Department of Physics Comprehensive Examination No. 86

March 29 and 30, 1999

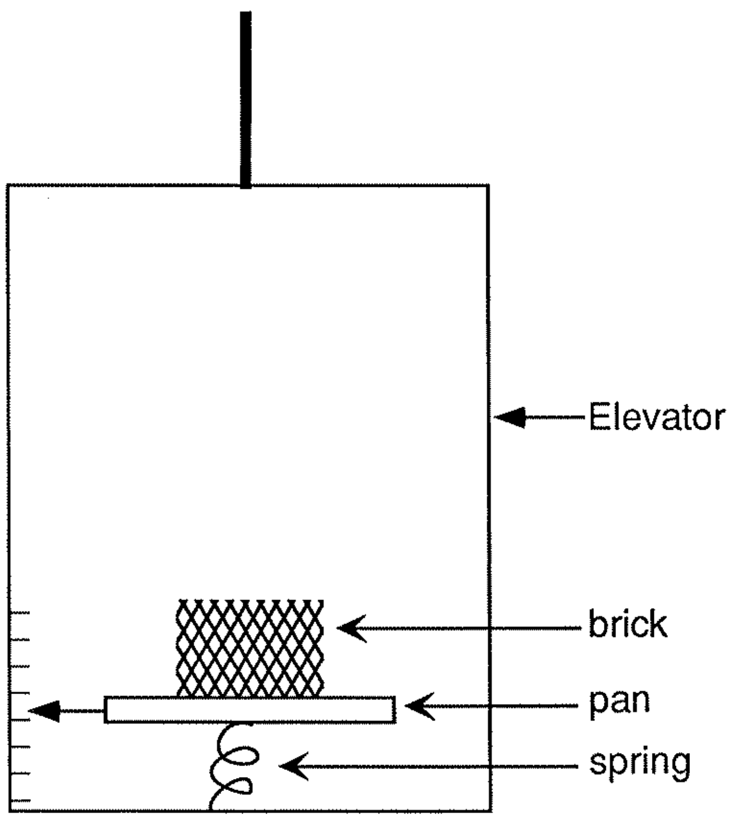
This Comprehensive Examination for Spring 1999 consists of eight problems each worth 20 points. The problems are grouped into four sessions, each of which lasts for three hours. Session One (problems 1 and 2) begins at 9:00 AM Monday 29 March. Session Two (problems 3 and 4) begins at 1:30PM Monday 29 March. Session Three (problems 5 and 6) begins at 9:00 AM Tuesday 30 March. Session Four (problems 7 and 8) begins at 1:30PM Tuesday 30 March.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it is possible to obtain partial credit, especially if you demonstrate conceptual understanding. Do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter, but not your name, is on the inside of the back cover of every bluebook. Be sure to remember your student letter for use in the remaining sessions of the examination.

If something is omitted from the statement of the problem or you feel there is an ambiguity, please ask your question quietly and privately, so as not to disturb the others. Only your bluebooks and the examination should be on the table before you. Any other items should be stored on the floor. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your notebooks for scratch work separated by at least one page from your solutions. Scratch work will not be graded.

A spring balance is constructed from a pan attached to a spring, as in the figure. The spring is such that an 80 kg mass placed on the pan compresses the spring by 1 cm when the pan comes to rest. The balance, with a 50 kg brick resting on top of the pan, is attached to the floor of an elevator at the top of a very tall building, and everything is allowed to come to rest. The elevator is very massive, so you can ignore the force exerted on the elevator's floor by the spring, as well as the mass of the spring and of the pan. At time $t = 0$, the elevator cable breaks and the elevator falls freely.



a) What is the length by which the spring is compressed at $t = 0$?

b) Solve for the motion of the brick relative to the elevator.

c) Determine what "weight" the balance reads as a function of time (it is read with the arrow and scale on the left side of the elevator).

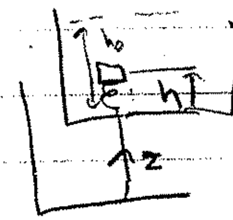
CM1 $\quad F = -kx \Rightarrow k = F/x = (80\,kg)(9.8\,m/s^2)/10^{-2}_{m} = 775 \times 10^{2}\ N/m$

a) $t=0 \quad x = F/k = \frac{50 \times 9.8}{80 \times 9.8} 10^{-2} m = \frac{5}{8} \times 10^{-2} m = \alpha = .0625\ m$

$\uparrow F_x$

$\downarrow W = Mg$

b) In elevator (free fall) no gravity ($\equiv$ prin)



