Department of Physics Comprehensive Examination # 93

Part I

30 September 2002

This Comprehensive Examination for Fall 2002 consists of eight problems each worth 20 points. The problems are grouped into four sessions:

Session 1	problems 1, 2	9-12 AM	Mon 30 Sept.
Session 2	problems 3, 4	1:30-4:30 PM	Mon 30 Sept.
Session 3	problems 5, 6	9-12 AM	Tues 1 Oct.
Session 4	problems 7, 8	1:30-4:30 PM	Tues 1 Oct.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it is possible to obtain partial credit, especially if you demonstrate conceptual understanding. Do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter, but not your name, is on the inside of the back cover of every bluebook. Be sure to remember your student letter for use in the remaining sessions of the examination. If something is omitted from the statement of the problem or you feel there is an ambiguity, please ask your question quietly and privately, so as not to disturb the others. Only your bluebooks and the examination should be on the table before you. Any other items should be stored on the floor. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for scratch work separated by at least one page from your solutions. Scratch work will not be graded.

1. Consider a particle of mass m and possessing charge e to be confined inside a rectangular box and subjected to a weak electric field, $\boldsymbol{E}$, acting parallel to the x-axis. The box is defined by having $V = 0$ for $0 \le x \le a$, $0 \le y \le b$, and $0 \le z \le c$ where $a \ne b \ne c$, and $V = \infty$ elsewhere. Determine the energies allowed to this charged particle.

$$\varphi_{pqr} = \sqrt{\frac{8}{abc}} \sin \frac{p\pi x}{a} \sin \frac{q\pi y}{b} \sin \frac{r\pi z}{c} \quad \left(\text{students need to show this}\right)$$

for infinitely high V outside box, and

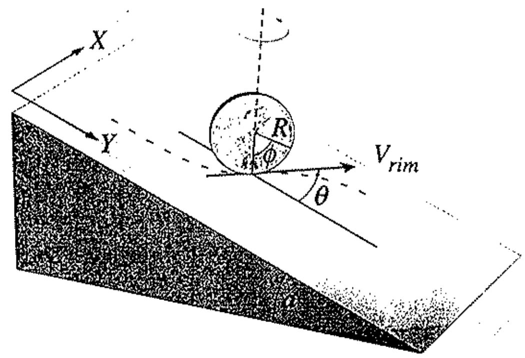
$$E_{pqr} = \frac{\hbar^2 \pi^2}{2m}\left(\frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2}\right) \quad \left(\text{students need to show use of } \mathcal{H}\varphi = E\varphi\right)$$

The perturbation appears as (to first order)

$$\Delta E_{pqr} = \langle \varphi_{pqr} | -e\mathcal{E}x | \varphi_{pqr} \rangle$$ where $\mathcal{E}$ is the electric field and $v_{\text{pert.}} = -e\mathcal{E}x$.

$$\Delta E_{pqr} = -\frac{e8\mathcal{E}}{abc} \underbrace{\int_0^a x \sin^2 \frac{p\pi x}{a}\, dx}_{\frac{a^2}{4}} \underbrace{\int_0^b \sin^2 \frac{q\pi y}{b}\, dy}_{\frac{b}{2}} \underbrace{\int_0^c \sin^2 \frac{r\pi z}{c}\, dz}_{\frac{c}{2}}$$

$$\therefore \quad E'_{pqr} = \frac{\hbar^2\pi^2}{2m}\left(\frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2}\right) - \frac{e\mathcal{E}a}{2}.$$



2. Suppose you stand a penny up on edge on an incline plane. Normally it would fall over, but let's just suppose that this time it doesn't. Perhaps the penny is a bit thicker than usual, and you were *very* careful in placing it; so rather than falling over, it rolls downhill. We assume that the penny cannot slide on the surface and that there is no friction of any kind.

   Think about a cylinder rolling downhill. The cylinder always rolls in the direction perpendicular to its axis of rotation. The penny does this too, but the penny can also spin about a *vertical* axis. The penny, therefore, does not follow a straight path, but at each instant it is rolling in a direction perpendicular to an axis perpendicular to its surface, *i.e.* at each instant the penny rolls in the direction it's pointing.

   Just so that we agree on notation, let's assume that the coin makes contact with the incline plane at a point (x, y). The y coordinate points down the incline plane. The penny turns around an axis perpendicular to its surface. This rotation is measured by the angle ϕ. The rotation about its vertical axis is measured by θ. The coordinate system is chosen so that $\theta = 0$ corresponds the coin pointing straight downhill. To put it another way, if you put the coin down with $\theta = 0$ and $\dot{\theta} = 0$, it would simply roll downhill with constant x.

   (a) What is the minimum number of variables required to describe the motion? How many degrees of freedom does the penny itself have? Find the equations of constraint.

   (b) Calculate the Lagrangian for the motion.

   (c) Find the equations of motion. Verify that they give the expected results when $\theta = \dot{\theta} = 0$

   (d) Suppose that at $t = 0$, $\theta = \phi = \dot{\phi} = 0$, and $\dot{\theta} = \omega$. Find the subsequent motion. You should be able to find $\phi(t)$ and $\theta(t)$ in closed form. You can leave $x(t)$ and $y(t)$ in the form of integrals over t.

