Department of Physics Comprehensive Examination # 95

Part I

29 September 2003

This Comprehensive Examination for Fall 2003 consists of eight problems each worth 20 points. The problems are grouped into four sessions:

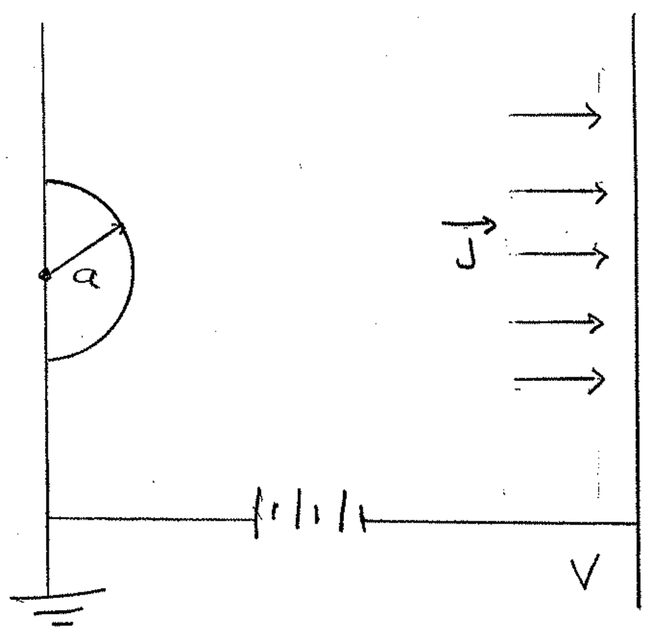
Session 1	problems 1, 2	9-12 AM	Mon 29 Sept.
Session 2	problems 3, 4	1:30-4:30 PM	Mon 29 Sept.
Session 3	problems 5, 6	9-12 AM	Tues 30 Sept.
Session 4	problems 7, 8	1:30-4:30 PM	Tues 30 Sept.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it is possible to obtain partial credit, especially if you demonstrate conceptual understanding. Do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter, but not your name, is on the inside of the back cover of every bluebook. Be sure to remember your student letter for use in the remaining sessions of the examination. If something is omitted from the statement of the problem or you feel there is an ambiguity, please ask your question quietly and privately, so as not to disturb the others. Only your bluebooks and the examination should be on the table before you. Any other items should be stored on the floor. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for scratch work separated by at least one page from your solutions. Scratch work will not be graded.

1.

A small hemispherical bump (of radius a) is raised on the inside of one conducting plate of a parallel-plate capacitor whose plates (each of infinite area) are separated by a distance $d \gg a$. The space between the plates is filled with a very leaky dielectric, which acts like an ohmic medium of conductivity g and permittivity ε. The plate with the bump is grounded; the other is held at potential V. A steady current flows between the plates; the current density at the far plate is uniform and constant as shown. Calculate the electrical force tending to pull the bump from the plate.



Solution

The current is steady, so $\frac{\partial \rho}{\partial t} = 0$, so $\nabla \cdot \mathbf{J} = 0$, but $\mathbf{J} = g\mathbf{E}$, so $\nabla \cdot \mathbf{E} = 0$. All of which is to say that this is an electrostatics problem despite the current. We need to solve $\nabla^2 \phi = 0$ for this configuration of conductors.

This resembles a problem you all have solved. Find the potential due to a conducting grounded sphere placed in an initially uniform electric field $\mathbf{E}_0$ pointing in the z direction. Without going into details

$$\phi = E_0 \left(\frac{a^3}{r^2} - r \right) \cos\theta$$

You can verify that $\phi = 0$ on all the grounded conductors, and $\mathbf{E}$ is perpendicular to the conductors as required by the boundary conditions. We can be sure that $\mathbf{E}$ has the right asymptotic value by setting $E_0 = V/d$. The uniqueness theorem says that this will be the correct and unique potential inside the region in question.

The charge distribution on the bump is given by

$$\sigma = \left. \frac{\mathbf{D} \cdot \hat{\mathbf{r}}}{4\pi} \right|_{r=a} = \left. \frac{\varepsilon}{4\pi} E_r \right|_{r=a} = -\left. \frac{\varepsilon}{4\pi} \frac{\partial \phi}{\partial r} \right|_{r=a} = \frac{3\varepsilon}{4\pi} \left(\frac{V}{d} \right) \cos\theta$$

All the components of the force cancel except the z component

$$F_z = \int_{\text{bump}} da\ \sigma E_z = 4\pi a^2 \int_0^{\pi/2} \sin\theta\, d\theta\, \sigma\, E_z$$

where

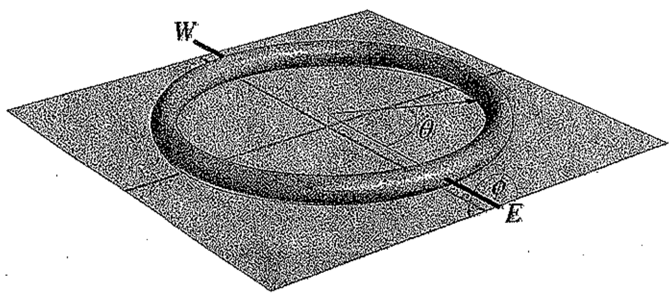
$$E_z = -\left. \frac{\partial \phi}{\partial r} \right|_{r=a} \cos\theta = 3 \frac{V}{d} \cos^2\theta$$

The integral gives $F_z = \frac{9\varepsilon a^2}{4} \left(\frac{V}{d} \right)^2$.

#2

Mechanics – Grad

Consider a hollow glass tube shaped like a donut as shown in the sketch. The inside of the tube is filled with water. The "donut" lies flat on a table and is then turned over by rotating it 180^0 around a diameter, so that it again lies flat on the table surface, which is horizontal. The result is that the water moves with a certain drift velocity around the tube after the donut has been rotated. If there were no friction or other losses, the water would continue to circulate indefinitely.



