Department of Physics Comprehensive Examination # 94

Part I

31 March 2003

This Comprehensive Examination for Spring 2003 consists of eight problems each worth 20 points. The problems are grouped into four sessions:

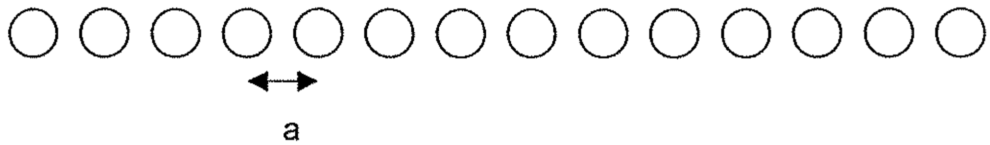
Session 1	problems 1, 2	9-12 AM	Mon 31 March
Session 2	problems 3, 4	1:30-4:30 PM	Mon 31 March
Session 3	problems 5, 6	9-12 AM	Tues 1 April
Session 4	problems 7, 8	1:30-4:30 PM	Tues 1 April

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it is possible to obtain partial credit, especially if you demonstrate conceptual understanding. Do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter, but not your name, is on the inside of the back cover of every bluebook. Be sure to remember your student letter for use in the remaining sessions of the examination. If something is omitted from the statement of the problem or you feel there is an ambiguity, please ask your question quietly and privately, so as not to disturb the others. Only your bluebooks and the examination should be on the table before you. Any other items should be stored on the floor. Please return all bluebooks and formula sheets at the end of the exam.

Scratch paper will be supplied for the exam. Scratch work will not be graded.

1.

Consider a long, linear chain of N hydrogen atoms in their 1s ground states which we denote by $|j\rangle$ for the j^{th} atom in the chain. The atoms are separated by a uniform distance a.



For periodic boundary conditions (atom # 1 is considered to be equivalent to atom # N), the states of the system are labeled by a wave vector k and can be expressed in the form

$$|k\rangle = N^{-1/2}\sum_{j=1}^{N} e^{ikx_j}|j\rangle$$

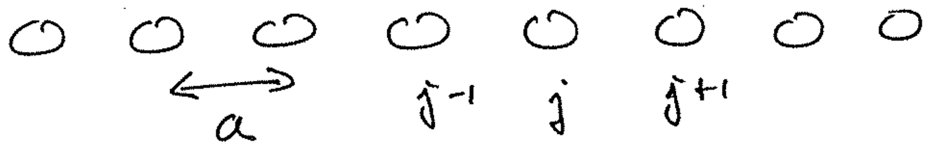
where x_j is the position of the j^{th} atom, and the atomic states $|j\rangle$ are assumed to form an orthonormal basis set (not quite true, but not a bad approximation).

(a) Apply the time-independent Schrödinger equation to obtain a general expression for the electron energy $E(k)$ in terms of the matrix elements of the Hamiltonian $H_{jj'}$.

(b) Consider the "nearest-neighbor" approximation in which the matrix elements of the Hamiltonian are given by $H_{jj'} = -t$ if $j' = j \pm 1$, $H_{jj} = \varepsilon$, and $H_{jj'} = 0$ otherwise. Show that $E(k) = \varepsilon - 2t\cos(ka)$ in this approximation.

(c) Show that in the limit $k << 1/a$, $E(k)$ reduces to a form resembling the kinetic energy of a free particle for which the de Broglie wavelength = $2\pi/k$. Relate the "effective mass" of these quasi-free particles to the parameters a and t of the model used in this problem.

QM2 (G)



$$|k\rangle = N^{-1/2} \sum_{j=1}^{N} e^{ikx_j} |j\rangle$$

(a) Derive $\varepsilon(k)$:

Schrödinger's Eqn: $H|k\rangle = \varepsilon(k)|k\rangle$

$$N^{-1/2} \sum_{j=1}^{N} e^{ikx_j} H|j\rangle = \varepsilon(k) N^{-1/2} \sum_{j=1}^{N} e^{ikx_j} |j\rangle$$

project onto a state $\langle j'|$:

$$N^{-1/2} \sum_{j=1}^{N} e^{ikx_j} \langle j'|H|j\rangle = \varepsilon(k) N^{-1/2} \sum_{j=1}^{N} e^{ikx_j} \langle j'|j\rangle$$

$\langle j'|j\rangle = \delta_{jj'}$ by orthonormality

so $$\sum_{j=1}^{N} e^{ikx_j} \langle j'|H|j\rangle = \varepsilon(k) e^{ikx_{j'}}$$

Solve for $$\varepsilon(k) = \sum_{j=1}^{N} e^{ik(x_j - x_{j'})} \langle j'|H|j\rangle$$

$$\Rightarrow \quad \varepsilon(k) = \sum_{j=1}^{N} H_{j'j}\, e^{ik(x_j - x_{j'})} \quad \Leftarrow$$