Grad Mechanics

1. The equations of constraint are

$$\dot{x} = R\dot{\phi}\sin\theta$$
$$\dot{y} = R\dot{\phi}\cos\theta$$

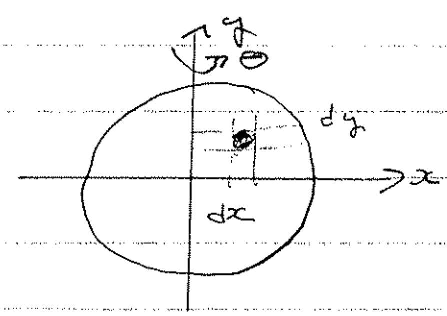
These involve velocities rather than coordinates, so they are an example of non-holonomic constraints. The penny has only 2 degrees of freedom θ & ϕ. A complete description requires x & y, but $\dot{x}$ & $\dot{y}$ are not independent as the above equations show. We can solve the equations of motion with only 3 variable

2. First find moments of inertia

$$dI_\phi = \frac{m}{\pi R^2}(2\pi r\,dr)\,r^2$$

$$I_\phi = \frac{mR^2}{2}$$

I_θ is a bit more complicated:



$$I_\theta = 4\int_0^R dy \int_0^{\sqrt{R^2-y^2}} \frac{m}{\pi R^2}\,x^2\,dx = \frac{mR}{4}$$

$T = T(\text{rotation}) + T(\text{translation})$

$$= \frac{1}{2}\left(\frac{mR^2}{2}\right)\dot{\phi}^2 + \frac{1}{2}\left(\frac{mR^2}{4}\right)\dot{\theta}^2 + \frac{mR^2}{2}\dot{\phi}^2$$

$$= \frac{3}{4}mR^2\dot{\phi}^2 + \frac{1}{8}mR^2\dot{\theta}^2$$

$$L = T - V = \frac{mR^2}{2}\left[\frac{3}{2}\dot{\phi}^2 + \frac{1}{4}\dot{\theta}^2\right] - mgy \sin\alpha$$

α is the incline plane angle.

3. The EOM require a slight "variation" on the usual Lagrange multiplier routine. If we make a small virtual displacement in ϕ then

$$\delta y = R\cos\theta \; \delta\phi$$

or $$\lambda\left(\delta y - R\cos\theta \; \delta\phi\right) = 0$$

$$\delta S = \int \left[\frac{\delta L}{\delta y}\delta y + \frac{\delta L}{\partial\theta}\delta\theta + \frac{\delta L}{\partial\phi}\delta\phi\right] dt$$

$$\Rightarrow \int \left[\left(\frac{\partial L}{\delta y} + \lambda\right)\delta y + \left(\frac{\delta L}{\delta\phi} - \lambda R\cos\theta\right)\delta\phi + \frac{\delta L}{\delta\theta}\delta\theta\right] dt$$

as usual $$\frac{\delta L}{\delta q} = \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

We require that $\delta S = 0$ for arbitrary $\delta y, \delta\phi, \delta\theta$ so the EOM are

$$-mg\sin\alpha + \lambda = 0$$

$$-\frac{3mR^2}{2}\ddot{\phi} - \lambda R\cos\theta = 0$$

$$\frac{mR^2}{4}\ddot{\theta} = 0$$

If $\theta = \dot{\theta} = 0$ then $\tau = -\frac{3m}{2} R \ddot{\phi} = mg \sin\alpha$

$$\ddot{\phi} = -\frac{2g \sin\alpha}{3mR}$$

The usual Ph 211 result.

4.

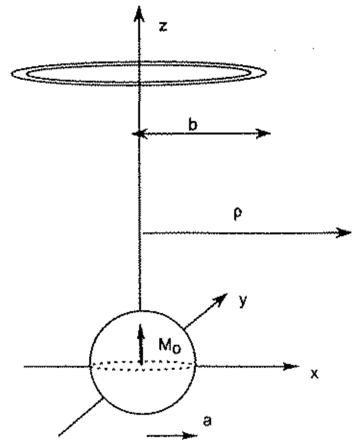
$\theta = \phi = 0 \quad \dot{\theta} = \omega \quad$ since $\ddot{\theta} = 0 \quad \theta = \omega t$

$$-\frac{3mR^2}{2} \ddot{\phi} = (mg \sin\alpha) R \cos(\omega t)$$

$$\ddot{\phi} = -\frac{2g \sin\alpha}{3mR} \cos(\omega t)$$

$$\dot{\phi} = -A \int \cos(\omega t)\, dt = -\frac{A}{\omega} \sin \omega t$$

$$\phi = \frac{A}{\omega^2} \left(\cos(\omega t) - 1 \right)$$



3. A sphere of radius a is uniformly magnetized, and has a constant magnetic dipole moment M_0 pointing in the positive z direction.

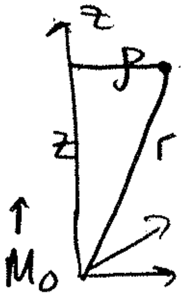
   (a) Show that, in the cylindrical coordinates indicated in the sketch, the vertical and radial components of the magnetic field are given by

$$\begin{aligned} B_\rho &= M_0 \frac{3\rho z}{(\rho^2 + z^2)^{5/2}} \\ B_z &= M_0 \frac{(2z^2 - \rho^2)}{(\rho^2 + z^2)^{5/2}}. \end{aligned}$$

   (b) A rigid ring of mass m, radius b, resistance R, a vertical axis, is located with its center a distance z_0 above the center of the sphere, and it is released from rest. Derive an expression for the current in the ring as a function of time, assuming that the ring falls for a short time with the acceleration due to gravity and that it has no self inductance.

