Department of Physics Comprehensive Examination # 91

Part I

27 August 2001

This Comprehensive Examination for Fall 2001 consists of eight Problems each worth 20 points. The Problems are grouped into four sessions:

Session 1	Problems 1, 2	8-11 AM	Monday 24 September
Session 2	Problems 3, 4	12-3 PM	Monday 24 September
Session 3	Problems 5, 6	8-11 AM	Tuesday 25 September
Session 4	Problems 7, 8	12-3 PM	Tuesday 25 September

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it is possible to obtain partial credit, especially if you demonstrate conceptual understanding. Do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter, but not your name, is on the inside of the back cover of every bluebook. Be sure to remember your student letter for use in the remaining sessions of the examination. If something is omitted from the statement of the problem or you feel there is an ambiguity, please ask your question quietly and privately, so as not to disturb the others. Only your bluebooks and the examination should be on the table before you. Any other items should be stored on the floor. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for scratch work separated by at least one page from your solutions. Scratch work will not be graded.

1. An unstable elementary particle decays at rest in the laboratory into a π^+ and a π^o meson. These two π mesons have the same mass, $m_\pi = 140\ MeV/c^2$. In the laboratory reference frame, the π^o is observed to have a speed $v = 0.825c$. Determine numerical answers to the following:

   (a) Calculate the rest mass of the unstable elementary particle.

   (b) The π^o meson, while moving in a straight line with speed $v = 0.825c$, decays into two photons (each with zero rest mass):

   $$\pi^o \rightarrow \gamma + \gamma$$

   As viewed *in the rest frame of the π^o meson*, the photons are emitted at right angles to the original line of flight of the π^o meson.

   Find the angle between the direction of motion of a photon and the line of flight of the π^o meson *in the laboratory frame of reference.*

   (c) While the π^o meson was decaying into two photons, the π^+ meson decayed into a μ meson (of mass 105 MeV/c^2) and a neutrino (of zero mass):

   $$\pi^+ \rightarrow \mu^+ + \nu$$

   The energy of the μ meson measured in the laboratory could have a range of values, depending on the angle at which it was emitted.

   Find the *minimum* energy which the μ meson could conceivably have. For this case, indicate its direction of motion relative to that of the π^+ meson in the laboratory.

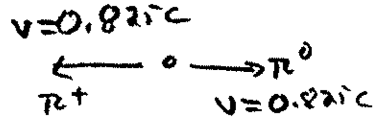
Problem # 1

1) 

$m_\pi = 140\,\text{MeV}/c^2$

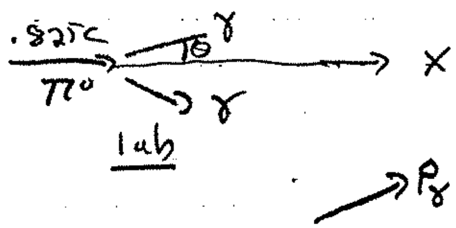
$v = 0.825c$

a) Since at rest initially, 4 momentum conservation $\Rightarrow$ Total Final momentum $= 0$

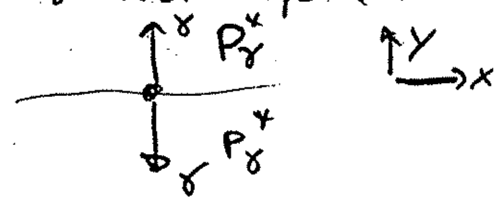


$$E_i = E_f$$
$$mc^2 = \sqrt{m_\pi^2 + p_\pi^2} + \sqrt{m_\pi^2 + p_\pi^2}$$

$$P_\pi = \gamma m v$$
$$= \frac{1}{\sqrt{1 - .825^2}}\, 140 \frac{\text{MeV}}{c^2} (0.825c)$$
$$= (1.769)(140)(.825)\frac{\text{MeV}}{c} = 204.4\,\text{MeV}/c$$
$$Mc^2 = 2\sqrt{\underbrace{m_\pi^2 + p_\pi^2}_{61369}} = \underline{495.5\,\text{MeV}/c^2}$$

b) 

in π_0 rest frame



Lorentz T.F. for Photon to lab

$$P_y = P_y^* = 70\,\text{MeV}/c$$
$$P_x = \gamma(P_x^* + \beta \frac{E_\pi^*}{c})$$
$$= \gamma\beta E_\pi^* = (1.769)(.825)\, 70\,\text{MeV}$$
$$= +102.16\,\text{MeV}/c$$

E conservation

$$E_\pi = E_\gamma^* + E_\gamma^* = 2E_\gamma^*$$
$$m_\pi c^2 = 2E_\gamma^*$$
$$E_\gamma^* = \tfrac{1}{2} m_\pi c^2 = 70\,\text{MeV}$$
$$E_\gamma^* = P_\gamma^* c \Rightarrow P_\gamma^* = 70\,\frac{\text{M}}{}$$

$$\tan\theta = P_y/P_x = \frac{70}{102.16} = 0.685$$

$$\underline{\theta = 34.4^\circ}$$

c) $\pi^+ \longrightarrow \mu$ (105 MeV/c^2) $+ \nu$ ($m=0$)

$v = 0.825c$

$E_\mu(\mathrm{MIN}) = ?$

"π^+ rest frame": If the μ went forward it would gain E
" " " backward " " lose E