$\uparrow kx$

$\downarrow W = mg$

$\Sigma F = ma$

$k(h_0 - h) = m\ddot{h}$

let $x = h - h_0 \qquad \omega =$

$\Rightarrow \ddot{x} = -\frac{k}{m} x = \omega^2 x \qquad \omega = \sqrt{k/m} = \sqrt{\frac{775 \times 10^2}{50}}$

$= 39.4$ rad/sec

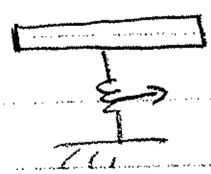
$\Rightarrow x = A\cos\omega t + B\sin\omega t$

$t = 0$, no vel $\Rightarrow B = 0$, $x = h - h_0 = -\alpha$

$h_0 - h = \alpha \cos\omega t$

$\underline{h = h_0 - \alpha\cos\omega t}$

c)

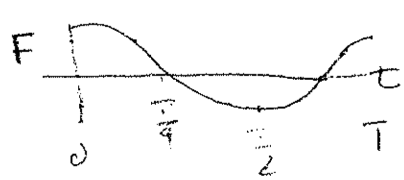


Scale reads normal force. Scale starts compressed, moves up and ejects the mass when v stop increasing. Only exerts force on mass while accelerating upwards

$F = ma = m\ddot{h} = \underline{m\alpha\omega^2\cos\omega t \quad \Theta(T/4 - t)}$

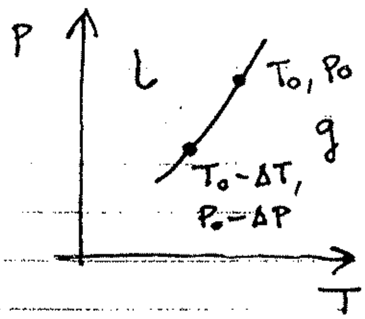
$T = \text{period} = \frac{2\pi}{\omega} = \frac{2\pi\ sec}{39.4} = 0.159\ sec$

$=$



Suppose that you go on a mountain hike. After reaching the top of a high mountain, you find that water boils at 95°C. You remember that the latent heat value for water boiling at standard sea-level atmospheric pressure (T_0 = 100 °C, P_0 = 0.10135 MPa) is L_0 = 2088 kJ/kg. You also remember the value of the gas constant: R = 8.314 J/K-mole, and that the molar mass of air is approximately 29 g. Having all that information, find the approximate height of the mountain.

We use the <u>Clapeyron Equation</u> for the slope of the coexistence curve between the liquid and gas phases:



$$\frac{\Delta P}{\Delta T} = \frac{\lambda}{T \, \Delta v} \qquad (*)$$

$$\Delta v = v_g - v_l \qquad (v \equiv \text{molar volume}).$$

l = liquid
g = gas

For water boiling at ~atmosperic pressure, $v_g \sim 1500 \, v_l$, so that $\Delta v \cong v_g$

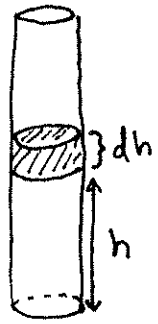
Approximate value of v_g can be obtained from the ideal gas equation:

$$vP = RT \quad \Rightarrow \quad v = \frac{RT}{P} \quad \text{and} \quad \Delta v \cong v_g = \frac{RT_0}{P_0} \qquad (**)$$

From (*) and (**) we obtain the relation between the pressure change ΔP, and the boiling temperature change:

$$\Delta P \cong \frac{\lambda_0 P_0}{R T_0^2} \Delta T \qquad (***)$$

We need to know, in addition, how the atmospheric pressure changes with altitude. Consider an imaginary vertical "cylinder" of air with cross section A. The volume of the air enclosed between h and h+dh is $dV = A\,dh$. The mass of the air in this volume element is $\rho A\,dh$ (ρ = density) and its weight is $-g\rho A\,dh$.