   (c) Again assume that the ring has no self inductance, but account for the magnetic force on the induced current. Find the differential equation for the position, velocity and acceleration of the ring for short times. Do not attempt to integrate the equation.

② EM #2, Rubin



ⓐ Treat sphere as dipole moment $M_0 \hat{e}_z$

$$\vec{B} = 3\frac{\hat{n}\cdot\vec{m}}{r^3}\hat{n} - \frac{\vec{m}}{r^3}$$

using cylindrical symmetry, we can confine problem to a plane

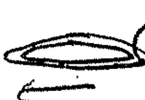
$$r = (\rho^2 + z^2)^{1/2} \qquad \hat{n} = \frac{z\hat{e}_z + \rho\hat{e}_\rho}{r}$$

$$\vec{m} = m_0 \hat{e}_z$$

$$\Rightarrow \quad \vec{B} = \frac{3 z m_0}{r^5}(z\hat{e}_z + \rho\hat{e}_\rho) - \frac{m_0 \hat{e}_z}{r^3}$$

$$B_z = \frac{3z^2 M_0}{r^5} - \frac{M_0}{r^3} = \frac{m_0(3z^2 - \rho^2 z^2)}{(\rho^2+z^2)^{5/2}} = \frac{M_0(2z^2 - \rho^2)}{(\rho^2+z^2)^{5/2}}$$

$$B_\rho = \frac{3M_0 z\rho}{(\rho^2+z^2)^{5/2}}$$



ⓑ (Gaussian Units) Falling $\Rightarrow$ Δ flux $\Rightarrow \mathcal{E} \Rightarrow I$

$$\mathcal{E} = -\frac{1}{c}\frac{\partial\phi}{\partial t} \quad > \quad \mathcal{E} = IR \Rightarrow I = \mathcal{E}/R$$

In order for induced $\mathcal{E}$ to oppose Δ, current flows <u>clockwise (look Dow</u>

$$\phi = \int \vec{B}\cdot d\vec{A} = 2\pi\int B_z\, \rho\, d\rho$$

$$= 2\pi M_0 \Big[2z^2 \int_0^b \frac{\rho\, d\rho}{(\rho^2+z^2)^{5/2}} - \int_0^b \frac{\rho^3\, d\rho}{(\rho^2+z^2)^{5/2}}$$

$$= \Big|_0^b \frac{-1}{3(\rho^2+z^2)^{3/2}} = \frac{-1}{3}\Big[\frac{1}{(b^2+z^2)^{3/2}} - \frac{1}{z^3}\Big]$$

$$= \Big|_0^b \frac{-1}{(\rho^2+z^2)^{1/2}} + \frac{z^2}{3(\rho^2+z^2)^{3/2}} = -\frac{1}{(b^2+z^2)^{1/2}} + \frac{2}{3z} + \frac{b^2}{3(b^2+z^2)^{3/2}}$$

$$\phi = 2\pi M_0 \left[\frac{-2z^2}{3r^3} + \frac{2}{3z} + \frac{3r^2}{3r^3} - \frac{3}{3z} - \frac{b^2}{3r^3} \right\}$$

$$\phi = 2\pi M_0 \left\{ \frac{2b^2+z^2}{3(b^2+z^2)^{3/2}} - \frac{1}{3z} \right\}$$

ring falls with $g \Rightarrow$

$$\ddot{z} = -g$$
$$\dot{z} = -gt + 0$$
$$z = -\frac{gt^2}{2} + z_0$$

$$I = -\frac{1}{Rc}\frac{\partial \phi}{\partial t} = -\frac{2\pi M_0}{Rc}\left[\frac{2z\dot{z}}{3(b^2+z^2)^{3/2}} - \frac{3}{2}\frac{(2b^2+z^2)2z\dot{z}}{(b^2+z^2)^{5/2}} + \frac{\dot{z}}{3z^2} \right\}$$

$$= -\frac{2\pi M_0}{Rc}\left[\frac{2g^2 t}{3(b^2+\frac{g^2}{4}t^4+z_0^2-gt^2 z_0)^{3/2}} \right]$$

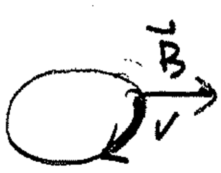
(c) No self inductance, $\ddot{z} = ?$
Magnetic force on current

$$\nabla \times E + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

$$I = -\frac{2\pi M_0}{Rc}\left[\underbrace{\frac{2z\dot{z}}{3(b^2+z^2)^{3/2}}}_{\text{neg.}} - 3\,\underbrace{\frac{(2b^2+z^2)z\dot{z}}{(b^2+z^2)^{5/2}}}_{\text{neg.}} + \underbrace{\frac{\dot{z}}{3z^2}}_{\text{neg.}} \right]$$

I is positive, clockwise flow.

$$\vec{F} = m\vec{a} \Rightarrow -m\ddot{z} = -mg + F_B$$



$F_B = \frac{e}{c}\vec{J}\times\vec{B}$: radial field causes upwards force for clockwise current

$$F_B = \frac{I}{c}\, d\vec{l}\times\vec{B}$$
$$dl = -b\,d\theta\,\hat{e}_\theta \qquad \left\{ d\vec{l}\times\vec{B} = -b\,d\theta\,B_\rho(\hat{e}_\theta\times\hat{e}_\rho) = b\,d\theta\,B_\rho \right.$$

$$F_B = \frac{I}{c}\int d\vec{l}\times\vec{B} = \frac{2\pi b I}{c}\,\frac{M_0\, 3bz}{(b^2+z^2)^{5/2}}$$

$$\Rightarrow \ddot{z} = g - \frac{12\pi^2 b^2}{Rc^2}M_0^2\left(\frac{z}{(b^2+z^2)^{5/2}}\right)\left[\frac{2z\dot{z}}{3(b^2+z^2)^{3/2}} - \frac{3(2b^2+z^2)z}{(b^2+z^2)^{5/2}} + \dot{z}/3\ldots\right.$$