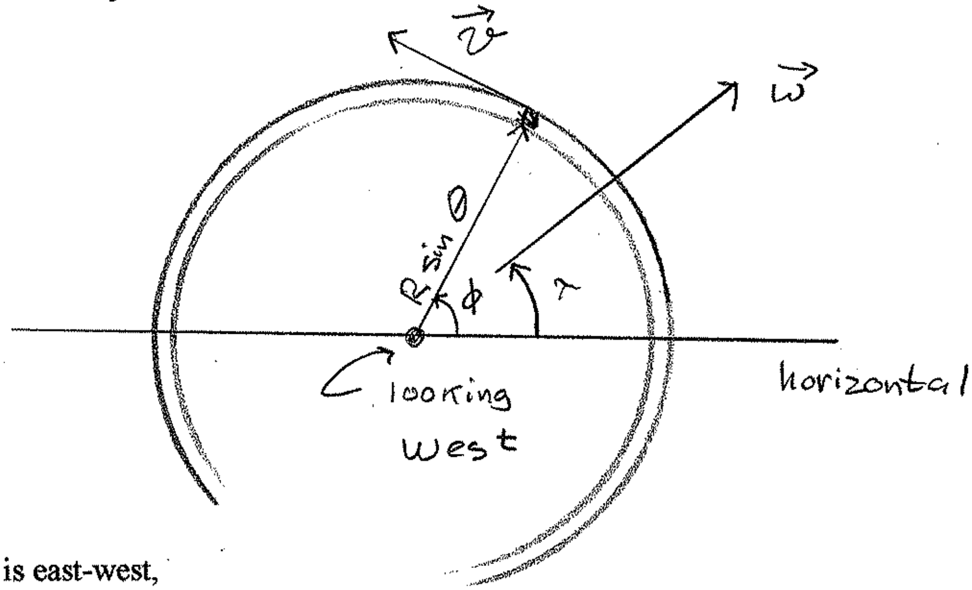
This is remarkable enough. Even more remarkable is the fact that by measuring this velocity, one can determine the latitude of the point on the earth where the experiment is performed. All you need to know is the radius of the tube and the number of seconds in a day. (Come to think of it, you also have to know which way is up and which way is west.)

Even more remarkable than this, is the fact that A. H. Compton invented this device when he was an undergraduate. He suspended some coal dust in the water so he could measure its velocity and determined the longitude to within 3%.

- Prove that the axis about which the donut is flipped should be oriented east-west to maximize the drift velocity of the water. What will happen if it is oriented north-south instead?
- Draw a sketch of the tube with little arrows to show the direction of the Coriolis force at different points around the tube. Just so everyone draws the same picture, assume that the donut has been rotated 90^0 and is still moving.
- Let the angle of rotation about the diameter be ϕ. Then ϕ starts at zero, and after the tube has flipped, $\phi = 180^0$. Calculate the change in the total tangential momentum of the water in the tube. Show that this is independent of $\phi(t)$, *i.e.* it doesn't matter how fast you flip the tube, it only depends of the total change in ϕ.
- Assume the water is equidistant from the center of the circle at some constant radius R. Calculate the drift velocity as a function of the earth's angular velocity and latitude.
- Suppose $R = 1$ m. What speed would you get in Corvallis?

Solution

The Coriolis force is $\mathbf{F} = -2m\omega \times \mathbf{v}$. The best way to visualize this cross product is to imagine looking along the axis of rotation so that you see all the motion projected on a plane perpendicular to the axis of rotation. In this plane, each small volume of water moves in a circle of radius $R\sin\theta$ with velocity $v = \dot{\phi}R\sin\theta$. If the axis points east-west, then the ω vector also lies in this plane. When $\phi = \lambda$, $\omega \times \mathbf{v}$ points east, and when $\phi = \lambda + \pi$, $\omega \times \mathbf{v}$ points west. This causes the water to flow around the tube. If we rotate the apparatus 90^0, these forces would be perpendicular to the plane of the tube and there could be no velocity.



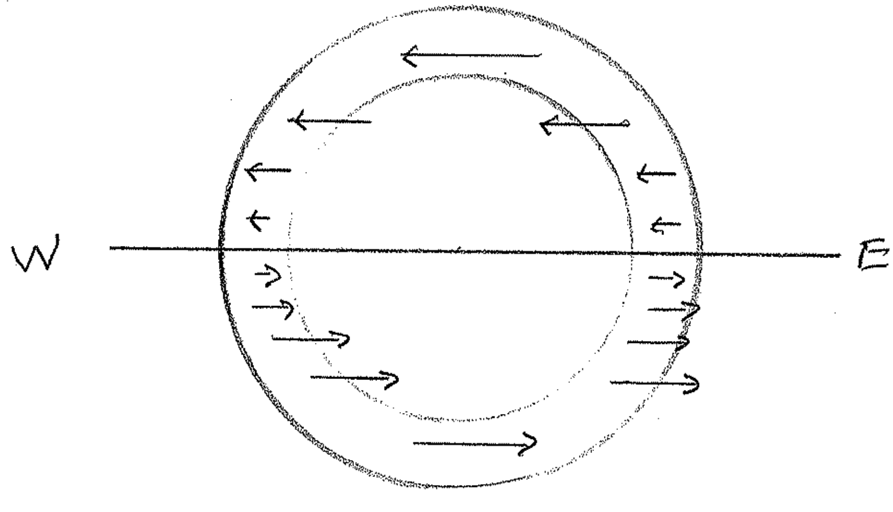
If the axis is east-west,

$$\omega \times \mathbf{v} = \mathbf{e}_\perp \omega v \sin(\pi/2 + \phi - \lambda)$$

where the unit vector $\mathbf{e}_\perp$ points perpendicular to the plane perpendicular to the rotation axis. The Coriolis force is then

$$\mathbf{F}_\perp = \mathbf{e}_\perp(-2m)\omega\,(\dot{\phi}R\sin\theta)\sin(\pi/2 + \phi - \lambda)$$

The following sketch shows the forces on the water at various points around the tube.



The component tangent to the tube is

$$\mathbf{F}_\theta = \mathbf{F}_\perp \sin\theta = -2m\omega R\dot{\phi}\ \sin^2\theta\ \sin(\pi/2+\phi-\lambda)$$

The m in the above equation should really refer to a small unit of mass located between θ and $\theta+\Delta\theta$, $dm = m\,d\theta/2\pi$. It will experience a change of momentum

$$d\dot{P} = -2\left(\frac{m}{2\pi}d\theta\right)\omega R\dot{\phi}\ \sin^2\theta\ \sin(\pi/2+\phi-\lambda)$$

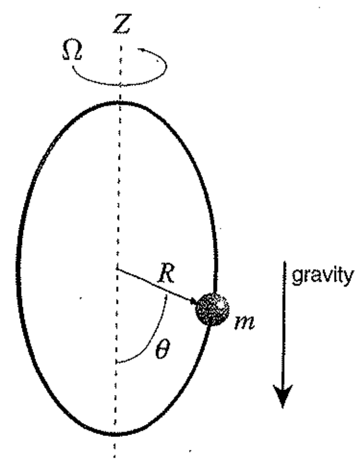
Integrate around the tube.

$$\dot{P} = -2m\omega R\dot{\phi}\quad \sin(\pi/2+\phi-\lambda)$$
$$dP = -2m\omega R\ \sin(\pi/2+\phi-\lambda)\,d\phi$$
$$P = 2m\omega R\cos\lambda = mv$$
$$v = 2\,\omega R\cos\lambda$$

Corvallis is almost at the 45^0 latitude. If $R = 1$ m, $v = 0.1$ mm/s. Compton used a microscope to measure the velocity.