(b) Nearest neighbor model:

$$H_{jj'} = -t \quad j \; j' = j \pm 1 \qquad H_{jj} = \varepsilon$$

otherwise $H_{jj'} = 0$

We have $\varepsilon(k) = \sum_{j=1}^{N} H_{j'j} \, e^{ik(x_j - x_{j'})}$

$j = j'+1, \; x_j - x_{j'} = a \qquad j = j'-1, \; x_j - x_{j'} = -a$

$j = j, \; x_j - x_j = 0$

$$\varepsilon(k) = -t e^{ika} - t e^{-ika} + \varepsilon$$

$$\varepsilon(k) = \varepsilon - 2t \cos ka$$

(c) $ka \ll 1$

$$\cos ka \simeq 1 - \tfrac{1}{2}(ka)^2$$

$$\varepsilon(k) = \varepsilon - 2t\left(1 - \tfrac{1}{2}(ka)^2\right) = \varepsilon_{min} + t a^2 k^2$$

So $\varepsilon(k) - \varepsilon_{min} = t a^2 k^2$

Free particle: momentum $p = \frac{h}{\lambda} = \frac{h}{2\pi} \cdot \frac{2\pi}{\lambda} = \hbar k$

$$K.E. = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

Compare with $\mathcal{E}(k) - \mathcal{E}_{min} = t a^2 k^2$, so relative to minimum electron energy, electron behaves like a free particle with $\frac{\hbar^2}{2m} \Leftrightarrow t a^2$ or an effective mass $m_{eff} = \frac{\hbar^2}{2 t a^2}$

2.

A material can exist in gas, liquid, and solid phases and in mixtures of these. The figure shows isotherms in the pressure-volume plane.

i) draw the boundaries between the phases.

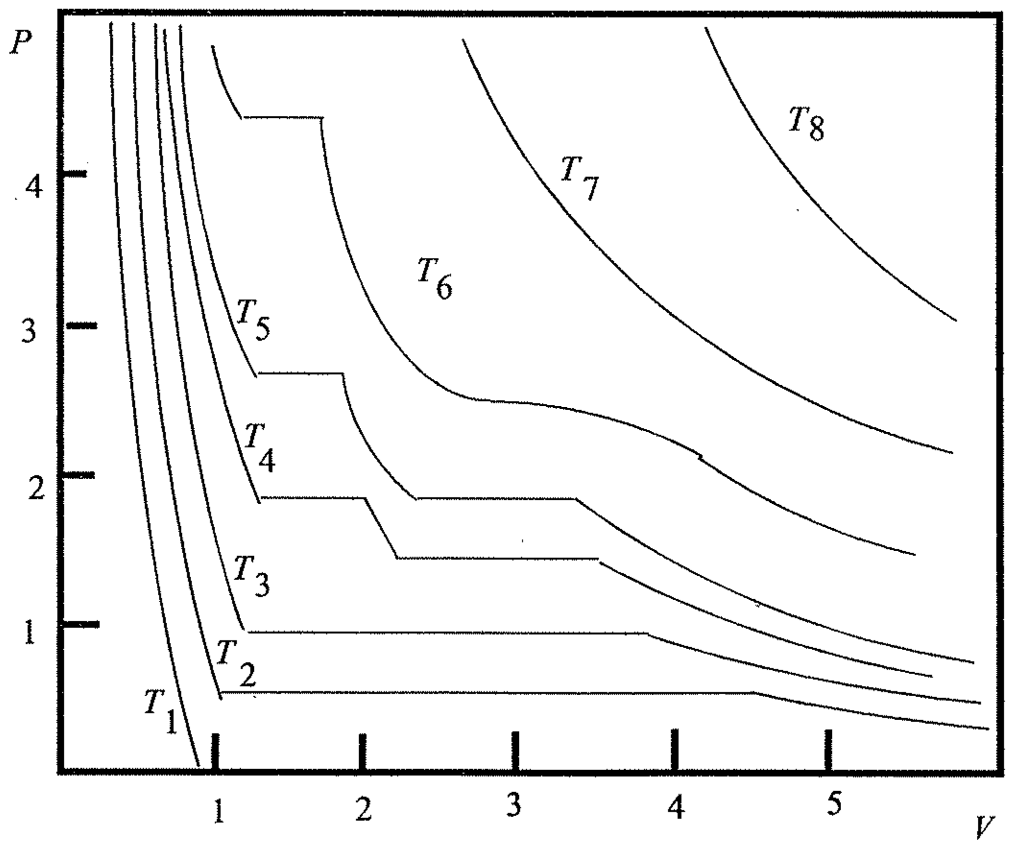
ii) which region of the PV diagram corresponds to