4. An electron gas essentially confined to two-dimensions can be realized in certain semiconductor structures. This problem concerns some of the properties of a gas of N non-interacting electrons of mass m confined to a 2-D square "box" of side L. The density of electrons is $n = N/L^2$.

   The wave functions of the particles in this "box" are of the form

$$\psi(x, y) = A\psi(k_x x)\psi(k_y y)$$

   and the energies are quantized according to $E = \hbar^2 k^2/2m$ where $k_x,\ k_y = 0,\ \pm 2\pi/L,\ \pm 4\pi/L,\ \cdots$ The density of electronic states per unit energy per unit area has the form

$$D(E) = C \text{ (a constant)} \quad \text{for } E > 0,$$
$$D(E) = 0 \text{ for } E < 0.$$

   (a) Derive an expression for the chemical potential as a function of n, C, the temperature T, and fundamental constants.

   (b) Determine the chemical potential at T = 0 (the "Fermi energy" E_F).

   (c) Determine the constant C in terms of fundamental constants and given quantities.

   (d) Suppose a 2-D electron gas experiment is carried out at T = 1 K with an electron density of $n \approx 10^{12}$ electrons/cm^2. Due to the effects of the crystal medium, the electrons behave as if they have a mass $m \approx 0.1 m_0$ where m_0 is the mass of a free electron. Would you expect the electron gas to be degenerate under these conditions? Do a "back of the envelope" order of magnitude calculation to justify your answer.

Integral:

$$\int \frac{dx}{1 + e^x} = \ln\left(\frac{e^x}{1 + e^x}\right)$$

SM / Thermo Grad Problem

Electrons are fermions and therefore obey Fermi-Dirac statistics with distribution function

$$f(E) = \frac{1}{e^{(E-\mu)/k_BT} + 1}$$

where k_B is the Boltzmann constant.

(1) Since $D(E)$ is the density of states per unit area we can write

$$n = \frac{N}{L^2} = \int_{-\infty}^{\infty} D(E)\, f(E)\, dE = C \int_0^{\infty} f(E)\, dE$$

Integral is $\displaystyle\int_0^{\infty} \frac{dE}{e^{(E-\mu)/k_BT} + 1} = k_BT \int_{-\mu/k_BT}^{\infty} \frac{dx}{e^x + 1}$

$$= k_BT \log_e \left(\frac{e^x}{1+e^x} \right) \Bigg|_{-\mu/k_BT}^{\infty} = -k_BT \log_e \left(\frac{e^{-\mu/k_BT}}{1 + e^{-\mu/k_BT}} \right)$$

$$\therefore\ n = k_BT\, C \log_e \left(\frac{1 + e^{-\mu/k_BT}}{e^{-\mu/k_BT}} \right)$$

Solve for $\displaystyle e^{-\mu/k_BT} = \frac{1}{\exp\left(\frac{n}{k_BTC}\right) - 1}$

or $\displaystyle \mu = k_BT \log_e \left[\exp\left(\frac{n}{k_BTC} \right) - 1 \right]$

(2) Take limit as $T \to 0$

$$\mu \to k_BT \log_e\left[\exp\left(\frac{n}{k_BTC}\right)\right] = k_BT \cdot \frac{n}{k_BTC} = \frac{n}{C}$$

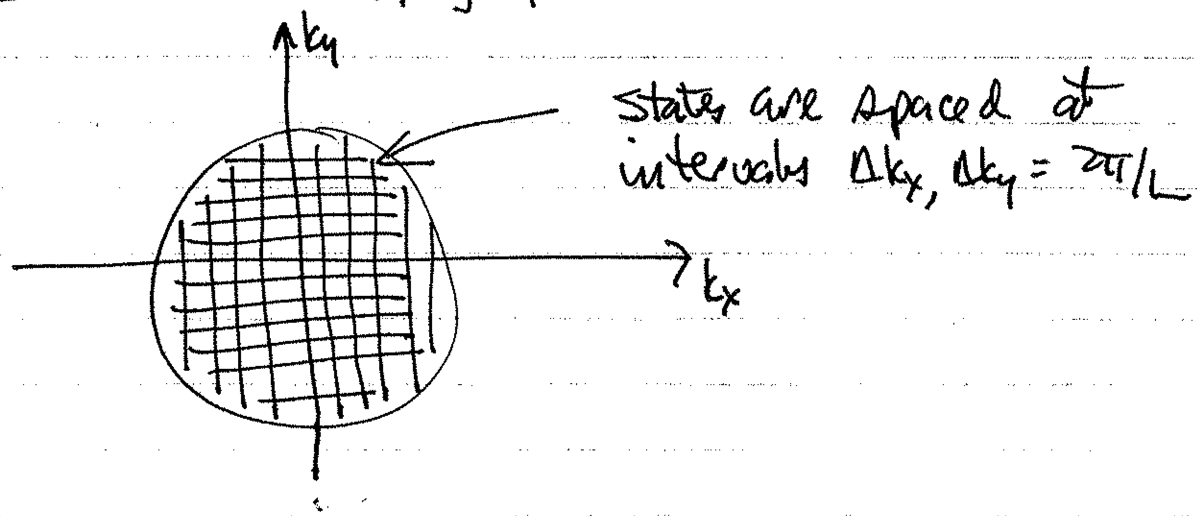
$$\mu(0) = E_F = n/C$$

OR use $n = \int_{-\infty}^{\infty} D(E) f(E)\,dE = \int_0^{E_F} C\,dE = CE_F$ @ $T=0$