Mechanics UG #3

Consider a bead of mass m moving without friction on a circular hoop of radius R, which rotates at angular frequency Ω about the Z axis as shown in the sketch.



There is one degree of freedom, described by the coordinate θ.

- Find the Lagrangian $L(\theta,\dot{\theta})$.
- Find the equilibrium points from the equation of motion. Which are points of stable equilibrium and which are unstable? How does this depend on Ω?
- Find the frequency of small oscillations about each of the stable equilibrium points.
- The bead is placed very near the bottom of the hoop ($\theta = 0$), and the hoop rotation speed is slowly increased from 0 to some large value. Describe in words what you would expect to see happen to the bead.

UG Mechanics – Solution

$$L = T - V = \tfrac{1}{2}m(R\Omega\sin\theta)^2 + \tfrac{1}{2}(R\dot{\theta})^2 - (-mgR\cos\theta)$$

The equation of motion comes from

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0$$

$$mR^2\ddot{\theta} - mR^2\Omega^2\sin\theta\cos\theta + mgR\sin\theta = 0$$

Equilibrium is given by $\ddot{\theta} = 0$. There are three ways to do this, $\theta = 0$, $\theta = \pi$, and $\cos\theta = g/R\Omega^2$. Put it another way, there is a critical speed, $\Omega_{\text{crit}} = \sqrt{g/R}$. The third equilibrium point only exists for $\Omega > \Omega_{\text{crit}}$.

To find the frequencies, we expand the Lagrangian about the equilibrium points and then find the equations of motion. Expanding about $\theta = 0$ gives

$$\ddot{\theta} + \left[\frac{g}{R} - \Omega^2\right]\theta = 0$$

The frequency is thus $\omega = \sqrt{\Omega_{\text{crit}}^2 - \Omega^2}$. The term with $-\Omega^2$ acts like an "anti-restoring" force. When $\Omega > \Omega_{\text{crit}}$ the $\theta = 0$ point becomes unstable. The $\theta = \pi$ point is always unstable.

In order to find the frequency for the third equilibrium point, we must expand about $\theta_0 = \cos^{-1}(g/R\Omega^2)$. Let $\theta = \theta_0 + \Delta\theta$. Then, for example,

$$\begin{aligned}\sin\theta = \sin(\theta + \Delta\theta) &= \sin\theta\cos\Delta\theta + \cos\theta\sin\Delta\theta \\ &\approx \left(1 - \tfrac{1}{2}\Delta\theta^2\right)\sin\theta_0 + \Delta\theta\cos\theta_0\end{aligned}$$

After much algebra, I get

$$\Delta\ddot{\theta} + \frac{\Omega_{\text{crit}}^2}{2}\left(\frac{\Omega^2}{\Omega_{\text{crit}}^2} - \frac{\Omega_{\text{crit}}^2}{\Omega^2}\right)\Delta\theta = 0$$

The new frequency is then

$$\omega = \sqrt{\frac{\Omega_{\text{crit}}^2}{2}\left(\frac{\Omega^2}{\Omega_{\text{crit}}^2} - \frac{\Omega_{\text{crit}}^2}{\Omega^2}\right)}$$

Now if we release the weight near $\theta = 0$ and gradually increase Ω, it will oscillate slower until $\Omega = \Omega_{\text{crit}}$. As Ω is increased further, the mass will move up the hoop according to $\theta_0 = \cos^{-1}(g / R\Omega^2)$ as it does so it oscillates faster and faster. In the limit $\Omega \to \infty$, $\theta_0 \to \pi / 2$, and $\omega \to \infty$.

Consider a collection of non-interacting particles subject to a Hamiltonian of the form $H_0 = AJ_z^2$ where J_z is the operator representing the component of angular momentum J in the z-direction. Denote the eigenvectors of J_z by $|J,m\rangle$.

(i) Find the energy eigenvalues of this Hamiltonian for the case $J = 3/2$. Indicate the degree of degeneracy, if any, of the eigenvalues.

(ii) Suppose now that the particles are subject to an additional interaction of the form $H_1 = B(J_z \cos\theta + J_x \sin\theta)$ where $B << A$. (This could represent, for example, a weak magnetic field directed at an angle θ to the z-axis in the x-z plane.)

Apply time-independent perturbation theory to find the energy eigenvalues in the presence of this additional interaction.

(iii) Find the normalized linear combinations of $|3/2,1/2\rangle$ and $|3/2,-1/2\rangle$ that are the eigenvectors of the perturbed states related to $|3/2,1/2\rangle$ and $|3/2,-1/2\rangle$ in the unperturbed system.

Useful relations:

$$J_+ = J_x + iJ_y \qquad J_- = J_x - iJ_y$$

$$\langle J,m+1\,|J_+|J,m\rangle = \hbar\,[(J-m)(J+m+1)]^{1/2} \qquad \langle J,m-1\,|J_-|J,m\rangle = \hbar\,[(J+m)(J-m+1)]^{1/2}$$

(i) The obvious choice of basis is the set $|J,m\rangle$ which are eigenvectors of both J_z and J_z^2:

$$J_z|J,m\rangle = \hbar m|J,m\rangle \qquad J_z^2|J,m\rangle = \hbar^2 m^2|J,m\rangle$$

The energy eigenvalues are $E_m = \langle Jm|AJ_z^2|Jm\rangle = A\hbar^2 m^2$

For $J = 3/2$, we have $J = 3/2, 1/2, -1/2, -3/2$.