The air pressure at h and h+dh is p and p+dp. The force due to the pressure acting on the volume element from below is Ap, and from above is $A(p+dp)$. Thus the difference is $A\,dp$. If the air in the cylinder is in equilibrium, $A\,dp = -\rho g A\,dh$.
From the ideal gas equation, $pv = RT$, where v is the molar volume. If m_M is the molar mas, $\rho = \frac{m_M}{v} = \frac{p\,m_M}{RT}$. Putting everything together, we obtain the equation:

$$\frac{dp}{p} = -\frac{g\,m_M}{RT}\,dh$$

Solving, we obtain:

$$p = p_0 \exp\left(-\frac{m_M g}{RT}h\right) \cong p_0 \exp\left(-1.15\times10^{-4}\,m^{-1}\,h\right)$$

For small arguments we can expand the exponential function, and we obtain:

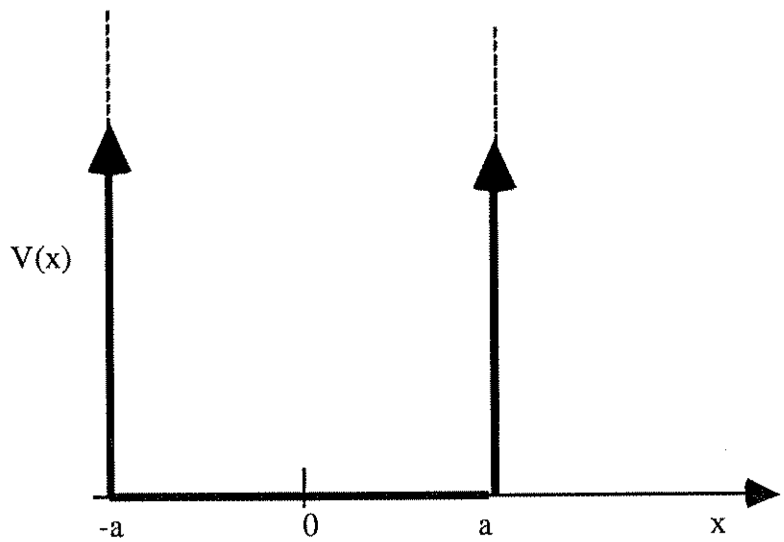
$$\Delta p \cong p_0\left(1.15\times10^{-4}\,m^{-1}\,h\right).$$

Combining this with the formula (***), we finally have:

$$h \cong \frac{10^4\,l_0}{1.15\,RT_0^2}\,\Delta T \approx 1400\ m$$

Note that the latent heat in the problem statement was in KJ/kg, whereas in the Clapeyron equation l_0 is the latent heat per mole. One mole of H_2O is 18g, so the conversion factor is 0.018.

Consider a particle of mass m placed in an infinitely deep, one-dimensional square well potential as shown below.



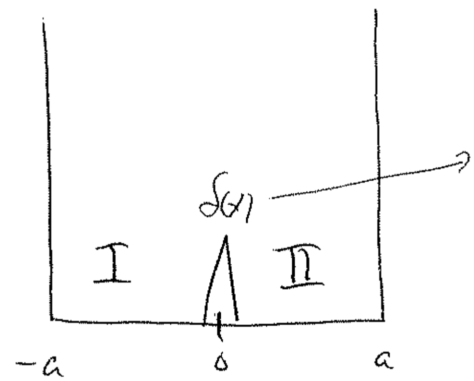
Now add a delta function potential barrier in the middle of the well. The repulsive delta function potential can be written as

$$V(x) = \lambda \frac{\hbar^2}{2ma} \delta(x),$$

where λ is a dimensionless parameter that characterizes the strength of the barrier and a is the same length as in the barrier diagram.

a) Determine the energy level spectrum of this system. You can expect to encounter one or more transcendental equations (as in the finite square well problem). Show how the eigenvalues can be found graphically.

b) Calculate explicitly the energies of the two lowest energy levels in this system for the case of very large λ (strong barrier). (Do this to first order in the relevant parameter.) Discuss your results.



$$V(x) = \lambda \frac{\hbar^2}{2ma} \delta(x)$$

a) Sch eqn: $H\psi = E\psi$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\psi(x)$$

Call region $(-a<x<0)$ I

$0<x<a$ II

To find condition on $\psi(x)$, $\psi'(x)$ at $x=0$ integrate Sch eqn over δ-fnc:

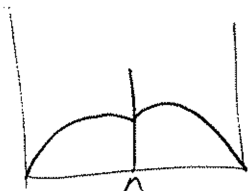
$\int_{-\epsilon}^{+\epsilon} dx$:
$$\int_{-\epsilon}^{+\epsilon}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \lambda\frac{\hbar^2}{2ma}\delta(x)\right)\psi(x)\, dx = \int_{-\epsilon}^{\epsilon} E\psi(x)dx$$

$$-\frac{\hbar^2}{2m}\left[\frac{d\psi(x)}{dx}\right]_{-\epsilon}^{+\epsilon} + \lambda\frac{\hbar^2}{2ma}\psi(0) = 0 \quad \text{for finite } \psi$$