-gas	-liquid	-solid
-gas/liquid mix	-liquid/solid mix	-gas-solid mix

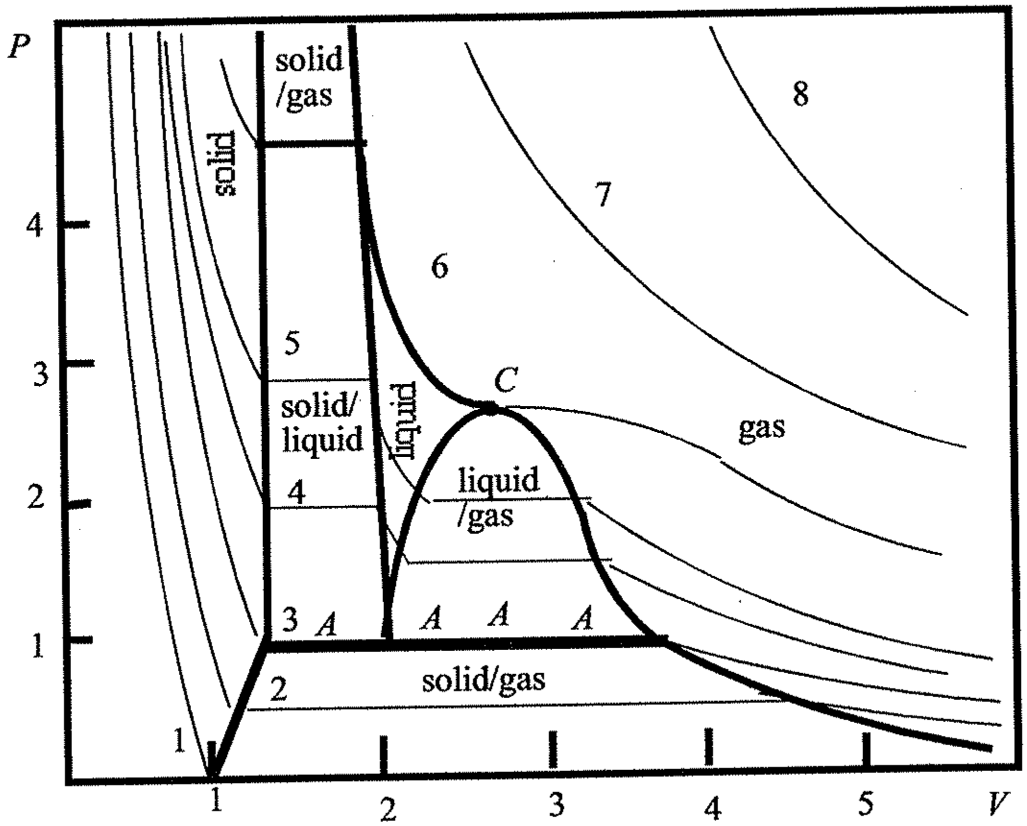
iii) where, on this diagram, are all three phases in equilibrium? label as A

iv) where is the critical point ? label as C

v) for each isotherm, estimate the heat of vaporization and/or the heat of melting.



SOLUTION



phase transitions: volume changes via mixed phase at constant P, T

energy to change phase = $P\,\Delta V$ at constant T

isotherm	heat of melting	heat of vaporization
1	0	0
2	$0.5 \times (4.5-1.2) = 1.6$	same transition
3	$1 \times (2-1.3) = 0.7$	$1 \times (3.7-2) = 1.7$
4	$1.9 \times (2-1.3) = 1.4$	$1.6 \times (3.3-2.3) = 1.6$
5	$2.8 \times (2-1.3) = 2$	$2 \times (3.2-2.4) = 1.6$
6	$4.5 \times (1.9-1.4) = 2.2$	same transition
7	can't read from graph	same transition
8	can't read from graph	same transition

Mechanics – Graduate

A particle with mass m moves in one dimension in the potential

$$V(x) = \frac{K}{2}(e^{x^2} - 1)$$

1. Plot some phase space trajectories for various values of the energy.
2. Find the frequency for small amplitude oscillations.
3. Find the action-angle variables appropriate to the small amplitude oscillations.
4. Use canonical perturbation theory to find the first order correction to the frequency.

Grad mechanics

1.) $H = \frac{p^2}{2m} + \frac{\kappa}{2}\left(e^{x^2}-1\right) = E$ (constant)

For small oscillations $H \approx \frac{p^2}{2m} + \frac{\kappa}{2}x^2 = E$

This is the harmonic oscillator $x = A \sin\omega t$

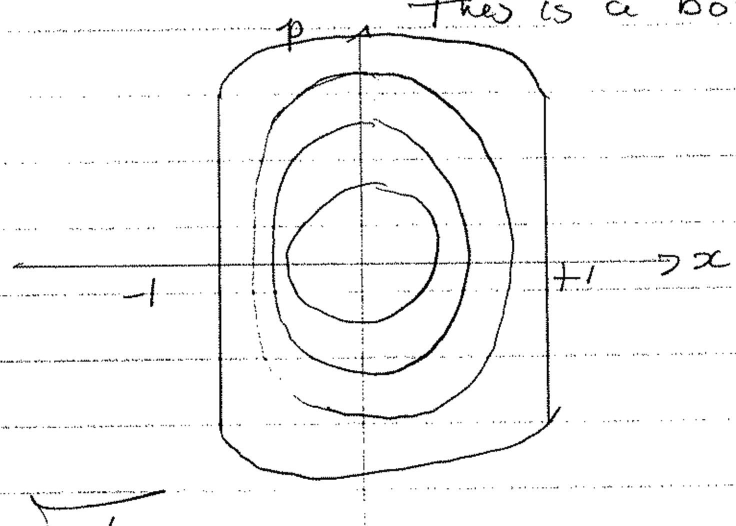
where $A^2 = 2E/\kappa$ & $\omega_0^2 = \kappa/m$; $A \ll 1$

For $A \gg 1$ there are two regimes:

$x < 1$ $\quad p \approx \infty \quad V \approx 0$

$x > 1$ $\quad p \approx 0 \quad V \approx \infty$

This is a box!