$E_{3/2} = E_{-3/2} = \frac{9}{4}A\hbar^2$ (doubly degenerate)

$E_{1/2} = E_{-1/2} = \frac{1}{4}A\hbar^2$ (doubly degenerate)

(ii) $H_1 = B(J_z\cos\theta + J_x\sin\theta) = B(J_z\cos\theta + \frac{1}{2}(J_+ + J_-)\sin\theta)$

Although the states $|3/2, 3/2\rangle$ and $|3/2, -3/2\rangle$ are degenerate, there are no matrix elements of H_1 connecting them. Thus

$$E'_{3/2} = E_{3/2} + \langle 3/2, 3/2|B(J_z\cos\theta + \tfrac{1}{2}(J_+ + J_-)\sin\theta|3/2, 3/2\rangle$$

$$= E_{3/2} + \tfrac{3}{2}B\hbar\cos\theta = \tfrac{9}{4}A\hbar^2 + \tfrac{3}{2}B\hbar\cos\theta$$

$$E'_{-3/2} = E_{-3/2} - \tfrac{3}{2}B\hbar\cos\theta = \tfrac{9}{4}A\hbar^2 - \tfrac{3}{2}B\hbar\cos\theta$$

However, the degenerate states $|3/2, 1/2\rangle$ and $|3/2, -1/2\rangle$ are connected by the perturbation. The matrix elements are:

$$\langle \tfrac{3}{2}\tfrac{1}{2}|H_1|\tfrac{3}{2}\tfrac{1}{2}\rangle = \tfrac{1}{2}B\hbar\cos\theta$$
$$\langle \tfrac{3}{2}-\tfrac{1}{2}|H_1|\tfrac{3}{2}-\tfrac{1}{2}\rangle = -\tfrac{1}{2}B\hbar\cos\theta$$

$$\langle \tfrac{3}{2},\tfrac{1}{2}|H_1|\tfrac{3}{2},-\tfrac{1}{2}\rangle = B\sin\theta \cdot \tfrac{1}{2}\langle \tfrac{3}{2}\tfrac{1}{2}|J_+|\tfrac{3}{2},-\tfrac{1}{2}\rangle = B\hbar\sin\theta$$
$$\langle \tfrac{3}{2},-\tfrac{1}{2}|H_1|\tfrac{3}{2},\tfrac{1}{2}\rangle = B\hbar\sin\theta$$

To find the energy eigenvalues, solve the secular equation

$$\begin{vmatrix} -\tfrac{1}{2}B\hbar\cos\theta - E & B\hbar\sin\theta \\ B\hbar\sin\theta & \tfrac{1}{2}B\hbar\cos\theta - E \end{vmatrix} = 0$$

$$E_\pm^* = \pm\tfrac{1}{2}B\hbar\cos\theta\,[1 + 4\tan^2\theta]^{1/2}$$

Total energies are

$$E'_{3/2} = \tfrac{9}{4}A\hbar^2 + \tfrac{3}{2}B\hbar\cos\theta$$

$$E'_{-3/2} = \tfrac{9}{4}A\hbar^2 - \tfrac{3}{2}B\hbar\cos\theta$$

$$E'_+ = \tfrac{1}{4}A\hbar^2 + \tfrac{1}{2}B\hbar\cos\theta\,[1 + 4\tan^2\theta]^{1/2}$$

$$E'_- = \tfrac{1}{4}A\hbar^2 - \tfrac{1}{2}B\hbar\cos\theta\,[1 + 4\tan^2\theta]^{1/2}$$

(iii) To find eigenvectors related to $|3/2, 1/2\rangle$ and $|3/2, -1/2\rangle$:

Eigenvalues are $E_\pm = \pm\frac{1}{2}B\hbar\cos\theta\left[1+4\tan^2\theta\right]^{1/2}$

(can drop common term $E_{1/2} = E_{-1/2} = A\hbar^2/4$)

Let $f \equiv \left[1+4\tan^2\theta\right]^{1/2} \Rightarrow E_\pm = \pm\frac{1}{2}B\hbar\cos\theta\, f$

For $|+\rangle$: $|+\rangle = a^+_{-1/2}|3/2,-1/2\rangle + a^+_{1/2}|3/2,1/2\rangle$ and

$$\left(-\tfrac{1}{2}B\hbar\cos\theta - E_+\right)a^+_{-1/2} + (B\hbar\sin\theta)\,a^+_{1/2} = 0$$

$$-\tfrac{1}{2}(1+f)\,a_{-1/2} + \tan\theta\, a_{1/2} = 0$$

$$\frac{a^+_{-1/2}}{a^+_{1/2}} = \frac{2\tan\theta}{1+f}$$

normalize: $\left[\left(\frac{4\tan^2\theta}{(1+f)^2}\right) + 1\right]|a^+_{1/2}|^2 = 1$

subst. $4\tan^2\theta = f^2-1 \Rightarrow a^+_{1/2} = \left(\frac{f+1}{2f}\right)^{1/2}$

$$a^+_{-1/2} = \left[\frac{f^2-1}{(1+f)^2}\right]^{1/2}\cdot a^+_{1/2} = \left[\frac{f^2-1}{(1+f)^2}\cdot\frac{f+1}{2f}\right]^{1/2} = \left(\frac{f-1}{2f}\right)^{1/2}$$

$$|+\rangle = \left(\frac{f-1}{2f}\right)^{1/2}|3/2,-1/2\rangle + \left(\frac{f+1}{2f}\right)^{1/2}|3/2,1/2\rangle$$

Easiest to get $|-\rangle$ by requiring $\langle -|+\rangle = 0$:

$$|-\rangle = -\left(\frac{f+1}{2f}\right)^{1/2}|3/2,-1/2\rangle + \left(\frac{f-1}{2f}\right)^{1/2}|3/2,1/2\rangle$$

#5

Statistical Mechanics – Grad

Consider a gas of electrons and positrons. Ignore the interactions between the particles, and assume that we can treat systems of the electrons and of the positrons each as an ideal gas. The only interaction we do take into account is pair creation and annihilation. The density of the electrons is n_- and of the positrons n_+. The mass of the electrons and positrons is m_0. The internal chemical potential of an ideal gas is $\mu_{int}(n,T) = k_B T \ln(\frac{n}{n_Q(T)})$, where k_B is the Boltzmann constant and $n_Q(T)$ is a function of temperature, called the quantum concentration. The volume of the system is constant, and the temperature is constant, only the particle numbers can change.

(1) What is the total chemical potential for the electrons?

(2) How are changes in the electron and positron density related?

(3) Which free energy do we need to describe the system?

(4) How does this free energy change when the electron density changes by an amount Δn_-?

(5) Which criterium gives us the stable state of the system?