So $\underline{E_F = n/C}$

(3) If $N(E)$ is the number of electronic states with energy $\leq E$, then $D(E)$ is $dN(E)/dE$. We have to find $N(E)$.

Since $E = \frac{\hbar^2 k^2}{2m}$, states of a given E lie on a <u>circle</u> in the k_x, k_y plane:



The total number of states in the circle is

$$N = \frac{\text{area of circle}}{\text{area per state}} \times 2 \longleftarrow \text{if we include elec-spin}$$

$$= \frac{2\pi k^2}{(2\pi/L)^2} = \frac{1}{2}\frac{Ak^2}{\pi} \qquad A = L^2$$

Since $k^2 = \frac{2mE}{\hbar^2}$, $N(E) = \frac{1}{2}\frac{A}{\pi}\cdot\frac{2mE}{\hbar^2} = \frac{AmE}{\pi\hbar^2}$

$$\frac{dN}{dE} = \frac{Am}{\pi\hbar^2} \Rightarrow D(E) = \frac{m}{\pi\hbar^2} \quad \text{(per unit area)}$$

i.e $C = \frac{m}{\pi\hbar^2}$ $\quad$ ($\frac{m}{2\pi\hbar^2}$ if we don't include spin)

(4) The condition for degeneracy of a Fermi gas is

$$T \ll E_F/k_B \qquad E_F = \frac{n}{C} = \frac{n\pi\hbar^2}{m} \qquad (m = 0.1\, m_0)$$

$$\frac{E_F}{k_B} = \frac{n\pi\hbar^2}{m k_B} \sim \frac{10^{12}\cdot\pi\cdot 10^{-54}}{10^{-1}\cdot 10^{-27}\cdot 10^{-16}} = 10^2\pi \cong \underline{300\,K}$$

@ $T = 1$ K, the 2-D e-gas will be highly degenerate.

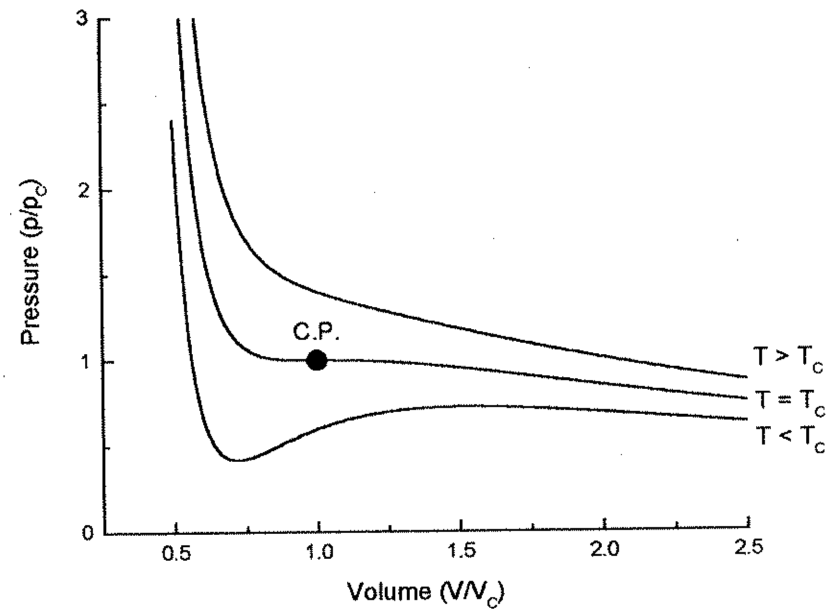
5. A free electron is at rest at the origin of the lab coordinate system. A relativistic deuteron flies past along the line $z = 0$, $y = b$. You may assume that the deuteron's velocity v does not change, and that the displacement of the electron during the collision is negligible.

   (a) Calculate the force on the electron as a function of time.

   (b) Calculate the direction and magnitude of the impulse given to the electron.

   (c) Repeat this same calculation non-relativistically and compare the results.

6. The plot below illustrates the van der Waals equation of state for a fluid of interacting particles. The axes are labelled in units of the critical pressure (p_C) and volume (V_C).



In the van der Waals approximation, the attractive and repulsive interactions are represented by constant parameters a and b, respectively. The Helmholtz free energy of a van der Waals fluid of N spinless particles of mass M can be written

$$F = -k_B T N \left\{ \ln \left[n_Q \left(\frac{V - Nb}{N} \right) \right] + 1 \right\} - \frac{N^2 a}{V} \tag{1}$$

where k_B is Boltzmann's constant and $n_Q \equiv (M k_B T / 2\pi\hbar^2)^{3/2}$.