MIN μ E
μ COM
(π rest frame)

$\mu \xleftarrow{p^*} \bullet \xrightarrow{p^*} \nu$

$E = \sqrt{m_\mu^2 + p^2}$ $\qquad E = p$

$$E_\mu + E_\nu = E_\pi = m_\pi c^2$$

$$\sqrt{p^2 + 105^2} + p^* = 140$$

$$\cancel{p^2} + 105^2 = 140^2 + \cancel{p^2} - 280p^*$$

$$p^* = \frac{140^2 - 105^2}{280} = 30.625$$

$$\underline{E_\mu^*} = E_\pi - E_\nu = 140 - 30.625 = 109.4\ \mathrm{MeV}$$

LT
$$\left\{\begin{aligned} E_\mu &= \gamma(E^* + \beta p^*) \\ &= (1.77)\left[109.4 - .825(30.6)\right] \\ \underline{E_\mu} &\underline{= 150\ \mathrm{MeV}} \end{aligned}\right.$$

$\xrightarrow{\pi^+}$ $\qquad \mu \longleftarrow \bullet \longrightarrow \nu$

2. Two noninteracting particles of equal mass m are in a one-dimensional potential well. The well has infinitely high walls separated by a distance a, and is represented by the potential function

$$V(x) = \begin{cases} 0, & \text{for } 0 \leq x \leq a, \\ \infty, & \text{for } x < 0, \text{ or } x > a. \end{cases}$$

(a) Write down the Hamiltonian describing this two-particle system. Use the notation that particle 1 has coordinate x_1 and particle 2 has coordinate x_2.

(b) Give the most general solution of the time-dependent Schrödinger equation for this problem.

(c) Show that the following wave function satisfies the two-particle time-independent Schrödinger equation for this potential,

typo

$$\Psi(x_1, x_2) = \frac{2}{a}\left[\frac{1}{\sqrt{3}} \sin\frac{3\pi x_1}{a} \sin\frac{3\pi x_2}{a} - \left(\sqrt{\frac{2}{3}}i\right) \sin\frac{3\pi x_1}{a} \sin\frac{\pi x_2}{a}\right].$$

(d) Determine the energy eigenvalue for this state and verify that $\Psi(x_1, x_2)$ is normalized.

(e) Does particle 2 have a well-defined kinetic energy? Explain.

(f) Suppose that the kinetic energy of particle 1 is measured, and that no measurement is made for particle 2. What are the possible measured values of the kinetic energy of particle 1? What are the probabilities associated with these measured values?

(g) If the preceding kinetic energy measurement yields the lowest allowed value, how would you write the normalized wave function for the system, after the measurement?

(h) After this measurement, what are the possible results of a subsequent measurement of the kinetic energy of particle 2?

(i) Explain how your answers to parts (e) and (h) are compatible. In particular, how can a measurement on particle 1 can affect the results for particle 2, even though the particles are noninteracting.

Two particles in square well: Problem # 2

For a single particle the energy eigenstates are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$ and the energy eigenvalues

are $$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}. \qquad n = 1, 2, 3, \cdots$$

(a) For 2 particles,

Hamiltonian

$$H = T_1 + T_2 = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right).$$

(b) Time-independent Schrödinger eq.: $H\Psi = i\hbar \frac{\partial \Psi}{\partial t}$.

General solution: $$\Psi(x_1, x_2, t) = \sum_{n_1, n_2} c_{n_1 n_2} \psi_{n_1 n_2}(x_1, x_2) e^{-i E \cdot t}$$

$$\Rightarrow \boxed{\Psi(x_1, x_2, t) = \sum_{n_1, n_2} c_{n_1, n_2} \frac{2}{a} \sin\frac{n_1\pi x_1}{a} \sin\frac{n_2 \pi x_2}{a} e^{-i\frac{(E_{n_1}+E_{n_2})}{\hbar}t}}$$

where E_{n_1} & E_{n_2} are given above;
c_{n_1, n_2} are arbitrary complex constants.

(c) The given wave function is

$$\Psi(x_1, x_2) = \frac{1}{\sqrt{3}} \psi_{n=1}(x_1) \cdot \psi_{n=3}(x_2) - \sqrt{\frac{2}{3}}\, i\, \psi_{n=3}(x_1) \cdot \psi_{n=1}(x_2)$$

Ψ is normalized.

(d) If H acts on this Ψ, the result is

$$H\Psi = \frac{1}{\sqrt{3}}(E_1 + E_3)\, \psi_1(x_1) \cdot \psi_3(x_2) - \sqrt{\frac{2}{3}}\, i\, (E_3 + E_1)\, \psi_3(x_1) \cdot \psi_1(x_2)$$

$$= (E_1 + E_3)\,\Psi$$

which is an eigenvalue equation = Time-independent Sch. eq.

The eigenvalue is $\underline{\underline{E_1 + E_3}} = \underline{\underline{\dfrac{5\pi^2\hbar^2}{ma^2}}}$

(e.) Particle #2 is described by a superposition of ψ_3 & ψ_1, which have different energies. Therefore particle #2 does not have a well-defined energy.