2) $\omega_0 = \sqrt{\kappa/m}$

3) $p^2 = 2m\left(E - \frac{\kappa}{2}x^2\right)$

$$I = \frac{1}{2\pi}\oint p\,dx = \frac{1}{2\pi}\oint \sqrt{2m\left(E-\frac{\kappa}{2}x^2\right)}\,dx$$

We define the angle variable so as to make the integral do-able. (There are more systematic ways of finding it, but this works.)

Let $x = \sqrt{\frac{2E}{K}} \sin\theta$. Then the loop integral $\oint$ means integrate over one cycle $0 \to \theta \to 2\pi$.

$$I = \frac{1}{2\pi}\sqrt{2mE}\sqrt{\frac{2E}{K}}\int_0^{2\pi}\cos^2\theta\, d\theta = \frac{E}{\omega_0}$$

This is the standard result for the harmonic osc.,

To put it the other way: I & θ are the action & angle and $p = \sqrt{2mI\omega_0}\cos\theta \qquad x = \sqrt{\frac{I}{m\omega_0}}\sin\theta$.

4) Expand the potential: $V = \frac{K}{2}\left(x^2 + \frac{x^4}{2} + \cdots\right)$

So take the perturbing term $H' = \frac{K}{4}x^4$

Substitute from above

$$H' = \frac{K}{4}\left(\frac{I}{m\omega_0}\right)^2 \sin^4\theta = \frac{I^2}{4m}\sin^4\theta$$

The first-order energy correction is the average of the

$$K_1(J) = \bar{H}_1(J) = \frac{1}{2\pi}\int_0^{2\pi} H' \, d\theta = \frac{1}{2\pi}\left(\frac{J^2}{4m}\right)\int_0^{2\pi} \sin^4\theta \, d\theta$$

$$K_1 = \frac{J^2}{16m} \qquad \omega^{(1)} = \frac{\partial K_1}{\partial J} = \frac{J}{8m}$$

Note: ① To first order we just replace $I \to J$

where J is the first-order corrected action variable.

② The integral is easy because

$$\sin^{2n} = \frac{1}{2^n} + \text{terms that integrate to zero.}$$

③ Once you see this trick it is unnecessary

to expand the potential. See next page.

One can make a better approximation as follows.

$$V = \frac{K}{2}x^2 + \left[\frac{K}{2}\left(e^{x^2}-1\right) - \frac{K}{2}x^2\right]$$

$$H' = \frac{K}{2}\left[e^{x^2} - (x^2+1)\right]$$

$$= \frac{K}{2}\left[\sum_{n=0}^{\infty}\frac{1}{n!}x^{2n} - (x^2+1)\right]$$

$$= \frac{K}{2}\left[\sum_{n}\frac{1}{n!}\left(\frac{I}{m\omega_0}\right)^n \sin^{2n}\theta - \left(\frac{I}{m\omega_0}\right)\sin^2\theta - 1\right]$$

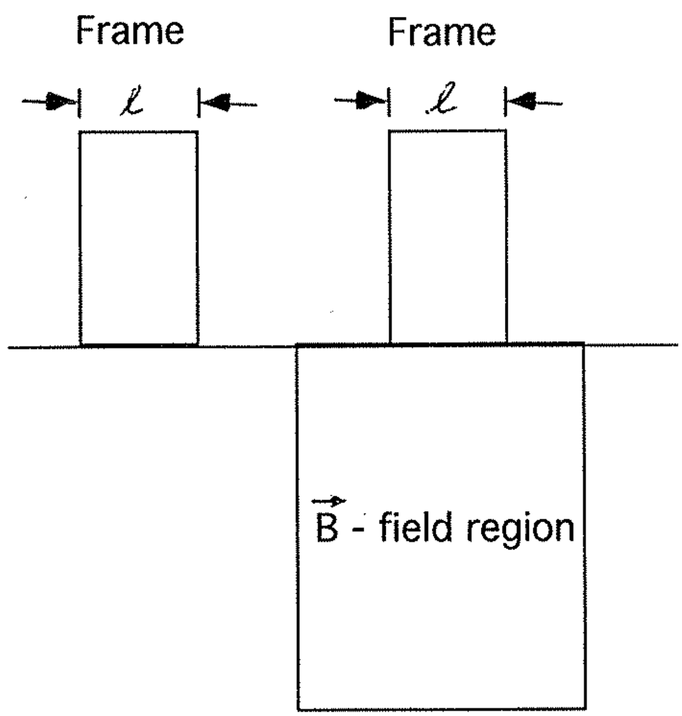
averaging $\sin^{2n}\theta = \frac{1}{2^n}$

$$\bar{V} = \frac{K}{2}\left[\sum_{n}\frac{1}{n!}\left(\frac{I}{m\omega_0}\right)^n\frac{1}{2^n} - \left(\frac{I}{2m\omega_0}\right) - 1\right]$$

$$= \frac{K}{2}\left[\exp\left(\frac{I}{2m\omega_0}\right) - \left(\frac{I}{2m\omega_0}\right) - 1\right]$$

4.