(6) Derive the relation $n_+ n_- = n_Q^2(T) e^{-\frac{2m_0c^2}{k_BT}}$ for the equilibrium state.

(7) At room temperature we have $k_BT \approx 0.025eV$. Twice the mass energy of the electrons is about $1MeV$. Assume that the density of the electrons is $10^{-10} n_Q(T)$, which is a realistic value. Show that in this case in thermal equilibrium the presence of positrons can be completely ignored.

Problem ~~1~~ #5

Consider a gas of electrons and positrons. Ignore the interactions between the particles, and assume that we can treat systems of the electrons and of the positrons each as an ideal gas. The only interaction we do take into account is pair creation and annihilation. The density of the electrons is n_- and of the positrons n_+. The mass of the electrons and positrons is m_0. The internal chemical potential of an ideal gas is $\mu_{int}(n,T) = k_B T \ln(\frac{n}{n_Q(T)})$, where k_B is the Boltzmann constant and $n_Q(T)$ is a function of temperature, called the quantum concentration. The volume of the system is constant, and the temperature is constant, only the particle numbers can change.

(1) What is the total chemical potential for the electrons?

$$\mu_{tot} = \mu_{int} + m_0 c^2$$

In order to create an electron we need to account for the kinetic energy, via the internal chemical potential, and for the rest mass energy.

(2) How are changes in the electron and positron density related?

Production happens in pairs:

$$\Delta n_+ = \Delta n_-$$

(3) Which free energy do we need to describe the system?

The basic parameters of the problem are T, V, N_+, and N_-. The last two are the total number of positrons and electrons. The free energy that corresponds to this situation is the Helmholtz free energy, $F = U - TS$.

(4) How does this free energy change when the electron density changes by an amount Δn_-?

Since $\Delta T = 0$ and $\Delta V = 0$ we have:

$$\Delta F = \mu_{tot}^- \Delta N_- + \mu_{tot}^+ \Delta N_+$$

$$\frac{1}{V}\Delta F = \left(\mu_{int}(n_-,T) + \mu_{int}(n_+,T) + 2m_0c^2\right)\Delta n_-$$

(5) Which criterium gives us the stable state of the system?

Minimum energy principle:

$$\Delta F = 0$$

(6) Derive the relation $n_+ n_- = n_Q^2(T) e^{-\frac{2m_0c^2}{k_BT}}$ for the equilibrium state.

$$\mu_{int}(n_-, T) + \mu_{int}(n_+, T) + 2m_0c^2 = 0$$

$$k_BT \ln(\frac{n_-}{n_Q(T)}) + k_BT \ln(\frac{n_+}{n_Q(T)}) = -2m_0c^2$$

$$\ln(\frac{n_- n_+}{n_Q^2}) = -\frac{2m_0c^2}{k_BT}$$

(7) At room temperature we have $k_BT \approx 0.025eV$. Twice the mass energy of the electrons is about $1MeV$. Assume that the density of the electrons is $10^{-10} n_Q(T)$, which is a realistic value. Show that in this case in thermal equilibrium the presence of positrons can be completely ignored.

$$10^{-10} n_Q n_+ \approx n_Q^2 e^{-4 10^7}$$

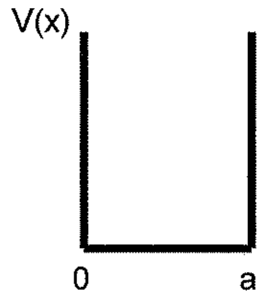
$$n_+ \approx n_Q 10^{10} e^{-4 10^7}$$

The last factor is much smaller than the first, which shows $n_+ \ll n_-$.

#6

QM – UGRAD

This problem concerns a particle of mass m in a one-dimensional infinite square well in which the potential $V(x) = 0$ for $0 \le x \le a$ and V is infinite elsewhere:



Recall that the energy eigenfunctions for this potential are of the form $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$ and the energy eigenvalues are $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

(i) Evaluate the root mean square deviation Δx of the particle's position for the state $\psi_n(x)$.

(ii) Show that for large values of the quantum number n, Δx approaches the value expected for a classical probability distribution for the particle's position.

(iii) What are the expectation values of the particle's position in the states ψ_1, ψ_2, *and* the state $\psi = \frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_2$?

(iv) Suppose that at some time $t = 0$, the system is in the state $\psi = \frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_2$. Derive an expression for the expectation value of the position at a later time t.

Some of these integrals may be useful:

$$\int x\sin^2 x\,dx = \frac{x^2}{4} - \frac{x\sin 2x}{4} - \frac{\cos 2x}{8} \qquad \int x^2\sin^2 x\,dx = \frac{x^3}{6} - \left(\frac{x^2}{4} - \frac{1}{8}\right)\sin 2x - \frac{x\cos 2x}{4}$$

$$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x \qquad \int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x \qquad \int \sin x\cos x\,dx = \frac{1}{2}\sin^2 x$$

$$\int \sin mx\,\sin nx\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} \qquad (m^2 \ne n^2)$$

$$\int \sin mx\,\cos nx\,dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} \qquad (m^2 \ne n^2)$$

QM – Ugrad

For the infinite square well, it was given that

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \quad \text{and} \quad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}.$$

(i) The r.m.s. deviation for x is $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$

where, for example, $\langle x^2\rangle = \langle n|x^2|n\rangle$

$$= \left(\frac{2}{a}\right)\int_0^a \sin\left(\frac{n\pi x}{a}\right) x^2 \sin\left(\frac{n\pi x}{a}\right) dx.$$

$$= \left(\frac{2}{a}\right)\left(\frac{a}{n\pi}\right)^3 \int_0^{n\pi} u^2 \sin^2 u \, du = \left(\frac{2}{a}\right)\left(\frac{a}{n\pi}\right)^3 \left[\frac{(n\pi)^3}{6} - \frac{n\pi}{4}\right]$$

$$\langle x^2\rangle = \langle n|x^2|n\rangle = a^2\left[\frac{1}{3} - \frac{1}{2(n\pi)^2}\right]$$

$$\langle x\rangle = \langle n|x|n\rangle = \left(\frac{2}{a}\right)\int_0^a x \sin^2\left(\frac{n\pi x}{a}\right) dx$$

$$= \left(\frac{2}{a}\right)\left(\frac{a}{n\pi}\right)^2 \int_0^{n\pi} u \sin^2 u \, du = \left(\frac{2}{a}\right)\left(\frac{a}{n\pi}\right)^2 \cdot \frac{(n\pi)^2}{4}$$

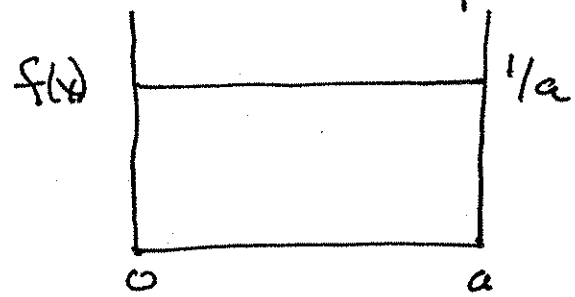
$\langle x\rangle = \langle n|x|n\rangle = a/2$ for all n

(We could have seen this from the form of $\psi_n(x)$ by symmetry.)