(f) The possible measured values of particle #1's kinetic energy are $E_1 = \dfrac{\pi^2\hbar^2}{2ma^2}$ $\left(\text{Probability} = \left|\frac{1}{\sqrt{3}}\right|^2 = \underline{\underline{\frac{1}{3}}}\right)$

and $E_3 = \dfrac{9\pi^2\hbar^2}{2ma^2}$ $\left(\text{Probability} = \left|-\sqrt{\frac{2}{3}}\, i\right|^2 = \underline{\underline{\frac{2}{3}}}\right)$.

(g) If the measurement in (d) yields E_1, then the system state following the measurement is the "reduced state" given by

$$\Psi_{\substack{AFTER\\MEASUREMENT}}(x_1, x_2) = \psi_1(x_1)\,\psi_3(x_2)$$

$$= \underline{\underline{\frac{2}{a}\sin\frac{\pi x_1}{a}\,\sin\frac{3\pi x_2}{a}}}$$

(h) Since $\Psi_{\substack{AFTER\\MEASUREMENT}}$ contains only $\psi_3(x_2)$ for particle #2, the subsequent measurement of the kinetic energy for #2 can yield only $\underline{\underline{E_3 = \frac{9\pi^2\hbar^2}{2ma^2}}}$.

(i) The measurements in (e) and (f) are correlated in accord with QM measurement theory and with conservation of total energy. Various interpretations are possible; see for example, D.J. Griffiths, Introd. to QM, pp. 374-385.

3. A two-dimensional quantum-mechanical rigid rotator consists of two particles having equal masses m, separated by a fixed distance $2a$, and constrained to rotate in the $x-y$ plane. In terms of the angular momentum L_z, the Hamiltonian is

$$H_o = \frac{L_z^2}{2I}$$

where $I = 2ma^2$.

(a) Find the energy eigenvalues, the corresponding normalized eigenfunctions, and the degree of degeneracy of each energy level.

(b) Suppose that the two masses have charges $\pm q$, and that a uniform static electric field $\vec{\mathbf{E}} = E\hat{\mathbf{x}}$ acts in the x-direction. Calculate the energy level shift for the lowest-energy state, using stationary perturbation theory through second order.

(c) Instead of the static field in part (b), a spatially uniform but time-dependent electric field $\vec{\mathbf{E}}(t) = E_o e^{-t/\tau}\hat{\mathbf{x}}$ is applied to the rotator, beginning at time $t = 0$. Assume that this field is weak, and that the system is initially in its ground state. Using first-order time-dependent perturbation theory, calculate the probability that the rotator will be in an excited state at time $t = \infty$.

(d) What criterion would you propose for the field in part (c) to be "weak"?

Rigid Rotator Problem #3

(a) $H_0 = \frac{L_z^2}{2I} = -\frac{\hbar^2}{4ma^2}\frac{\partial^2}{\partial\phi^2}$

$H_0\psi = E\psi \longrightarrow \frac{\partial^2\psi}{\partial\phi^2} + \underbrace{\left(\frac{4ma^2}{\hbar^2}E\right)}_{\equiv m^2}\psi = 0$ eigenvalue eq

Solutions: $\psi = Ae^{im\phi}$

where $m \equiv \sqrt{\frac{4ma^2E}{\hbar^2}}$.

Continuity conditions: $\begin{cases}\psi(2\pi) = \psi(0) \\ \psi'(2\pi) = \psi'(0)\end{cases} \Rightarrow m = \text{integer} = 0, \pm1, \pm2, \cdots$

Normalization: $\int_0^{2\pi}|\psi|^2 d\phi = 1 \Rightarrow A = \frac{1}{\sqrt{2\pi}}$

Eigenfunctions: $\boxed{\psi(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}}$ $m = 0, \pm1, \pm2, \cdots$

Energy eigenvalues: $\boxed{E = \frac{m^2\hbar^2}{4ma^2}}$

The $m=0$ level (ground state) is nondegenerate; all excited levels are doubly degenerate.

(b) $\vec{\mathcal{E}} = \mathcal{E}\hat{x} \longrightarrow \Delta V = -q\mathcal{E}x + q\mathcal{E}(-x)$ interaction

$= -2q\mathcal{E}a\cos\phi$ potential

Level shift:

First order $\Delta E^{(1)} = \langle 0|\Delta V|0\rangle = -2q\mathcal{E}a \cdot \frac{1}{2\pi}\int_0^{2\pi}\cos\phi\, d\phi$

$= 0.$

Second order $\Delta E^{(2)} = \sum_{\substack{m \\ (m\neq 0)}} \frac{|\langle 0|\Delta V|m\rangle|^2}{E_0 - E_m}$

Matrix element:

$\langle 0|\Delta V|m\rangle = -2q\mathcal{E}a \cdot \frac{1}{2\pi}\int_0^{2\pi}\cos\phi\, e^{im\phi}\, d\phi$

$\cos m\phi + i\sin m\phi$

$= -\frac{q\mathcal{E}a}{\pi} \cdot 2\left(\frac{\pi}{2}\delta_{m,\pm 1} + i\cdot 0\right)$ using integral given.