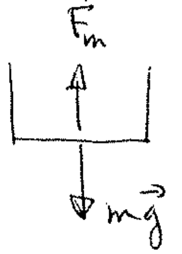
Consider two identical conducting wire frames as shown in the figure below. They are both released from rest at the initial moment. Show that for very short distances traveled, the frame falling in the field-free region falls an amount $\Delta y \approx \dfrac{B_0^6 \ell^6 t^3}{6m^3 R^3}$ further than its twin falling in the $\vec{B}$-field region. The direction of the $\vec{B}$-field is perpendicular to the plane of the figure. Here t is the time since release of the frames, B_0 is the constant magnetic field, ℓ is the width of each frame, m is the mass of each frame, and R is the electrical resistance of each frame.



Consider the frame falling in the B-field region.

$$\mathcal{E} = B_0 l \frac{dy}{dt} \quad ; \quad I = \frac{\mathcal{E}}{R} = \frac{B_0 l}{R}\frac{dy}{dt}$$

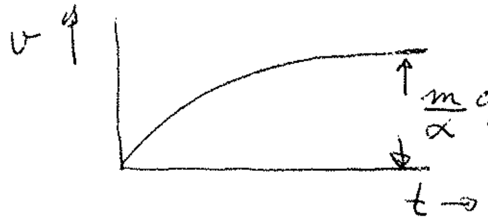
$$F_m = I l B = \frac{B_0^2 l^2}{R}\frac{dy}{dt} \equiv \alpha \frac{dy}{dt}$$



$$mg - F_m = m\frac{d^2y}{dt^2}$$

or $$\frac{dv}{dt} + \frac{\alpha}{m} v = g.$$

Given that $v(0) = 0$; $$v(t) = \frac{mg}{\alpha}\left(1 - e^{-\frac{\alpha}{m}t}\right)$$



A further integration yields $$y(t) = \int v(t)\,dt + C$$

which for $y = 0$ at $t = 0$, yields

$$y(t) = \frac{mg}{\alpha} t + \left(\frac{m}{\alpha}\right)^2 g \left[e^{-\alpha t/m} - 1\right].$$

For $\frac{\alpha t}{m} \ll 1$ (short times after release); $\left[\text{Recall } \alpha = \frac{B_0^2 l^2}{R}\right]$

$$y \approx \frac{1}{2} g t^2 - \frac{1}{6}\left(\frac{B_0^2 l^2}{mR}\right)^3 t^3, \quad \text{for small } t.$$

Thus, $$\Delta y = y(t)_{\substack{B\text{ free}\\ \text{region}}} - y(t)_{\substack{B\\ \text{region}}} = \frac{1}{6}\left(\frac{B_0^2 l^2}{mR}\right)^3 t^3.$$

5.

A quantum particle of mass m moves in a 3-dimensional harmonic oscillator potential with classical frequency $\omega = \sqrt{k/m}$. It is in thermal equilibrium with a heat bath of temperature T.

a. Find the partition function Z

b. Find the average energy U. Compute its limiting value for $k_B T >> \hbar\omega$. Does the zero-point energy contribute in this limit?

c. Find the heat capacity C_V.

6.

A plumb bob is hung from the tower of Pisa. What angle does it make with the vertical? In what direction does it deviate from the vertical? By "vertical" we mean a line drawn through the center of the earth. You will need to know the latitude of Pisa, which is about 45^0, and the radius of the earth $R = 6.4 \times 10^6 m$.

SOLUTION

a. States specified by integers n_x, n_y, n_z, Energy of each state

$$E(n_x, n_y, n_z) = (n_x + n_y + n_z + 3/2)\,\hbar\omega.$$

partition function $Z = \sum_{\text{states}} \exp(-E/k_B T) =$

$$\sum_{n_x=0}^{\infty} \sum_{n_y=0}^{\infty} \sum_{n_z=0}^{\infty} \exp -(n_x + n_y + n_z + 3/2)\hbar\omega/k_B T$$

$$= e^{-3\hbar\omega\beta/2} \left(\sum_{n=0}^{\infty} e^{-n\hbar\omega\beta} \right)^{\beta} \qquad \text{where } \beta \equiv 1/k_B T$$

$$Z = \left(\frac{e^{-\hbar\omega\beta/2}}{1 - e^{-\hbar\omega\beta}} \right)^{\beta}$$

b. $$U = -\frac{\partial}{\partial\beta}(\ln Z) = \frac{3}{2}\hbar\omega - 3\frac{\partial}{\partial\beta}\ln\left(1 - e^{-\hbar\omega\beta}\right) = \frac{3}{2}\hbar\omega + 3\frac{\hbar\omega\, e^{-\hbar\omega\beta}}{1 - e^{-\hbar\omega\beta}}$$

$$= 3\,\hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\hbar\omega\beta} - 1}\right)$$

$$\rightarrow \frac{3}{2}\hbar\omega + \frac{3\hbar\omega}{1 + \hbar\omega\beta + (\hbar\omega\beta)^2/2 - 1} = \frac{3}{2}\hbar\omega + \frac{3\,k_B T}{1 + \hbar\omega\beta/2}$$