Now, $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{a^2\left[\frac{1}{3} - \frac{1}{2(n\pi)^2}\right] - a^2/4}$

$$\Delta x = a\sqrt{\frac{1}{12} - \frac{1}{2(n\pi)^2}}$$

(ii) The classical probability distribution is $f(x) = 1/a$:



since $\int_0^a f(x)\,dx = 1$

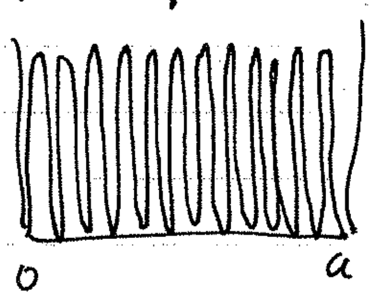
$$\langle x^2 \rangle = \int_0^a \left(\frac{1}{a}\right) x^2\,dx = \frac{a^2}{3} \qquad \langle x \rangle = \frac{a}{2}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a\sqrt{\frac{1}{3} - \frac{1}{4}} = a\sqrt{\frac{1}{12}}$$

The quantum result $\Delta x = a\sqrt{\frac{1}{12} - \frac{1}{2(n\pi)^2}} \xrightarrow[n\to\infty]{} a\sqrt{\frac{1}{12}}$

At high values of n, the quantum probability distribution

$|\psi_n(x)|^2 \rightarrow$



(approx. uniform)

(iii) We have already found the expectation values for ψ_1 and ψ_2:

$$\langle 1|x|1\rangle = \langle 2|x|2\rangle = a/2$$

Because of interference effects $\langle \psi|x|\psi\rangle \neq a/2$:

$$\langle \psi|x|\psi\rangle = \frac{1}{2}\left[\langle 1|x|1\rangle + \langle 2|x|2\rangle + \langle 1|x|2\rangle + \langle 2|x|1\rangle\right]$$

We need the matrix elements $\langle 1|x|2\rangle = \langle 2|x|1\rangle$

$$= \left(\frac{2}{a}\right)\int_0^a \sin\left(\frac{\pi x}{a}\right) x \sin\left(\frac{2\pi x}{a}\right) = \left(\frac{2}{a}\right)\left(\frac{a}{\pi}\right)^2 \int_0^\pi u \sin u \sin 2u \, du$$

integrate by parts:

$$\int_0^\pi u \sin u \sin 2u = \underbrace{u \cdot \left(\frac{\sin u}{2} - \frac{\sin 3u}{6}\right)\Big|_0^\pi}_{=0} - \int_0^\pi \left(\frac{\sin u}{2} - \frac{\sin 3u}{6}\right) du$$

$$= \left[\frac{1}{2}\cos u - \frac{1}{6}\cdot\frac{1}{3}\cos 3u\right]_0^\pi = -8/9$$

$$\langle 1|x|2\rangle = \langle 2|x|1\rangle = \left(\frac{2}{a}\right)\left(\frac{a}{\pi}\right)^2\left(-8/9\right) = -\frac{16a}{9\pi^2}$$

$$\langle \psi|x|\psi\rangle = \frac{1}{2}\left[\frac{a}{2} + \frac{a}{2} - \frac{16a}{9\pi^2} - \frac{16a}{9\pi^2}\right]$$

$$\langle \psi|x|\psi\rangle = \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\right]$$

(iv) At time t,

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-iE_1t/\hbar}|1\rangle + \frac{1}{\sqrt{2}} e^{-iE_2t/\hbar}|2\rangle$$

$$\langle\psi(t)|x|\psi(t)\rangle =$$

$$\frac{1}{2}\left[e^{iE_1t/\hbar}\langle 1| + e^{iE_2t/\hbar}\langle 2|\right] x \left[e^{-iE_1t/\hbar}|1\rangle + e^{-iE_2t/\hbar}|2\rangle\right]$$

$$= \frac{1}{2}\Big[\langle 1|x|1\rangle + \langle 2|x|2\rangle + e^{i(E_2-E_1)t/\hbar}\langle 2|x|1\rangle$$
$$+ e^{-i(E_2-E_1)t/\hbar}\langle 1|x|2\rangle\Big]$$

$$= \frac{1}{2}\left[\frac{a}{2} + \frac{a}{2} - \left(\frac{16a}{9\pi^2}\right)e^{i\omega_{21}t} - \left(\frac{16a}{9\pi^2}\right)e^{-i\omega_{21}t}\right]$$

$$\omega_{21} \equiv \frac{E_2 - E_1}{\hbar} = 3\frac{\pi^2\hbar}{2ma^2}$$

$$= \frac{1}{2}\left[a - \left(\frac{16a}{9\pi^2}\right)\left(e^{i\omega_{21}t} + e^{-i\omega_{21}t}\right)\right]$$

$$\langle\psi(t)|x|\psi(t)\rangle = a\left[\frac{1}{2} - \left(\frac{16}{9\pi^2}\right)\cos\omega_{21}t\right]$$

#7

E&M Grad

Calculate the time-average power radiated per unit solid angle in nonrelativistic motion of a particle with charge e, moving

- along the z axis with instantaneous position $z = a\cos\omega t$.
- In a circle of radius R in the x - y plane with angular frequency ω.

Sketch the angular distribution of the radiation and determine the total power radiated in each case. You may assume that the observation point is a great distance from the source and make any other approximations necessary to get a simple (but meaningful) answer. Be sure to justify your approximations. You will need the formulas for the Leinard-Wiechert fields.

$$\mathbf{E}(\mathbf{x},t) = e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{\text{ret}} + \frac{e}{c}\left[\frac{\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{\text{ret}}$$

$$\mathbf{B} = [\mathbf{n}\times\mathbf{E}]_{\text{ret}}$$

Standard notation calls $\mathbf{R}$ the vector from the particle to the field point, and $\hat{\mathbf{n}}$ the unit vector in the same direction.