Only $m = \pm 1$ terms contribute to $\Delta E^{(2)}$:

$\Delta E^{(2)} = (-q\mathcal{E}a)^2 \frac{1+1}{0 - \frac{\hbar^2}{4ma^2}}$

$$\boxed{\Delta E^{(2)} = -\frac{8q^2ma^4}{\hbar^2}\mathcal{E}^2}$$

(c) Now $\vec{\mathcal{E}}(t) = \mathcal{E}_0 e^{-t/\tau}\,\hat{x}$.

The interaction potential is

$$V'(t) = -2q\mathcal{E}_0 a \cos\phi\, e^{-t/\tau}.$$

$$H(t) = H_0 + V'(t) \; ; \quad H(t)|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle.$$

$|\psi(0)\rangle = |0\rangle$ ground state. $\qquad E_0 = 0.$

$$|\psi(t)\rangle = \sum_m C_m(t)\, e^{-i\frac{E_m}{\hbar}t}\,|m\rangle \quad \text{expansion}$$

$$\dot{C}_m \cong -\frac{i}{\hbar}\langle m|V'(t)|0\rangle\, e^{-i\frac{E_m}{\hbar}t} \quad \text{in } 1^{ST} \text{ order.}$$

$$C_m(t) \cong -\frac{i}{\hbar}\left(-q\mathcal{E}_0 a\right)\delta_{m,\pm1}\int_0^t e^{-\left(\frac{1}{\tau}+i\frac{E_m}{\hbar}t\right)}dt.$$

using matrix element from part (b).

$$C_{\pm1}(\infty) = +i\,\frac{q\mathcal{E}_0 a}{\hbar}\cdot\frac{1}{\left(\frac{1}{\tau} \pm i\frac{\hbar}{4ma^2}\right)}$$

Probability that final state is an excited state ($m=\pm1$)

$$P_{EXC.} = P_{m=1} + P_{m=-1} = |C_{+1}(\infty)|^2 + |C_{-1}(\infty)|^2$$

$$\boxed{P_{EXC.} = \frac{2q^2a^2\mathcal{E}_0^2}{\hbar^2\left(\frac{1}{\tau^2} + \frac{\hbar^2}{(4ma^2)^2}\right)}}$$

(d) $P_{EXC.} \ll 1$ "Weak field" criterion. $\quad \mathcal{E}_0 \ll \dfrac{\hbar\sqrt{\frac{1}{\tau^2} + \frac{\hbar^2}{(4ma^2)^2}}}{qa}$

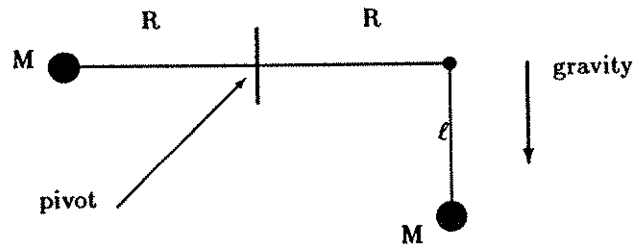


Figure 1: Side view of rotator. The 2R bar rotates in the horizontal plane perpendicular to the page. The pendulum swings in a plane perpendicular to the plane of the pivot and the 2R bar.

4. As shown in the figure, two equal masses M are connected to a very light horizontal beam. One of the masses is connected directly to the end of the beam, and the other is connected to the beam by a very light vertical rod of length ℓ. The horizontal beam of length $2R$ is pivoted at its center such that it always remains horizontal but is otherwise free to rotate. The pendulum is pivoted such that it always moves in a vertical plane whose normal is along the direction of the beam. The pivots are both frictionless.

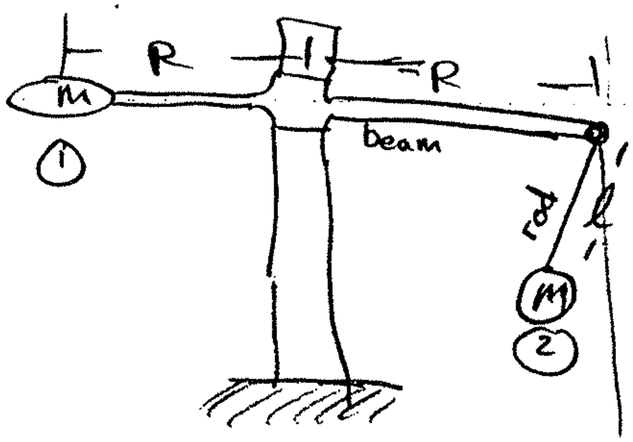
   (a) What are the equations of motion for each mass?

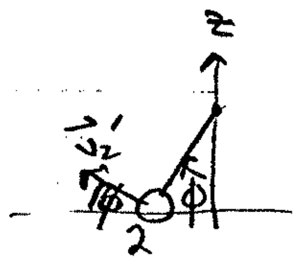
   (b) If the displacement of the pendulum is small, what are the general solutions of the equations of motion for each mass?

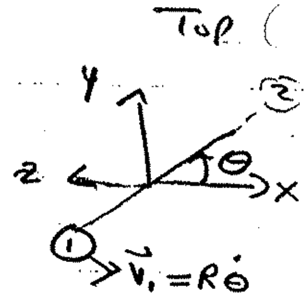
   (c) The system, initially at rest, is disturbed by a small, fast, horizontal, and tangential blow to the mass on the vertical rod. Determine and sketch the resulting angular velocity of the beam.