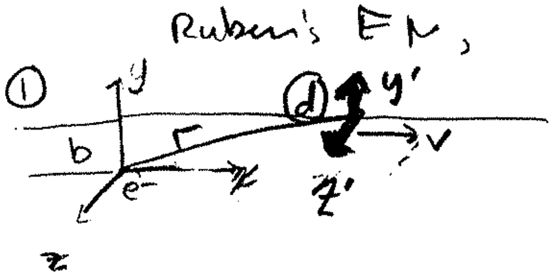
   (a) In a few sentences, describe the physical consequences of the local maximum and minimum exhibited by the p versus V curve shown in the plot for $T < T_C$.

   (b) Starting with the free energy expression of Eq. (1), derive the van der Waals equation of state.

   (c) As shown in the plot, the first two derivatives of the critical isotherm ($T = T_C$) vanish at the critical point (C.P.). Use this information to obtain expressions for the critical parameters p_C, V_C and $k_B T_C$ in terms of the van der Waals constants a and b.

   (d) Derive an expression for the entropy of the van der Waals fluid in terms of the given quantities.

Rubin's EM, rough Solution



$\vec{F} = -e(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$ on electron

Since electron remains at rest, the $\vec{B}$ field does not affect it

In O' the deuteron's at rest and so it produces only $\vec{E}$ field

let $t = 0$ correspond to $r = b$

$$F = -\frac{e^2}{|\vec{r}|^3}\,\vec{r}\,'$$

$$r' = \sqrt{b^2 + v^2 t'^2}$$

$$\begin{cases} F_x' = -\dfrac{e^2}{(b^2+v^2t'^2)^{3/2}}(\cdot vt') = \dfrac{e^2 vt'}{(b^2+v^2t'^2)^{3/2}} \\ F_y' = \dfrac{-e^2}{(b^2+v^2t'^2)^{3/2}}(-b) = \dfrac{e^2 b^2}{(b^2+v^2t'^2)^{3/2}} \\ F_z' = 0 \end{cases} \quad , \vec{B} = 0$$

now T.F. to unprimed frame (here field not force)

$$E_y = \gamma[E_y' + \beta B_x'] , \quad B_y = \gamma[B_y' - \beta E_x']$$

$$E_z = E_z' , \quad B_z = B_z'$$

$$E_x = \gamma[E_x' - \beta B_y'] , \quad B_x = \gamma[B_x' + \beta E_y']$$

$$\Rightarrow E_y = \frac{-eb\gamma}{(b^2+v^2\gamma^2t^2)^{3/2}} \qquad B_x = \gamma\frac{v}{c}\left(\frac{-eb}{\sqrt{b^2+v^2\gamma^2t^2}}\right)$$

$$E_z = \frac{-ev\gamma t}{(b^2+v^2\gamma^2t^2)^{3/2}}$$

$$\Rightarrow \vec{F} = \frac{e^2 b\gamma}{(b^2+v^2\gamma^2t^2)^{3/2}}\,\hat{e}_y + \frac{e^2 v^2\gamma t}{(b^2+v^2\gamma^2t^2)^{3/2}}\,\hat{e}_z$$

b) clearly, $I_x = \int F_x \, dt = 0$

$$\vec{I} = \int \vec{F} dt = \int_{-\infty}^{\infty} \frac{e^2 b \gamma}{(b^2+v^2\gamma^2t^2)^{3/2}} dt \, \hat{e}_y + \underbrace{\int_{-\infty}^{\infty} \frac{e^2 v \gamma t}{(b^2+v^2\gamma^2t^2)^{3/2}} dt \, \hat{e}_z}_{\text{odd} \therefore \text{ vanish}}$$

$$= \frac{e^2 b \gamma}{v^3\gamma^3} \int_{-\infty}^{\infty} \left[\left(\frac{b}{\gamma v}\right)^2 + t^2\right]^{3/2} dt \, \hat{e}_y$$

$$= \frac{e^2 b}{v^3\gamma^2} \left(\left(\frac{v\gamma}{b}\right)^2 \frac{t}{\left(\frac{b^2}{v^2\gamma^2}+t^2\right)^{1/2}} \right) \Bigg|_{-\infty}^{\infty} \hat{e}_y$$

$$\underline{\vec{I} = \frac{2e^2}{bv} \hat{e}_y}$$

c) non relativ $c \to \infty$, $\gamma \to 1$, $t' \to t$

$$\Rightarrow E_y = E_y' = \frac{-eb}{(b^2+v^2t^2)}, \quad E_z = E_z' = \frac{-evt}{(b^2+v^2t^2)^{3/2}}$$

$$B = B' = 0, \ E_x = E_x' = 0$$

$$\Rightarrow \vec{F} = \frac{e^2 b^2}{(b^2+v^2t^2)^{3/2}} \hat{e}_y + \frac{e^2 v^2 t}{(b^2+v^2t^2)^{3/2}} \hat{e}_z \quad \checkmark$$