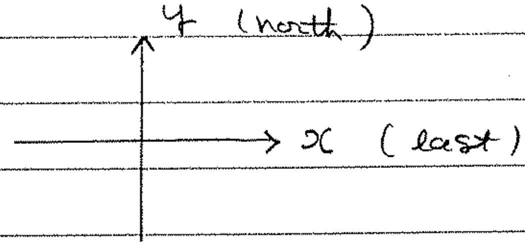
$$\approx \frac{3}{2}\hbar\omega + 3k_B T(1 - \hbar\omega\beta/2) = 3k_B T$$

c. $$C_V = \frac{\partial U}{\partial T} = 3k_B\,(\hbar\omega\beta)^2 \frac{e^{\hbar\omega\beta}}{(e^{\hbar\omega\beta} - 1)^2}$$

Mechanics Problem

The plumb bob is acted on by the force of gravity $\vec{F}_g = m\vec{g}$ and the centrifugal "force" $F_c = -m\vec{\omega}\times(\vec{\omega}\times\vec{r})$

$\vec{\omega}$ must be in the body centered system. Choose a c.s.



The z axis goes through the center of the earth

$$\vec{\omega} = |\omega|(0, \sin\theta, \cos\theta)$$

where θ is the usual polar angle (colatitude)

$$\vec{\omega}\times\vec{r} = \omega R \sin\theta \;\hat{x}$$

$$-\vec{\omega}\times(\vec{\omega}\times\vec{r}) = -\omega^2 R\left(\sin\theta\cos\theta\;\hat{y} - \cos^2\theta\;\hat{z}\right)$$

The second term has the effect of decreasing gravity, but

$$\frac{\omega^2 R \cos^2 45^\circ}{g} \approx 2\times10^{-3}$$

and we neglect it. The first term pulls in the $-\hat{y}$ (south) direction

$$\tan\theta = \frac{-\omega^2 R\sin\theta\cos\theta}{-g} \quad \text{or} \quad \theta \approx 1.73\times10^{-3} \text{ radians}$$

7.

Consider a poor electrical conductor to comprise an infinite half-space given by $z > 0$. Electromagnetic radiation in the form of a plane wave characterized by $\vec{E} = \vec{i}\, E_0 \exp\{-i(\omega t - kz)\}$ is incident on this half-space at $z=0$.

(a) Show there is a phase shift between the electric field ($\vec{E}$) and the magnetic field($\vec{B}$) in the medium that can be approximated by $\varphi \approx \cos^{-1}\left(1 - \dfrac{\sigma^2}{2\varepsilon^2\omega^2}\right)$, where, σ is the electrical conductivity of the medium, and where ε and μ are its permittivity and permeability, respectively.

(b) Show that the distance from the surface (ℓ) at which the power per unit area (in the x-y plane) associated with these fields has fallen to one-half its surface value, can be given by $\ell = \dfrac{\ln 2}{\sigma}\sqrt{\dfrac{\varepsilon}{\mu}}$.

8.

Consider a two-state quantum mechanical system that can be described in terms of an orthonormal basis $|+\rangle$ and $|-\rangle$. In terms of this basis, the Hamiltonian of the system has matrix elements: $H_{++} = 6\varepsilon_0$, $H_{+-} = H_{-+} = 2\varepsilon_0$, and $H_{--} = 3\varepsilon_0$.

(a) Find the energy eigenvalues and corresponding normalized eigenvectors expressed in terms of the basis $|+\rangle$ and $|-\rangle$.

(b) Show that your eigenvectors are orthogonal.

(c) Suppose that the system is initially in the state $|+\rangle$. Find an expression for the state of the system at some later time t in terms of the basis $|+\rangle$ and $|-\rangle$.

(d) Calculate the probabilities $p_+(t)$ and $p_-(t)$ that the system will still be in the states $|+\rangle$ and $|-\rangle$, respectively at time t. Verify that $p_+(t) + p_-(t) = 1$.

(a) MAXWELL #4: $\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$

where $\vec{D} = \varepsilon \vec{E}$, $\vec{J} = \sigma \vec{E}$, $\vec{B} = H/\mu$.

and for $\vec{E} = \vec{E}_0 e^{-i(\omega t - kz)}$ we have $\vec{E} = \frac{i}{\omega} \frac{\partial \vec{E}}{\partial t}$

Thus MAX 4 becomes $\vec{\nabla} \times \vec{B} = \mu \varepsilon \left[1 + \frac{i\sigma}{\varepsilon \omega} \right] \frac{\partial \vec{E}}{\partial t}$

[In free space, the pre-factor in front of $\frac{\partial \vec{E}}{\partial t}$ is $\mu_0 \varepsilon_0$.]