   (d) What are the frequencies of the normal modes of oscillation?

2) Problem # 4







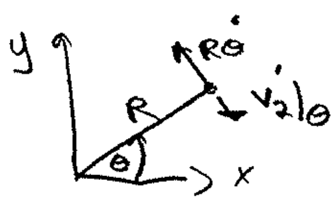
In frame of pivot

a) ① $V_1 = R\dot\theta\,\hat{e}_\theta$

② $|V_2'| = l\dot\phi$

$V_2')_z = l\dot\phi\sin\phi$

$V_2')_\theta = -l\dot\phi\cos\phi$ (in xy plane)



$\vec{V}_2 = \vec{V}_2' + \vec{V}_{beam}$

$V_2)_z = l\dot\phi$

$V_2)_\theta = R\dot\theta - l\dot\phi\cos\phi$

$PE = V_2 = V = +mgh = mg(l - l\cos\phi) = mgl(1-\cos\phi)$

$KE = \frac{1}{2}m(v_1^2+v_2^2) = \frac{m}{2}(R^2\dot\theta^2 + l^2\dot\phi^2\sin^2\phi + R^2\dot\theta^2 + l^2\dot\phi^2\cos^2\phi - 2Rl\dot\theta\dot\phi\cos\phi)$

$= m(R^2\dot\theta^2 + \frac{1}{2}l^2\dot\phi^2 - Rl\dot\theta\dot\phi\cos\phi)$

$L = T - V = m[R^2\dot\theta^2 + l^2\dot\phi^2/2 - Rl\dot\theta\dot\phi\cos\phi + gL\cos\phi - \cancel{gl}$

Eqn Mtn

i) $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0$

ii) $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right) - \frac{\partial L}{\partial\phi} = 0$

$\partial L/\partial\dot\theta = m[2R^2\dot\theta - Rl\dot\phi\cos\phi]$

$\partial L/\partial\dot\phi = m[l^2\dot\phi - Rl\dot\theta\cos\phi]$

$\partial L/\partial\theta = 0$, $\frac{\partial L}{\partial\phi} = m\ell\sin\phi\{R\dot\theta\dot\phi - g\}$

a) i) $2R^2\ddot\theta - R\ell\ddot\phi\cos\phi + R\ell\dot\phi^2\sin\phi = 0$

ii) $\ell^2\ddot\phi - R\ell\ddot\theta\cos\phi + \cancel{R\ell\dot\theta\dot\phi\sin\phi - \ell\sin\phi R\dot\theta\dot\phi} + \ell\sin\phi g$

$\Rightarrow$ $\boxed{\begin{array}{l} \text{i) } 2R\ddot\theta - \ell\ddot\phi\cos\phi + \ell\dot\phi^2\sin\phi = 0 \\ \text{ii) } \ell\ddot\phi - R\ddot\theta\cos\phi + g\sin\phi = 0 \end{array}}$

b) Small oscillations:

i) $2R\ddot\theta - \ell\ddot\phi + \ell\dot\phi^2\phi = 0$

ii) $\ell\ddot\phi - R\ddot\theta + g\phi = 0$

get $\dot\phi, \phi$ are small , so drop $\dot\phi^2\phi$

i) $2R\ddot\theta = \ell\ddot\phi$

ii) $\ell\ddot\phi - \frac{1}{2}\ell\ddot\phi + g\phi = 0$

$\frac{\ell}{2}\ddot\phi = -g\phi$

$\ddot\phi = -\frac{2g}{\ell}\phi = -\omega^2\phi$

ii) $\boxed{\begin{array}{l} \phi = A\sin\omega t + B\cos\omega t, \quad \omega = \sqrt{\frac{2g}{\ell}} \\ \phi = \frac{\dot\phi_0}{\omega}\sin\omega t + \phi_0\cos\omega t \end{array}}$

i) $\ddot\theta = \frac{\ell}{2R}\ddot\phi$

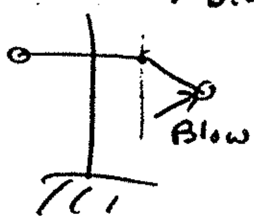
$\dot\theta = \frac{\ell}{2R}\dot\phi + C$

$\theta = \frac{\ell}{2R}\phi + Ct + D$ $= \frac{\ell}{2R}\phi(t) + [\dot\theta(0) - \frac{\ell}{2R}\dot\phi(0)]t + (\theta_0 - \frac{\ell}{2R}\phi_0)$

$\theta(0) = \frac{\ell}{2R}\phi(0) + D$

$\dot\theta(0) = \ell/_{\cdot}\,\dot\phi(0) + C$

c) blow causes velocity, but no displacement or velo
of be

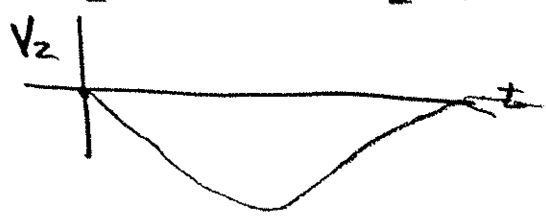


$\Theta_0 = \phi_0 = \dot{\Theta}_0 = 0$

ii) $\phi(t) = \frac{\dot{\phi}_0}{\omega} \sin\omega t$

i) $\Theta(t) = \frac{L}{2R} \frac{\dot{\phi}_0}{\omega} \sin\omega t - \frac{L}{2R} \dot{\phi}_0 t$

$$V = R\dot{\Theta}(t) = \frac{L}{2} \dot{\phi}_0 \cos\omega t - \frac{\ell}{2}\dot{\phi}_0 = \frac{\ell}{2}\dot{\phi}_0 (\cos\omega t - 1)$$