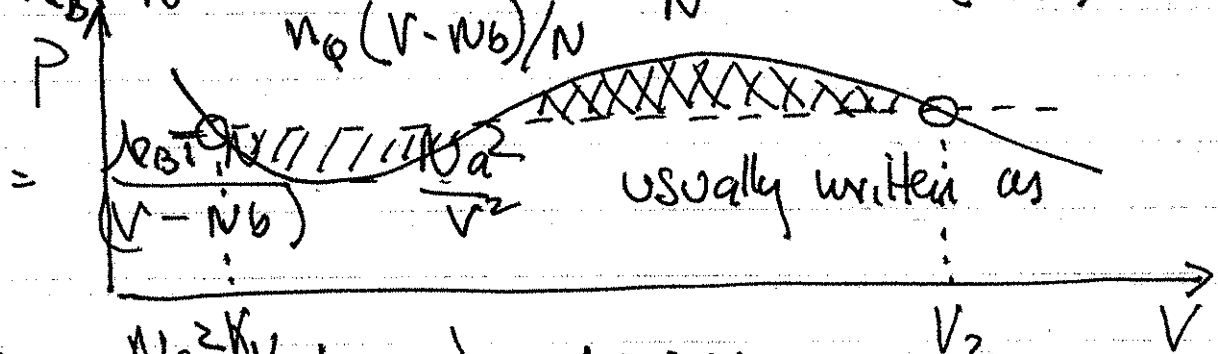
(just let $\gamma \to$

The impulse does <u>not</u> change! γ & $\frac{1}{\gamma}$ cancel

$5(a)$ Thermo: $F = -k_B T N \log[n_Q (V - Nb)/N] + \ldots - N^2 a/V$

(1) To get the equation of state: is there a relation between the separation of the fluid into two phases of different volume (liquid and gas). To see this consider the change of the Gibbs free energy between points of volume V_1 and V_2:

$$= k_B T N \cdot \frac{1}{n_Q (V - Nb)/N} \cdot \frac{n_Q}{N} + N^2 a \left(-1/V^2\right)$$



$$= \frac{k_B T N}{(V - Nb)} - \frac{N^2 a}{V^2}$$

usually written as

$$\left(P + \frac{N^2 a}{V^2}\right)(V - Nb) = k_B T N$$

(3) At constant temperature and number of particles,

$$dG = -S\,dT + V\,dp + \mu\,dN = V\,dp$$

$$\int_1^2 dG = \int_1^2 V\,dp$$

$$\frac{dp}{dV} = -\frac{k_B T N}{(V_c - Nb)^2} + \frac{2N^2 a}{V_c^3} = 0 \quad \text{(i)}$$

//// and XXXX areas are equal. Thus, two phases will be in equilibrium at the same T and p, but with volumes V_1 and V_2.

$$\frac{d^2 p}{dV^2} = \frac{2 k_B T N}{(V_c - Nb)^3} - \frac{6 N^2 a}{V_c^4} = 0 \quad \text{(ii)}$$

From Eq (i), $k_BTN = \frac{2N^2a}{V^3}(V-Nb)^2$

Then (ii) gives,

$$\frac{2}{(V-Nb)^3}\cdot\frac{2N^2a}{V^3}(V-Nb)^2 = \frac{6N^2a}{V^4}$$

$$\frac{4}{V-Nb} = \frac{6}{V} \Rightarrow \boxed{V = V_c = 3Nb}$$

From (i) again,

$$k_BT_cN = \frac{2N^2a}{27N^3b^3}\cdot(2Nb)^2 = \frac{8}{27}\frac{a}{b}$$

i.e $\boxed{T_c = \frac{1}{k_B}\cdot\frac{8}{27}\frac{a}{b}}$

$$P_c = \frac{k_BT_cN}{(V_c-Nb)} - \frac{N^2a}{V_c^2} = \frac{\frac{8}{27}\frac{a}{b}\cdot N}{2Nb} - \frac{N^2a}{9N^2b^2}$$

$$= \frac{4}{27}\frac{a}{b^2} - \frac{1}{9}\frac{a}{b^2} \Rightarrow \boxed{P_c = \frac{1}{27}\frac{a}{b^2}}$$

(4) To get the entropy, we need $S' = -\left.\frac{\partial F}{\partial T}\right)_{V,N}$

$$F = -k_B T N \{\log_e [n_Q (V-Nb)/N] + 1\} - N^2 a/V$$

$$\frac{\partial F}{\partial T} = -k_B N \log_e [n_Q (V-Nb)/N]$$

$$- k_B T N \cdot \frac{1}{n_Q (V-Nb)/N} \cdot \frac{V-Nb}{N} \frac{dn_Q}{dT}$$

$$- k_B N$$

$$\frac{dn_Q}{dT} = \frac{3}{2}\left(\frac{M k_B T}{2\pi\hbar^2}\right)^{1/2} \cdot \frac{M k_B}{2\pi\hbar^2}$$

Then, $\frac{\partial F}{\partial T} = -k_B N \log_e [n_Q (V-Nb)/N] - \frac{3}{2} k_B N - k_B N$

and $S = -\left.\frac{\partial F}{\partial T}\right)_{V,N} = k_B N \{\log_e [n_Q (V-Nb)/N] + 5/2\}$

7. A free electron moving in a uniform magnetic field $\boldsymbol{B}$ is described by the following Hamiltonian

$$H = \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A})^2$$

where $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. Consider $\boldsymbol{B}$ to be in the positive z-direction, and that $\boldsymbol{A}$ has components $(-yB,\ 0,\ 0)$.

Answer the following questions using non-relativistic quantum mechanics.

(a) Determine the energy eigenvalues for the electron.

(b) Describe the form of the corresponding energy eigenfunctions.

8. Two identical pendulums of length l and mass m are suspended side by side. They are coupled together with a spring that has the spring constant k. The spring is connected halfway up the pendulums so the force between the two is

$$F = k\frac{l}{2}(\theta_1 - \theta_2)$$

Assume that the spring is massless and that the amplitude of oscillations is sufficiently small that the equations of motion are linear.