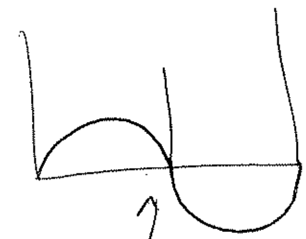
$$-\frac{\hbar^2}{2m}\left[\psi_{II}'(0) - \psi_I'(0)\right] + \lambda\frac{\hbar^2}{2ma}\psi(0) = 0$$

$$\Rightarrow \underline{\psi_{II}'(0) - \psi_I'(0) = +\frac{\lambda}{a}\psi(0)}$$

$\psi(x)$ must still go to zero at $\pm a$,
so expect sin or cos solns that look like:



$\psi(x)$ continuous
$\psi'(x)$ not



$\psi(x), \psi'(x)$
continuous
ψ' ok, since $\psi(0)=0$

These are also even & odd type solns
that are expected since potential symmetric.

$\Rightarrow$ try: <u>even</u>: $\psi_I(x) = \sin(kx + ka)$

$$\psi_{II}(x) = \sin(-kx + ka)$$

$$\Rightarrow \psi_I(-a) = \psi_{II}(a) = 0$$

$$\psi_I(0) = \psi_{II}(0) = \sin ka$$

ψ': $\psi_I'(x) = k \, \overset{\cos}{\cancel{\sin}}(kx + ka)$

$$\psi_{II}'(x) = -k \, \overset{\cos}{\cancel{\sin}}(kx + ka)$$

⇒ discontinuity in slope eqn gives:

$$-k\,\cancel{\sin}\cos ka - k\,\cancel{\sin}\cos ka = +\frac{\lambda}{a}\sin ka$$

$$-2k\cos ka = \frac{\lambda}{a}\sin ka$$

$$\boxed{-\tan ka = \frac{2ka}{\lambda}}$$ even

odd solns: $\psi_I(x) = -\sin(kx)$

$\psi_{II}(x) = +\sin(kx)$

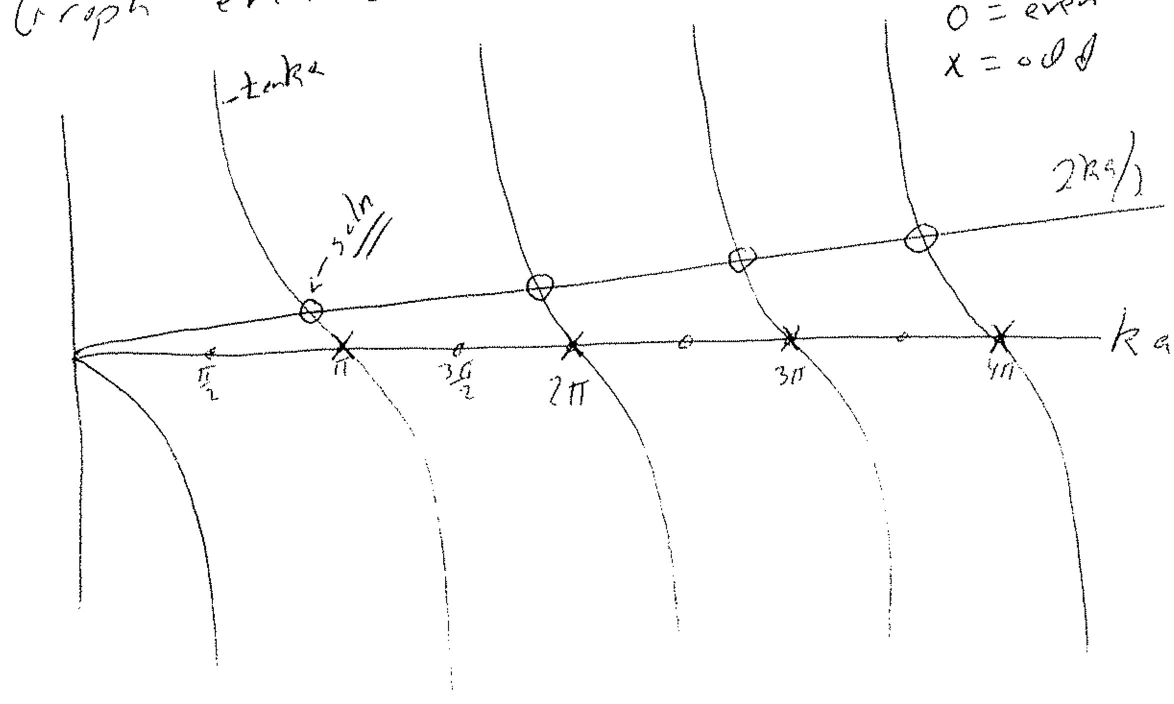
$\psi_I(0) = \psi_{II}(0) = 0 \Rightarrow \psi'$ continuous