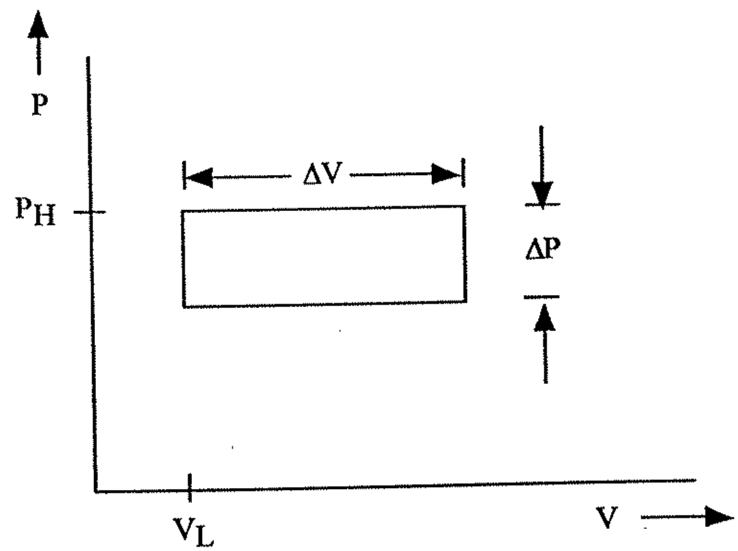


d) Normal modes

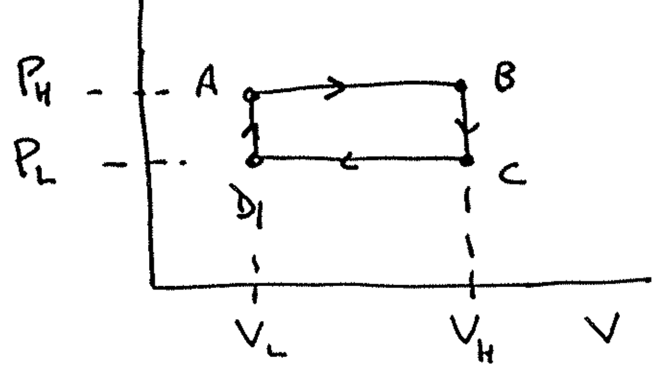
$\omega = 0$, rotation of beam

$\omega = \sqrt{g/\ell}$,

5. One mole of an ideal monatomic gas, used as the working substance in a heat engine goes through the cycle indicated in the diagram below. In the limit that $\frac{P_H}{\Delta P} >> 1$, while $\frac{V_L}{\Delta V}$ is not, show that the efficiency, η, of this engine can be approximated by $\eta \approx \frac{0.4\Delta P}{P_H}$.



Problem #5



$$W = (P_H - P_L)(V_H - V_L)$$
$$W = \Delta P \Delta V$$

Need to determine net heat in to System

since $\eta_{eff} = \dfrac{W}{Q_{in}}$

$A \to B$: $Q_{AB} = (U_B - U_A) + P_H(V_B - V_A)$

$$= \frac{3}{2}R(T_B - T_A) + R(T_B - T_A)$$

$$= \frac{5}{2}R(T_B - T_A) = \frac{5}{2}P_H(V_H - V_L) > 0 \ (i$$

$B \to C$: $Q_{BC} = (U_C - U_B) = \frac{3}{2}R(T_C - T_B) < 0$ (ou

$C \to D$: $Q_{CD} = (U_D - U_C) + P_L(V_D - V_C)$

$$= \frac{5}{2}P_L(V_L - V_H) < 0 \text{ (out)}.$$

$D \to A$: $Q_{DA} = U_A - U_D = \frac{3}{2}R(T_A - T_D)$

$$= \frac{3}{2}(P_H V_L - P_L V_L) = \frac{3}{2}(P_H - P_L)V_L > \text{: (i}$$

$$\therefore \eta = \frac{\Delta P \Delta V}{\frac{5}{2}P_H \Delta V + \frac{3}{2}\Delta P V_L} = \frac{1}{\frac{5}{2}\frac{P_H}{\Delta P} + \frac{3}{2}\frac{V_L}{\Delta V}}$$ QE

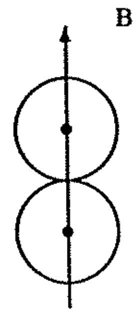


Figure 2: Alignment of the Spheres

6. A paramagnetic sphere of radius R and magnetic susceptibility χ_m is placed in a uniform magnetic field $\mathbf{H_0}$ and magnetic induction $\mathbf{B_0}$. Note that paramagnetic susceptibilities are normally very small.

   (a) Calculate the magnetic field $\mathbf{H}$ and magnetic induction $\mathbf{B}$ produced by the sphere. Sketch the fields and explain any differences between the two.

   (b) Find the total force and torque that the external field exerts on the sphere.