(a) Calculate the kinetic and potential energy in terms of the angular displacements of the two pendulums, θ_1 and θ_2.

(b) Find the equations of motion. Use the dimensionless coupling parameter

$$\eta = \frac{kl}{4mg}$$

and the natural frequency $\omega_0 = \sqrt{g/l}$ to eliminate all reference to g, l, m, and k from the equations.

(c) Find the normal modes of oscillation. Calculate the frequencies and describe the corresponding motion.

(d) Use the symbols Θ_1 and Θ_2 for the amplitudes of the motion of the two pendulums. Calculate the ratio Θ_1/Θ_2 for the normal modes.

G Quantum

(a) $$\mathcal{H} = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 = \frac{1}{2m}\left(p_x + eBy\right)^2 + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}.$$

$\mathcal{H}\varphi = E\varphi$. Since $\mathcal{H}$ doesn't depend on x or z,

$$\varphi_{k_x k_z} = e^{i(k_x x + k_z z)}$$ and, thus

$$\varphi = f(y)\,\varphi_{k_x k_z} = e^{i(k_x x + k_z z)} f(y).$$

$\therefore$ $\mathcal{H}\varphi = E\varphi$ becomes

$$\frac{1}{2m}\left[\left(p_x + eBy\right)^2 + p_y^2 + p_z^2\right] e^{i(k_x x + k_z z)} f(y) = E\, e^{i(k_x x + k_z z)} f(y)$$

The left hand side can be written as

$$\left\{\left[\frac{1}{2m}\underbrace{\left(\hbar^2 k_x^2 + 2eB\hbar k_x y + e^2B^2y^2\right)}_{e^2B^2\left(y + \frac{\hbar k_x}{eB}\right)^2} + \frac{p_y^2}{2m}\right] + \frac{\hbar^2 k_z^2}{2m}\right\} e^{ik_x x + ik_z z} f(y)$$

Let $y_0 = -\dfrac{\hbar k_x}{eB}$, and so we end up with

$$\left[\frac{p_y^2}{2m} + \frac{e^2B^2}{2m}(y - y_0)^2\right] f(y) = \left[E - \frac{\hbar^2 k_z^2}{2m}\right] f(y)$$

which should be recognized as the Schroedinger Equation for a SHO of displaced origin.

$$\therefore \quad E_n = \underbrace{\hbar\omega\left(n + \tfrac{1}{2}\right)}_{\text{Landau levels}} + \frac{\hbar^2 k_z^2}{2m},$$

where $\omega = \dfrac{eB}{m}$.

(b) The eigenfunctions will have the form of the usual Hermite polynomials with Gaussian factor and also cisoidal dependences on k_x and k_z; i.e.,

$$\varphi_n = A_n \underbrace{H_n\left[\sqrt{\frac{m\omega}{\hbar}}\,(y-y_0)\right]}_{\substack{n\text{-th order} \\ \text{Hermite polynomial}}} \exp\left[\underbrace{-\frac{1}{2}\sqrt{\frac{m\omega}{\hbar}}\,(y-y_0)^2}_{\substack{\text{Gaussian} \\ \text{centered on} \\ y = y_0}} + \underbrace{i\left(k_x x + k_z z\right)}_{\substack{\text{cisoidal} \\ \text{dependences.}}}\right]$$

UG Mechanics

1. $T = \frac{1}{2} m l^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2)$

$$V = \frac{mgl}{2}(\theta_1^2 + \theta_2^2) + \frac{K}{2}\left(\frac{l}{2}\right)^2 (\theta_2 - \theta_1)^2$$

2.

$L = T - V$, as usual $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$

So

$$\ddot{\theta}_1 + \frac{g}{l}\theta_1 + \frac{K}{4m}(\theta_1 - \theta_2) = 0$$

$$\ddot{\theta}_2 + \frac{g}{l}\theta_2 + \frac{K}{4m}(\theta_2 - \theta_1) = 0$$

with $\eta = Kl/4mg$

$$\ddot{\theta}_1 + \omega_0^2(1+\eta)\theta_1 - \omega_0^2 \eta \theta_2 = 0$$

$$\ddot{\theta}_2 + \omega_0^2(1+\eta)\theta_2 - \omega_0^2 \eta \theta_1 = 0$$

3. Write this as a matrix eqn.

$$\frac{d^2}{dt^2}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} + \omega_0^2 \begin{pmatrix} 1+\eta & -\eta \\ -\eta & 1+\eta \end{pmatrix}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} = 0$$

Look for a soln $\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} = \begin{pmatrix}\Theta_1 \\ \Theta_2\end{pmatrix} e^{i\omega t}$

& let $\lambda = \omega^2/\omega_0^2$

$$\begin{pmatrix} -\lambda+1+\eta & -\eta \\ -\eta & -\lambda+1+\eta \end{pmatrix} \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} = 0$$

There will be a soln. if $\det(\ ) = 0$

$$(-\lambda+1+\eta)^2 - \eta^2 = 0 \qquad \lambda_1 = 1$$

$$\lambda_2 = 1+2\eta$$

or $\omega_1 = \omega_0 \qquad \omega_2 = \omega_0 \sqrt{1+2\eta}$

with $\lambda_1 = 0 \qquad \dfrac{\Theta_1}{\Theta_2} = 1$

$\lambda = 1+2\eta \qquad \dfrac{\Theta_I}{\Theta_2} = -1$

In mode 1 the pendulums swing in parallel like free pendulums. In mode 2 they are 180° out of phase.