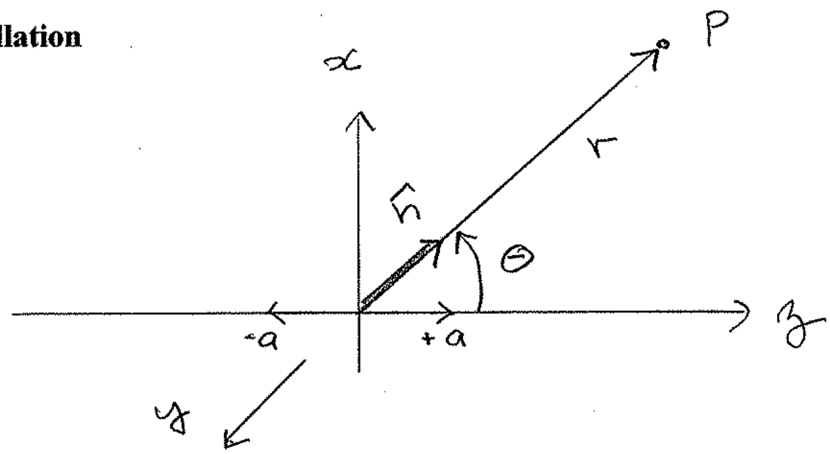
Solution

Since the field point is far from the source, only the $1/R$ term contributes to the radiated power. Make the following approximations:

$$R \approx r, \qquad (\mathbf{n} - \beta) \approx \mathbf{n}, \qquad (1\text{-}\beta \cdot \mathbf{n}) \approx 1$$

Under the circumstances, we can assume that there is no significant difference in the propagation time of radiation from different points along the particle's trajectory, so we can neglect the complications arising from evaluating the fields at retarded time. Just remember that all the radiation reaches the field point a time r/c later than it was radiated.

Linear Oscillation



$$\dot{\beta} = \frac{1}{c}\frac{d^2 z(t)}{dt^2}\hat{\mathbf{z}} = -\frac{a\omega^2}{c}\hat{\mathbf{z}}\cos\omega t$$

$$\mathbf{E} = \frac{e}{c}\frac{\mathbf{n} \times (\mathbf{n} \times \dot{\beta})}{r}$$

Assume that the observation point lies in the x - z plane, so that $\hat{\mathbf{n}} \times \hat{\mathbf{z}} = -\hat{\mathbf{y}}$.

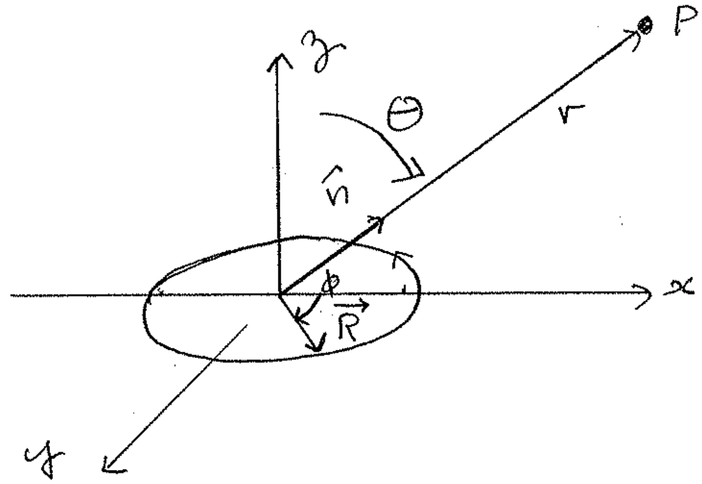
$$\mathbf{E} = \hat{\mathbf{n}} \times \hat{\mathbf{y}} \frac{ea\omega^2}{rc^2}\cos\omega t \sin\theta$$

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H} = \frac{c}{4\pi}|\mathbf{E}|^2\,\hat{\mathbf{n}}$$

$$\frac{\text{Power}}{\text{Solid angle}} = r^2 S = \frac{c}{4\pi}\left(\frac{ea\omega^2\cos\omega t\sin\theta}{rc^2}\right)^2 r^2 = \frac{e^2a^2\omega^4\sin^2\theta\cos^2\omega t}{4\pi c^3}$$

$$\left[\frac{\text{Power}}{\text{Solid angle}}\right]_{\text{Average}} = \frac{e^2 a^2 \omega^4 \sin^2\theta}{8\pi c^3}$$

Circular Motion



Assume that the observation point lies in the x - z plane.

$$\dot{\beta} = -\frac{\omega^2}{c}\mathbf{R}$$

$$\mathbf{E} = \frac{e}{c}\frac{\mathbf{n}\times(\mathbf{n}\times\dot{\beta})}{r} = \left(\frac{e}{rc}\right)\left[\mathbf{n}(\mathbf{n}\cdot\dot{\beta}) - \dot{\beta}\right]$$

We need to find the angle between $\mathbf{n}$ and $\mathbf{R}$, call it Ω. The angles are related by spherical trig.

$$\cos\Omega = \cos 90^0 \cos\theta + \sin 90^0 \sin\theta\cos\phi = \sin\theta\cos\phi$$

$$\mathbf{E} = \left(\frac{e}{rc}\right)\left(\frac{\omega^2 R}{c}\right)\left[\hat{\mathbf{R}} - \hat{\mathbf{n}}\cos\Omega\right] = \left(\frac{e\omega^2 R}{rc^2}\right)\left[\hat{\mathbf{R}} - \hat{\mathbf{n}}\sin\theta\cos\phi\right]$$

$$\mathbf{S} = \frac{c}{4\pi}|\mathbf{E}|^2\,\hat{\mathbf{n}} = \hat{\mathbf{n}}\left(\frac{\mathbf{e}^2\omega^4 R^2}{4\pi r^2 c^3}\right)\sin^2\theta\cos^2\phi$$

$$\left[\frac{\text{Power}}{\text{Solid angle}}\right]_{\text{Average}} = \frac{e^2 R^2 \omega^4 \sin^2\theta}{8\pi c^3}$$

Statistical Mechanics
Undergrad

Problem ~~2~~ 8

Consider a system at temperature T with volume V. The particles in this system are independent. There are N particles. Each particle has energy states $\epsilon_n = n\hbar\omega$ for $n = 0, 1, \cdots$. Calculate the heat capacity per particle in the following two cases: (1) the particles are distinguishable, (2) the particles are identical fermions.