   (c) Now suppose that two identical spheres are touching one another and lined up along the field lines as shown in the figure. Calculate the force between them. Note that several ways of doing this result in integrals that are hopeless to evaluate. Be careful to choose a method with some chance of success. Even if you are not able to finish the calculation, set up your integral carefully and explain your notation and your method.

Problem #6

E & M Problem - Paramagnetic Spheres.

(a) This is easiest to do as a boundary value problem with the magnetic potential.

$\vec{H} = -\vec{\nabla}\Phi_m \qquad \vec{M} = \chi_M \vec{H} \qquad \vec{B} = \mu \vec{H} = (1+4\pi\chi_M)$

$B_\perp$, $H_{\parallel}$, & Φ_m are continuous.

The uniform field can be written

$\vec{B_0} = B_0 \hat{z} \qquad B_r = B_0 \cos\theta = -\mu \frac{\partial \Phi_0}{\partial r}$

so $\Phi_0 = -\frac{B_0}{\mu} r\cos\theta$

$$\Phi_1 = \sum_l A_l r^l P_l \qquad \text{(inside)}$$

$$\Phi_2 = \sum_l C_l r^{-(l+1)} P_l - \frac{B_0}{\mu} r\cos\theta \qquad \text{(outside)}$$

Since $\Phi_0 \sim \cos\theta$, only $l=1$ terms survive.

$$\Phi_1(r=R) = A_1 R P_1 = C_1 R^{-2} P_1 - \frac{B_0}{\mu} R\cos\theta = \Phi_2(r=R)$$

$$A_1 = C_1 R^{-3} - B_0/\mu$$

Since B_r is continuous

$$+\mu \left.\frac{\partial \Phi_1}{\partial r}\right|_{r=R} = + \left.\frac{\partial \Phi_2}{\partial r}\right|_{r=R}$$

$$+\mu A = -2CR^{-3} - \frac{B_0}{\mu}$$

$$2A = 2CR^{-3} - 2B_o/\mu$$
$$\mu A = -2CR^{-3} - B_o/\mu$$

$$(2+\mu)A = -3B_o/\mu$$

$$A = -3B_o/\mu(2+\mu)$$

$$C = (A + B_o/\mu)R^3 = \left[\frac{\mu-1}{\mu(2+\mu)}\right] R^3 B_o$$

$$\phi_1 = Ar\cos\theta \qquad \vec{H}_1 = -\vec{\nabla}\phi_1 = -A\hat{z}$$

$$\phi_2 = \frac{C}{r^2}\cos\theta - \frac{B_o}{\mu} r\cos\theta$$

$$\vec{H}_2 = -\vec{\nabla}\phi_2 = \frac{2C}{r^3}\hat{r}\cos\theta + \frac{C}{r^3}\hat{\theta}\sin\theta + \frac{B_o}{\mu}\hat{z}$$

the easiest way to find to force is "coulomb's law

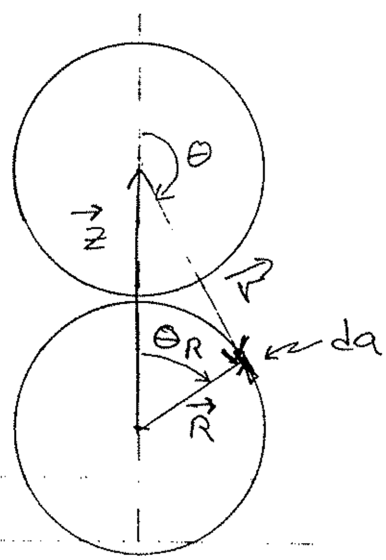
$$\vec{F} = \int_S (\vec{M} \cdot \hat{n})\, \vec{B}\, da$$

Since $\vec{F}$ is in the z direction we only need B_z

$$B_z = B_r \cos\Theta + B_\Theta \sin\Theta$$

$$= \frac{\mu c}{r^3}\left(2\cos^2\Theta + \sin^2\Theta\right) = \frac{\mu c}{r^3}\left(1 + \cos^2\Theta\right)$$

$$\vec{M} = \chi_M \vec{H} = -A\, \chi_m\, \hat{z}$$



$$F_z = \int M \cos\Theta_R \; B_z(r,\Theta)\, R^2\, d\Omega$$

This can be done with a few standard tricks using Legendre polynomials.

7. Quantized spin waves are called *magnons*. Magnon energies in the lowest approximation can be expressed as $\hbar\omega \approx 2SJa^2k^2$, where S is the maximum spin value at each site, k is the wave number, J is the exchange integral, and a is the lattice constant. Given that magnons are bosons, determine the temperature dependence of the specific heat at constant volume, C_V, for a ferromagnet represented as a collection of N magnons.

8. A simple model for the propagation of electromagnetic waves through matter assumes that the electrons behave like damped, driven harmonic oscillators.

   (a) Suppose that a single electron has a "spring constant" $m\omega_o^2$, a damping force $-m\gamma\mathbf{v}$, and is acted on by an electric field $\mathbf{E}(\mathbf{x}, t)$. Write the equation of motion. Explain the physical interpretation of the various terms in the equation.

   (b) Let $\mathbf{E}(\mathbf{x}, t)$ be the usual time harmonic field

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}(\mathbf{x})e^{-i\omega t}$$

   Solve the equation of motion for the displacement $\mathbf{x}(t)$.