$\psi_I(a) = -\sin(ka) = 0 \Rightarrow \boxed{ka = n\pi}$ odd

Graph even solns:

O = even

X = odd



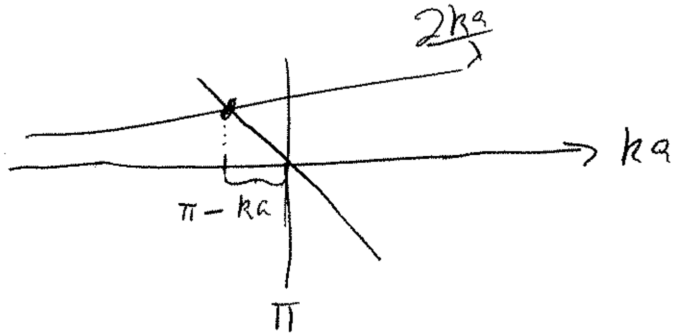
b) E_1 is soln w/ $ka < \pi$

E_2 is $ka = \pi$ (odd) soln

for $\lambda \gg 1$, $\frac{2ka}{\lambda}$ very close to ka axis

$\Rightarrow -\tan ka$ linear near π, w slope $= 1$

$-\tan ka \simeq -ka + \pi$



$$\Rightarrow \frac{2ka}{\lambda} = \pi - ka$$

$$ka\left(1 + \frac{2}{\lambda}\right) = \pi$$

$$ka = \pi\left(1 + \frac{2}{\lambda}\right)^{-1}$$

$$ka \simeq \pi\left(1 - \frac{2}{\lambda}\right) \quad \text{for } \lambda \gg 1$$

$$E = \frac{\hbar^2 k^2}{2m}$$

$$\Rightarrow E_1 = \frac{\hbar^2}{2m}\frac{\pi^2}{a^2}\left(1 - \frac{2}{\lambda}\right)^2$$

$$\boxed{E_1 \simeq \frac{\hbar^2 \pi^2}{2ma^2}\left(1 - \frac{4}{\lambda}\right)}$$

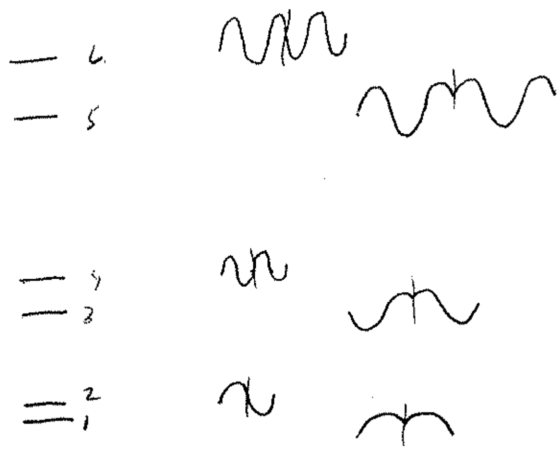
$$E_2 = \frac{\hbar^2}{2m}\frac{\pi^2}{a^2}$$

$$\boxed{E_2 = \frac{\hbar^2 \pi^2}{2ma^2}}$$

$\Rightarrow$ odd, even solns split by delta function.

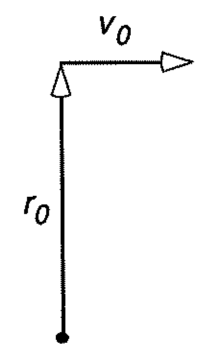
For $\lambda \to \infty$, solns degenerate.

Spectrum looks like:



odd solns unaffected by $\delta(x)$

There is an attractive central force field of constant magnitude mC with center at the origin. A particle of mass m is injected into this force field with a velocity v_0 at right angles to the initial position vector $\vec{r}_0$. Let β be the ratio of twice the initial kinetic energy to the initial potential energy measured with respect to the origin.



a) For what value of β will the ensuing motion be circular?

For all the next parts, consider β to be *slightly* less than the value for circular motion in a).

b) What is the half period of the motion, i.e., the time to go from the largest to the smallest distance from the origin? What is the corresponding angle through which the particle's radius vector has turned?

c) Determine the equation whose solution is the distance of closest approach. Solve it, keeping in mind that an algebraic equation of order n can easily be reduced to one of order $n-1$ if one root is known.

