1$ and $n+1$ only. The string is very long, so the effects of the end points can be ignored.

(a) Give Newton's equations of motion for this system.

(b) Calculate the normal modes of this system, and calculate the frequency spectrum.

(c) Show that the normal modes are travelling waves, and determine the phase velocity.

(d) The central mass is now replaced by a different mass $m + \Delta$. Consider the following situation. A wave comes in from the left with frequency ω. At the central mass this wave is partly transmitted and partly reflected. Calculated the transmission and reflection coefficients.

$$m_n \ddot{x}_n = -\kappa(2x_n - x_{n+1} - x_{n-1})$$

But $m_n = m$. Try solutions

$$x_n(t) = e^{\imath(n\phi - \omega t)}$$

$$-m\omega^2 e^{\imath(n\phi-\omega t)} = -\kappa(2 - e^{\imath\phi} - e^{-\imath\phi})e^{\imath(n\phi-\omega t)}$$

$$-m\omega^2 = -\kappa(2 - 2\cos(\phi))$$

$$\omega^2 = \frac{\kappa}{m} 4\sin^2(\frac{1}{2}\phi)$$

$$\omega = 2\sqrt{\frac{\kappa}{m}}|\sin(\frac{1}{2}\phi)|$$

These are the normal modes.
For the phase velocity we need to find the solutions for

$$x_{n+1}(t + \Delta t) = x_n(t)$$

This gives

$$\phi = \omega \Delta t$$

and hence the phase velocity is ω.
Now we change the central mass into $m + \Delta$. We can combine all normal modes with the same frequency, hence with the corresponding values ϕ and $-\phi$. As a solution, we are asked to try:
For $n < 0$:

$$x_n(t) = Ae^{\imath(n\phi - \omega t)} + Be^{\imath(-n\phi - \omega t)}$$

For $n = 0$:

$$x_0(t) = Ce^{-\imath\omega t}$$

For $n > 0$:

$$x_n(t) = De^{\imath(n\phi - \omega t)}$$

The equations for $n \neq 0$ are unchanged, and in order for this to be a solution we need (follows from applying the equation for $n = -1$ and $n = +1$):

$$A + B = C$$

$$C = D$$

The central equation is different:

$$(m + \Delta)\ddot{x}_0 = -\kappa(2x_0 - x_{+1} - x_{-1})$$

$$(m + \Delta)C(-\omega^2)e^{-\imath\omega t} = -\kappa(2Ce^{-\imath\omega t} - De^{\imath(\phi - \omega t)} - (Ae^{\imath(-\phi - \omega t)} + Be^{\imath(\phi - \omega t)}))$$

$$(m + \Delta)C(-\omega^2) = -\kappa(2C - De^{\imath\phi} - (Ae^{-\imath\phi} + Be^{\imath\phi}))$$

$$\Delta C\omega^2 = \kappa(2C - Ce^{\imath\phi} - (Ae^{-\imath\phi} + Be^{\imath\phi})) - m\omega^2 C$$

But

$$m\omega^2 C = \kappa(2C - Ce^{\imath\phi} - Ce^{-\imath\phi})$$

Hence

$$\Delta C\omega^2 = \kappa(Ce^{-\imath\phi} - (Ae^{-\imath\phi} + Be^{\imath\phi}))$$

$$\Delta C\omega^2 = \kappa((A + B)e^{-\imath\phi} - (Ae^{-\imath\phi} + Be^{\imath\phi}))$$

$$\Delta C\omega^2 = \kappa(Be^{-\imath\phi} - Be^{\imath\phi})$$

$$\Delta C\frac{\kappa}{m}4\sin^2(\frac{1}{2}\phi) = -\kappa 2\imath B\sin(\phi)$$

$$C\frac{\Delta}{m}\sin(\frac{1}{2}\phi) = -\imath B\cos(\frac{1}{2}\phi)$$

$$B = \imath C\frac{\Delta}{m}\tan(\frac{1}{2}\phi)$$

Now express this in terms of A, the incoming wave:

$$A = C - B = C(1 - \imath\frac{\Delta}{m}\tan(\frac{1}{2}\phi))$$

Hence for the transmitted wave:

$$D = A\left(1 - \imath\frac{\Delta}{m}\tan(\frac{1}{2}\phi)\right)^{-1}$$

and the transmission coefficient follows from $\frac{|D|^2}{|A|^2}$. For the reflected wave:

$$B = A\imath\frac{\Delta}{m}\tan(\frac{1}{2}\phi)\left(1 - \imath\frac{\Delta}{m}\tan(\frac{1}{2}\phi)\right)^{-1}$$

Note that $|B|^2 + |D|^2 = |A|^2$, conservation of energy.