[where $\frac{1}{c^2} = \mu_0 \varepsilon_0$]. In our medium, we have

$\frac{1}{v^2} = \mu \varepsilon \left(1 + \frac{i\sigma}{\varepsilon \omega} \right)$. Now, $\frac{1}{v^2} = \frac{n^2}{c^2}$, and so

we have $n = c \sqrt{\mu \varepsilon \left(1 + \frac{i\sigma}{\varepsilon \omega} \right)}$

Now, $k = \frac{n\omega}{c} = \omega \sqrt{\mu \varepsilon \left(1 + \frac{i\sigma}{\varepsilon \omega} \right)}$

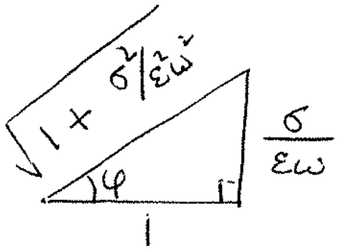
The medium, being a poor conductor, has $\frac{\sigma}{\varepsilon \omega} \ll 1$,

and so $k \approx \omega \sqrt{\mu \varepsilon} \left(1 + \frac{i\sigma}{2\varepsilon \omega} \right)$

Thus, $\vec{E} = E_0 \hat{x} \exp\left(-i\omega t + i\omega\sqrt{\mu\varepsilon}\, z - \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\, z \right)$

while $\vec{B} = \frac{n}{c} E_0 \hat{y} \exp\left(\cdots\cdots \right)$

Thus, $\vec{B} = \sqrt{\mu\varepsilon(1 + i\sigma/\varepsilon\omega)}\; E_0\, \hat{y} \exp\left(- - - - - \right)$



$$\therefore \varphi = \cos^{-1}\left(1 + \frac{\sigma^2}{\varepsilon^2\omega^2}\right)^{-1/2}$$

which for $\frac{\sigma}{\varepsilon\omega} << 1$, becomes $\varphi = \cos^{-1}\left[1 - \frac{\sigma^2}{2\varepsilon^2\omega^2}\right]$.

(b) $\vec{S} = \vec{E} \times \vec{H} \sim e^{-\sigma\sqrt{\frac{\mu}{\varepsilon}}\, z}$

$$\frac{S(\ell)}{S(0)} = \frac{e^{-\sigma\sqrt{\frac{\mu}{\varepsilon}}\,\ell}}{e^0} = \frac{1}{2}$$

$$\therefore \quad -\sigma\sqrt{\frac{\mu}{\varepsilon}}\,\ell = -\ln 2$$

or $$\ell = \frac{\ln 2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$$

QM1 (UG)

Basis: $|+\rangle$ and $|-\rangle$

$H_{++} = 6\varepsilon_0$ $\quad H_{+-} = H_{-+} = 2\varepsilon_0$ $\quad H_{--} = 3\varepsilon_0$

(a) Energy eigenvalues and eigenvectors

Hamiltonian matrix:

$$H = \varepsilon_0 \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}$$

Diagonalize to get eigenvalues:

$$\begin{vmatrix} 6\varepsilon_0 - \lambda & 2\varepsilon_0 \\ 2\varepsilon_0 & 3\varepsilon_0 - \lambda \end{vmatrix} = 0$$

$$(6\varepsilon_0 - \lambda)(3\varepsilon_0 - \lambda) - 4\varepsilon_0^2 = 14\varepsilon_0^2 - 9\varepsilon_0\lambda + \lambda^2 = 0$$

$$(7\varepsilon_0 - \lambda)(2\varepsilon_0 - \lambda) = 0 \Rightarrow \lambda = 2\varepsilon_0, \ 7\varepsilon_0$$

eigenvectors:

$\lambda = 2\varepsilon_0 \equiv \varepsilon_1$

$$\varepsilon_0 \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2\varepsilon_0 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$6a + 2b = 2a \Rightarrow a = b/2$$

normalize: $|\varepsilon_1\rangle = \frac{1}{\sqrt{5}}\left[|+\rangle - 2|-\rangle\right]$

$\lambda = 7\varepsilon_0 \equiv \varepsilon_2$

$$\varepsilon_0 \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 7\varepsilon \begin{pmatrix} a \\ b \end{pmatrix}$$

$6a + 2b = 7a \Rightarrow a = 2b$

normalize: $|\varepsilon_2\rangle = \frac{1}{\sqrt{5}}\left[2|+\rangle + |-\rangle\right]$

(b) Orthogonality

$$\langle \varepsilon_1 | \varepsilon_2 \rangle = \frac{1}{5}\left[\langle + | - 2\langle - |\right]\left[2|+\rangle + |-\rangle\right]$$
$$= \frac{1}{5}\left[2 - 2\right] = 0$$