CM2 $F = mc$

$$V = PE = \int F\,dr = mcr$$

a) Circular motion : straight forward way

$$F = ma$$

$$mc = m\frac{v^2}{r_0} = \frac{2\,KE(0)}{PE(0)/mc} = mc\beta \Rightarrow \boxed{\beta_0 = 1}$$

b) $\underline{\beta < \beta_0 \quad \tau/2 = ?}$

So we have a perturbed circular orbit

$$E = KE + PE = \tfrac{1}{2}m(r^2\dot\theta^2 + \dot r^2) + mcr$$

yet $L = \text{const} = mr^2\dot\theta = m r_0 v_0$

$$E = \underbrace{\frac{L^2}{2mr^2} + mcr}_{V_{eff}(r)} + \tfrac{1}{2}m\dot r^2$$

circular when $F = \frac{-\partial V_{eff}}{\partial r} = 0 = \frac{+2L^2}{2mr^3} - mc = 0 \Rightarrow mc = \frac{L^2}{mr^3} = \frac{m^2r^4\dot\theta^2}{mr^3}$

or $mc = \frac{mv^2}{r}$ ✓ same as above ✓

look for harmonic restoring force for perturbation $r = r_0 + x$

$$F = \frac{L^2}{mr^3} - mc = \frac{L^2}{m(r_0+x)^3} - m = \frac{L^2}{mr_0^3(1+\frac{x}{r_0})^3} - m$$

$$F \simeq \underbrace{\frac{L^2}{mr_0^3} - m}_{0} - \frac{L^2}{mr_0^3}\left(3\frac{x}{r_0}\right) = -\frac{3L^2}{mr_0^4}\,x$$

$$F = ma$$

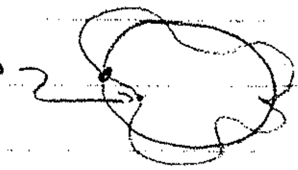
$$\frac{-3L^2}{mr_0^4} = m\ddot x \Rightarrow \ddot x = \frac{-3L^2}{m^2r_0^4}\,x = -\omega_0^2\,x \Rightarrow \omega_0 = \sqrt{3}\,\frac{L}{mr_0^2}$$

$$\omega_0 = \sqrt{3}\,\frac{L}{mr_0^2} = \sqrt{3}\,\frac{m r_0 v_0}{m r_0^2} = \sqrt{3}\,\frac{v_0}{r_0}$$

c) $L = mr^2\dot{\theta} = \text{const} \Rightarrow \dot{\theta} \simeq \frac{L}{mr^2} \leq \frac{m r_0 v_0}{m r_0^2} = \frac{v_0}{r_0}$

So $\Delta\theta = \dot{\theta}\,\Delta t = \frac{v_0}{r_0}\,\frac{\pi r_0}{\sqrt{3}\,v_0} = \frac{\pi}{\sqrt{3}}$

d) Distance of closest approach: when $\underline{KE} = 0$, $\dot{r} = 0$



$E = \frac{L^2}{2mr^2} + mcr + \frac{1}{2}m\dot{r}^2$

$E = \frac{L^2}{2mr^2} + mcr$ need solve for r

$2mr^2$:

$2m^2cr^3 - 2mr^2E + L^2 = 0$; yet $r = r_0$ is a root ∴

$(2m^2cr^3 - 2mr^2E + L^2) = (r - r_0)(2m^2cr^2 + \alpha r + \sigma)$ remove root

$= 2m^2cr^3 + \alpha r^2 + r\sigma - 2m^2cr_0r^2 - \alpha r r_0 - r_0\sigma$ now solve α, σ

r^3: $2m^2c = 2m^2c$ ✓

r^2: $-2mE = \alpha - 2m^2cr_0 \Rightarrow \underline{\alpha = 2m(mcr_0 - E)}$ } $\alpha < 0$, $\sigma < 0$

r: $\sigma - \alpha r_0 = 0 \Rightarrow \sigma = \alpha r_0 = 2m(cr_0 - E)r_0$

const: $L^2 = -r_0\sigma$ $\quad \sigma = -L^2/r_0$

root of $2m^2cr^2 + \alpha r + \sigma = 0$

$$r = \frac{-\alpha \pm \sqrt{\alpha^2 - 8m^2c\sigma}}{8m^2c\sigma}$$

OSU Physics Comprehensive Exam No. 86 29-30 March 1999 **Problem 5**

The pulsar within the Crab Nebula, about 6500 light years away, is a neutron star with a radius of about 10 km and a rotational frequency of 30/s. The mass is greater than that of our sun, and it produces a magnetic field perhaps as great as 10^{12} gauss at the magnetic polar surfaces. The magnetic dipole moment is inclined 30° from the rotational axis. The pulsar radiates electromagnetic waves through a number of different mechanisms.

a. A relatively meager amount of power is radiated as the magnetic dipole moment rotates. Calculate the power radiated per steradian.

b. A much greater power is radiated as charged particles orbiting the star interact with the magnetic field. Derive an expression for the power radiated from the vicinity of a magnetic pole for a single charged particle orbiting the star. These particles are actually moving at relativistic speeds, but a nonrelativistic analysis will suffice.