There are several ways to attack the problem. For part (1), for example, we can first calculate the one-particle partition function $\mathcal{Z}_1(T, V)$:

$$\mathcal{Z}_1(T,V) = \sum_{n=0}^{\infty} e^{-\frac{n\hbar\omega}{k_B T}}$$

$$\mathcal{Z}_1(T,V) = \frac{1}{1 - e^{-\frac{\hbar\omega}{k_B T}}}$$

The total partition function is

$$\mathcal{Z}(T,V,N) = \mathcal{Z}_1^N(T,V)$$

No factor $N!$ because the particles are distinguishable. The internal energy follows from:

$$U = k_B T^2 \left(\frac{\partial \ln(\mathcal{Z})}{\partial T}\right)_{V,N}$$

which can be derived easily:

$$\left(\frac{\partial \mathcal{Z}}{\partial T}\right)_{V,N} = \sum_s e^{-\frac{\epsilon_s}{k_B T}} \left(\frac{\epsilon_s}{k_B T^2}\right)$$

$$\left(\frac{\partial \mathcal{Z}}{\partial T}\right)_{V,N} = \frac{1}{k_B T^2} \sum_s \epsilon_s e^{-\frac{\epsilon_s}{k_B T}}$$

$$\left(\frac{\partial \mathcal{Z}}{\partial T}\right)_{V,N} = \frac{1}{k_B T^2} < \epsilon_s > \mathcal{Z}$$

This gives:

$$U = k_B T^2 \left(\frac{\partial \log(\mathcal{Z}_1)}{\partial T}\right)_V$$

$$U = -N k_B T^2 \frac{-\frac{\hbar\omega}{k_B T^2} e^{-\frac{\hbar\omega}{k_B T}}}{1 - e^{-\frac{\hbar\omega}{k_B T}}}$$

$$U = N\hbar\omega \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1}$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_{V,N} = N\hbar\omega \frac{\frac{\hbar\omega}{k_B T^2} e^{\frac{\hbar\omega}{k_B T}}}{\left(e^{\frac{\hbar\omega}{k_B T}} - 1\right)^2}$$

$$C_V = N k_B \frac{\hbar^2\omega^2}{4k_B^2 T^2} \sinh^{-2}(\frac{\hbar\omega}{2k_B T})$$

which approaches Nk_B at high temperatures and approaches zero exponentially at low temperature.

For fermions we know that only one particle can occupy one state. One approach is to use Fermi-Dirac statistics, because the particles are independent. This gives:

$$U(T,V,\mu) = \sum_{n=0}^{\infty} n\hbar\omega f_{FD}(n\hbar\omega : T,\mu)$$

$$N(T,V,\mu) = \sum_{n=0}^{\infty} f_{FD}(n\hbar\omega : T,\mu)$$

$$U(T,V,\mu) = \sum_{n=0}^{\infty} n\hbar\omega \frac{1}{e^{\frac{n\hbar\omega - \mu}{k_B T}} + 1}$$

$$N(T,V,\mu) = \sum_{n=0}^{\infty} \frac{1}{e^{\frac{n\hbar\omega - \mu}{k_B T}} + 1}$$

We need to solve the last equation for $\mu(N,T,V)$ and insert the result in the second to last equation to get $U(N,T,V)$ and then the heat capacity. Some observations that can give credit:

Replacing the equations by Boltzmann statistics gives:

$$U(T,V,\mu) = \sum_{n=0}^{\infty} n\hbar\omega e^{\frac{\mu - n\hbar\omega}{k_B T}}$$

$$N(T,V,\mu) = \sum_{n=0}^{\infty} e^{\frac{\mu - n\hbar\omega}{k_B T}} = e^{\frac{\mu}{k_B T}} \mathcal{Z}_1$$

or:

$$U(T,V,\mu) = N(T,V,\mu)\mathcal{Z}_1^{-1} \sum_{n=0}^{\infty} n\hbar\omega e^{-\frac{n\hbar\omega}{k_B T}}$$

which indeed gives the expression derived before. Note that there is a difference of a factor $N!$ in the partition function, but that does not affect the internal energy.

For the Fermi problem we need to do the sums. The only way we can do that is by replacing the sums by integrals. That works as long as $k_BT \gg \hbar\omega$. The frequency ω is a function of volume, and for large volumes it tends to be small. Assume that we can replace the sum by an integral. That gives, using $z = e^{\frac{\mu}{k_BT}}$:

$$U(T,V,\mu) = \frac{k_B^2T^2}{\hbar\omega}\int_0^\infty dx \frac{x}{z^{-1}e^x + 1}$$

$$N(T,V,\mu) = \frac{k_BT}{\hbar\omega}\int_0^\infty dx \frac{1}{z^{-1}e^x + 1}$$

This is also equivalent to:

$$U(T,V,\mu) = z\frac{k_B^2T^2}{\hbar\omega}\int_0^\infty dx \frac{x}{e^x + z}$$

$$N(T,V,\mu) = z\frac{k_BT}{\hbar\omega}\int_0^\infty dx \frac{1}{e^x + z}$$

Suppose $z \ll 1$, then we have:

$$U(T,V,\mu) \approx z\frac{k_B^2T^2}{\hbar\omega}\int_0^\infty dx x e^{-x}$$

$$N(T,V,\mu) \approx z\frac{k_BT}{\hbar\omega}\int_0^\infty dx e^{-x}$$

$$U(T,V,\mu) \approx z\frac{k_B^2T^2}{\hbar\omega}$$

$$N(T,V,\mu) \approx z\frac{k_BT}{\hbar\omega}$$

$$U \approx Nk_BT$$

$$C_V \approx Nk_B$$

which is indeed the correct high temperature limit.
In the general case, the integral for N can be done. Replace $y = e^x$ and we get:

$$N(T,V,\mu) = z\frac{k_BT}{\hbar\omega}\int_1^\infty dy \frac{1}{y(y+z)} = \frac{k_BT}{\hbar\omega}\ln(1+z)$$

which can be inverted to give:

$$z = e^{\frac{N\hbar\omega}{k_BT}} - 1$$

and hence we get:

$$U(T,V,N) = (e^{\frac{N\hbar\omega}{k_BT}} - 1)\frac{k_B^2T^2}{\hbar\omega}\int_0^\infty dx \frac{x}{e^x + e^{\frac{N\hbar\omega}{k_BT}} - 1}$$