$\therefore \langle \varepsilon_1 | \varepsilon_2 \rangle = 0 \Rightarrow$ orthogonality

(c) State of the system @ time t:

$|\psi(0)\rangle = |+\rangle$

We need to express $|+\rangle$ in terms of the energy eigenvectors $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$

$$|\varepsilon_1\rangle = \frac{1}{\sqrt{5}}\left[|+\rangle - 2|-\rangle\right]$$
$$|\varepsilon_2\rangle = \frac{1}{\sqrt{5}}\left[2|+\rangle + |-\rangle\right.$$

$$|\varepsilon_1\rangle + 2|\varepsilon_2\rangle = \frac{1}{\sqrt{5}}\left[|+\rangle - 2|-\rangle + 4|+\rangle + 2|-\rangle\right]$$
$$= \frac{1}{\sqrt{5}}\cdot 5|+\rangle = \sqrt{5}\,|+\rangle$$
$$|+\rangle = \frac{1}{\sqrt{5}}\left[|\varepsilon_1\rangle + 2|\varepsilon_2\rangle\right] = |\psi(0)\rangle$$

Then,
$$|\psi(t)\rangle = \frac{1}{\sqrt{5}}\left[e^{-i\varepsilon_1 t/\hbar}|\varepsilon_1\rangle + 2e^{-i\varepsilon_2 t/\hbar}|\varepsilon_2\rangle\right]$$

where $\varepsilon_1 = 2\varepsilon_0$ and $\varepsilon_2 = 7\varepsilon_0$

(d) Probabilities $P_+(t)$ and $P_-(t)$

The probability that the system will be in state $|i\rangle$ @ time t is $P_i(t) = |\langle i|\psi(t)\rangle|^2$

$$\langle +|\psi(t)\rangle = \frac{1}{\sqrt{5}}\left[\langle\varepsilon_1| + 2\langle\varepsilon_2|\right]\cdot\frac{1}{\sqrt{5}}\left[e^{-i\varepsilon_1 t/\hbar}|\varepsilon_1\rangle + 2e^{-i\varepsilon_2 t/\hbar}|\varepsilon_2\rangle\right]$$

$$= \frac{1}{5}\left[e^{-i\varepsilon_1 t/\hbar} + 4e^{-i\varepsilon_2 t/\hbar}\right]$$
$$= \frac{1}{5}e^{-i\varepsilon_1 t/\hbar}\left[1 + 4e^{-i(\varepsilon_2-\varepsilon_1)t/\hbar}\right]$$

$$\frac{\varepsilon_2 - \varepsilon_1}{\hbar} = \frac{5\varepsilon_0}{\hbar} \equiv 5\omega_0$$

$$\langle + | \psi(t) \rangle = \frac{1}{5} e^{-i\varepsilon_1 t/\hbar} \left[1 + 4 e^{-i 5\omega_0 t} \right]$$

$$P_+(t) = |\langle + | \psi(t) \rangle|^2$$

$$= \frac{1}{25} \left(1 + 4e^{-i5\omega_0 t}\right)\left(1 + 4e^{i5\omega_0 t}\right)$$

$$= \frac{1}{25} \left[1 + 16 + 4 \underbrace{\left(e^{i5\omega_0 t} + e^{-i5\omega_0 t} \right)}_{2\cos 5\omega_0 t} \right]$$

$$P_+(t) = \frac{1}{25} \left[17 + 8 \cos 5\omega_0 t \right]$$

$$\langle - | \psi(t) \rangle = \frac{1}{\sqrt{5}} \left[-2\langle \varepsilon_1 | + \langle \varepsilon_2 | \right] \frac{1}{\sqrt{5}} \left[e^{-i\varepsilon_1 t/\hbar} | \varepsilon_1 \rangle + 2 e^{-i\varepsilon_2 t/\hbar} | \varepsilon_2 \rangle \right]$$

$$= \frac{1}{5} \left[-2 e^{-i\varepsilon_1 t/\hbar} + 2 e^{i\varepsilon_2 t/\hbar} \right] = -\frac{2}{5} \left[1 - e^{-i(\varepsilon_2 - \varepsilon_1)t/\hbar} \right] e^{-i\varepsilon}$$

$$= -\frac{2}{5} e^{-i\varepsilon_1 t/\hbar} \left[1 - e^{-i5\omega_0 t} \right]$$

$$P_-(t) = |\langle - | \psi(t) \rangle|^2 = \frac{4}{5} \left[1 + 1 - e^{-i5\omega_0 t} - e^{i5\omega_0 t} \right]$$

$$P_-(t) = \frac{1}{25} \left[8 - 8 \cos 5\omega_0 t \right]$$

$$P_+(t) + P_-(t) = \frac{1}{25}\left[17 + 8\cos 5\omega_0 t\right]$$

$$+ \frac{1}{25}\left[8 - 8\cos 5\omega_0 t\right]$$

$$= \frac{1}{25}\left[17 + 8\right] = 1 \quad \checkmark$$