Consider a system of two distinquishable spin 1/2 particles. A set of basis kets for this system is $|++\rangle, |+-\rangle, |-+\rangle$, and $|--\rangle$, where the first (second) label in each ket denotes whether the spin projection of the first (second) particle along the z-axis is + or - $\hbar/2$. At time $t = 0$, the quantum state of the system is

$$|\psi(0)\rangle = \tfrac{1}{2}|++\rangle \;+\; \tfrac{1}{\sqrt{2}}|+-\rangle \;+\; \tfrac{1}{2}|--\rangle$$

a) At time $t = 0$, the spin projection of the first particle along the z-axis (S_{1z}) is measured. What is the probability of obtaining the result $-\hbar/2$? What is the quantum state vector after this measurement? What is the probability that a subsequent measurement of S_{1x} yields the value $+\hbar/2$?

b) What is the probability that the initial ($t = 0$) measurement of S_{1z} yields the result $+\hbar/2$? What is the quantum state vector after this measurement? What is the probability that a subsequent measurement of S_{1x} yields the value $+\hbar/2$?

c) Instead of performing the above measurements, consider placing the system in an external magnetic field $\vec{\mathbf{B}} = B_0\,\hat{\mathbf{z}}$ at $t = 0$. The magnetic moments of the particles can be written as $\vec{\mathbf{M}}_1 = \alpha\,\vec{\mathbf{S}}_1$ and $\vec{\mathbf{M}}_2 = \beta\,\vec{\mathbf{S}}_2$. Find the eigenvalues and eigenstates of the Hamiltonian. What is the state vector $|\psi(t)\rangle$ at a later time t?

You may want to know that

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$|\psi(0)\rangle = \frac{1}{2}|++\rangle + \frac{1}{\sqrt{2}}|+-\rangle + \frac{1}{2}|--\rangle$$

a) measure S_{1z}

$$P(S_{1z} = -\tfrac{\hbar}{2}) = \sum_{\varepsilon_2} |\langle -\varepsilon_2|\psi(0)\rangle|^2 \quad ; \quad \varepsilon_2 = +,-$$

$$= |\langle -+|\psi(0)\rangle|^2 + |\langle --|\psi(0)\rangle|^2$$

$$= \quad 0 \quad + \quad (\tfrac{1}{2})^2$$

$$\boxed{P(S_{1z} = -\tfrac{\hbar}{2}) = \frac{1}{4}}$$

after measurement project onto subspace

new ket $|\phi\rangle$.

$$|\phi\rangle = \frac{P_{1-}|\psi(0)\rangle}{\sqrt{\langle\psi(0)|P_{1-}|\psi(0)\rangle}}$$

projection postulate

w/ $P_{1-} = |-+\rangle\langle -+| + |--\rangle\langle --|$

$$P_{1-}|\psi(0)\rangle = \tfrac{1}{2}|--\rangle$$

$$\Rightarrow \boxed{|\phi\rangle = |--\rangle}$$

Now measure S_{1x}

recall that $|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$$P(S_{1x} = +\tfrac{\hbar}{2}) = \sum_{\epsilon_2} |\langle (+)_x\ \epsilon_2 | \phi\rangle|^2$$

$$= |\langle (+)_x + | \phi\rangle|^2 + |\langle (+)_x - | \phi\rangle|^2$$

$$= \left|\frac{1}{\sqrt{2}}(\langle ++| + \langle -+|)|\phi\rangle\right|^2 + \left|\frac{1}{\sqrt{2}}(\langle +-| + \langle --|)|\phi\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}(0+0)\right|^2 + \left|\frac{1}{\sqrt{2}}(0+1)\right|^2$$

$$\boxed{P(S_{1x} = +\tfrac{\hbar}{2}) = \frac{1}{2}}$$

b) $$P(S_{1z} = +\tfrac{\hbar}{2}) = \sum_{\epsilon_2} |\langle + \epsilon_2 | \Psi(0)\rangle|^2$$