   (c) If we assume there are N such electrons per unit volume in the solid that behave like this (presumably all the other electrons are tightly bound to atoms and don't move), then all the basic properties of conductors and dielectrics can be expressed as complex functions of ω, ω_o, m, N, and γ. Find the conductivity Γ, the electric susceptibility χ_e, the permittivity ϵ, and the dielectric constant n.

   (d) If the electrons are not bound, *i.e.* if $\omega_o = 0$ and ω is not too large, then this model describes an ordinary conductor that obeys ohm's law. The conductivity is real in this case; we usually call it σ. Derive a simple formula for σ.

   (e) When the electromagnetic wave propagates through this conductor it has a complex wave number k. Derive a formula for the real and imaginary parts of k assuming that σ is large. The following approximate formula ($y \gg x$) will be useful.

$$\sqrt{x + iy} \approx (1 + i)y/2$$

   (f) Suppose a plane-polarized electromagnetic wave travelling through vacuum is incident on a conductor of the sort we have just described. State the appropriate boundary conditions that must be satisfied at the surface of the conductor.

   (g) In order to simplify the problem, assume that the wave is normally incident on the conductor. Calculate the phase and intensity of the reflected wave relative to the incident wave. Calculate the skin depth. Express your result in terms of the real and imaginary parts of n (or k) using the results of (e) above.

Problem # 7

$$U = \sum_{\underline{k}} \hbar\omega \langle n_{\underline{k}} \rangle$$

$$U = \sum_{\underline{k}} \frac{2SJa^2k^2}{e^{\hbar\omega/k_BT} - 1}$$

$$U = \frac{V}{8\pi^3} \int_0^\infty \frac{2SJa^2k^2\, 4\pi k^2\, dk}{e^{2SJa^2k^2/k_BT} - 1}$$

$$U = \frac{SJa^2V}{\pi^2} \int \frac{k^4\, dk}{e^{(\;)} - 1}$$

Putting $x = \dfrac{2SJa^2k^2}{k_BT}$,

$$U = \frac{N}{2\pi^2}\left(\frac{1}{2SJ}\right)^{3/2} (k_BT)^{5/2} \int_0^\infty \frac{x^{3/2}\, dx}{e^x - 1}.$$

where $N = V/a^3$.

$$\therefore C_V = \left(\frac{\partial U}{\partial T}\right)_V \propto T^{3/2}.$$

Problem #8

E & M Problem – SHO Model (cgs units)

(a) $$m\left[\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^2\vec{x}\right] = e\vec{E}(\vec{x},t)$$

This is just $\vec{F} = m\vec{a}$

(b) assume $\vec{x}(t) = \vec{x}e^{-i\omega t}$

$$\vec{x} = \frac{e}{m}\left(\omega_0^2 - \omega^2 - i\omega\gamma\right)^{-1}\vec{E}$$

(c) The polarization / unit volume is P

$$\vec{P} = N\vec{p} = Ne\vec{x} = \frac{Ne^2}{m}\left(\omega_0^2 - \omega^2 - i\omega\gamma\right)^{-1}\vec{E}$$

$$= \chi_e\vec{E} \quad \text{this defines } \chi_e.$$

$$\epsilon = 1 + 4\pi\chi_e = 1 + \frac{4\pi Ne^2}{m}\left(\omega_0^2 - \omega^2 - i\omega\gamma\right)^{-1}$$

$$n = \sqrt{\epsilon}$$

Finally $$\vec{J} = Ne\dot{\vec{x}} = -i\omega Ne\vec{x} = \Gamma\vec{E}$$

so
$$\Gamma = \frac{-i\omega Ne^2}{m\left(\omega_0^2 - \omega^2 - i\gamma\omega\right)}$$

(d) $$\sigma = \frac{Ne^2}{m\gamma}$$ (ohm's law).

e) $k = \frac{\omega}{c} n = \frac{\omega}{c}\sqrt{\epsilon}$

$$= \frac{\omega}{c}\sqrt{1 + \frac{4\pi i}{\omega}\sigma}$$

$$\approx \frac{\omega}{c}\sqrt{\frac{2\pi\sigma}{\omega}}\,(1+i)$$

f) $B_\perp$, $E_\parallel$, $D_\perp$, $H_\parallel$ are continuous as usual.

(g) $\vec{B} = n\,\hat{k}\times\vec{E}$ (in cgs units)

Let E = incident wave
E' = transmitted "
E'' = reflected "

$$\vec{E} + \vec{E}'' = \vec{E}'$$
$$\vec{B} + \vec{B}'' = \vec{B}' \quad \text{or} \quad \vec{E} - \vec{E}'' = n\vec{E}'$$

$$\frac{E''}{E} = \frac{1-n}{1+n} = -1 \quad \text{if } \sigma \text{ is large.}$$

$$e^{ikx} = \exp\left[i\frac{\omega}{c}\sqrt{\frac{2\pi\sigma}{\omega}}\right]\exp\left[-\frac{\omega}{c}\sqrt{\frac{2\pi\sigma}{\omega}}\right]$$

so skin depth $\delta = \frac{\omega}{c}\sqrt{\frac{2\pi\sigma}{\omega}}$