$$= |\langle ++ | \Psi(0)\rangle|^2 + |\langle +- | \Psi(0)\rangle|^2$$

$$= \left|\frac{1}{2}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2$$

$$\boxed{P(S_{1z} = +\tfrac{\hbar}{2}) = \frac{3}{4}}$$

After measurement, new ket $|\alpha\rangle$

$$|\alpha\rangle = \frac{P_{1+} |\Psi(0)\rangle}{\sqrt{\langle \Psi(0) | P_{1+} | \Psi(0)\rangle}} \qquad ; \quad P_{1+} = |++\rangle\langle ++| + |+-\rangle\langle +-|$$

$$P_{1+}|\psi(0)\rangle = \frac{1}{2}|++\rangle + \frac{1}{\sqrt{2}}|+-\rangle$$

$$\langle\psi(0)|P_{1+}|\psi(0)\rangle = \frac{3}{4}$$

$$\Rightarrow \boxed{|\alpha\rangle = \frac{1}{\sqrt{3}}|++\rangle + \frac{\sqrt{2}}{\sqrt{3}}|+-\rangle}$$

Now measure S_{1x}

$$P(S_{1x} = +\tfrac{\hbar}{2}) = \left|\tfrac{1}{\sqrt{2}}(\langle++| + \langle-+|)|\alpha\rangle\right|^2 + \left|\tfrac{1}{\sqrt{2}}(\langle+-| + \langle--|)|\alpha\rangle\right|^2$$

$$= \left|\tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{3}} + 0\right)\right|^2 + \left|\tfrac{1}{\sqrt{2}}\left(\tfrac{\sqrt{2}}{\sqrt{3}} + 0\right)\right|^2$$

$$= \frac{1}{6} + \frac{1}{3}$$

$$\boxed{P(S_{1x} = +\tfrac{\hbar}{2}) = \frac{1}{2}}$$

c) $\vec{M}_1 = \alpha \vec{S}_1$, $\vec{M}_2 = \beta \vec{S}_2$, $\vec{B} = B_0 \hat{z}$

$$H = -\vec{M}\cdot\vec{B} = -\alpha B_0 S_{1z} - \beta B_0 S_{2z}$$

$\Rightarrow$ eigen states of energy same as eigenstates of S_{1z}, S_{2z} : $|\pm\pm\rangle$

$$\Rightarrow H|++\rangle = -\alpha B_0 \tfrac{\hbar}{2} - \beta B_0 \tfrac{\hbar}{2}$$

$$H|+-\rangle = -\alpha B_0 \tfrac{\hbar}{2} - \beta B_0 (-\tfrac{\hbar}{2})$$

$$H|-+\rangle = -\alpha B_0 (-\tfrac{\hbar}{2}) - \beta B_0 \tfrac{\hbar}{2}$$

$$H|--\rangle = -\alpha B_0 (-\tfrac{\hbar}{2}) - \beta B_0 (-\tfrac{\hbar}{2})$$

Eigenstates	Energy
$\|++\rangle$	$-\frac{B_0\hbar}{2}(\alpha+\beta)$
$\|+-\rangle$	$-\frac{B_0\hbar}{2}(\alpha-\beta)$
$\|-+\rangle$	$\frac{B_0\hbar}{2}(\alpha-\beta)$
$\|--\rangle$	$\frac{B_0\hbar}{2}(\alpha+\beta)$

Sch. evolution:

$$|\psi(t)\rangle = \frac{1}{2} e^{-i\frac{E_{++}}{\hbar}t}|++\rangle + \frac{1}{\sqrt{2}} e^{-i\frac{E_{+-}}{\hbar}t}|+-\rangle + \frac{1}{2} e^{-i\frac{E_{--}}{\hbar}t}|-$$

$$\Rightarrow |\psi(t)\rangle = \frac{1}{2} e^{i\frac{B_0 t}{2}(\alpha+\beta)}|++\rangle + \frac{1}{\sqrt{2}} e^{i\frac{B_0 t}{2}(\alpha-\beta)}|+-\rangle$$

$$+ \frac{1}{2} e^{-i\frac{B_0 t}{2}(\alpha+\beta)}|--\rangle$$

Two rigid classical dipoles, with dipole moments $\vec{m}_1$ and $\vec{m}_2$, are in thermal equilibrium with a heat bath at temperature T. Their centers are located on the x axis at points x_1 and x_2, respectively. The dipoles can rotate about the x axis in the yz plane (i.e., the plane perpendicular to the x axis). Calculate the mean force acting between these dipoles as a function of the distance between their centers (r_{12}):

(a) in the low-temperature limit (i.e., for $T \to 0$) **(3 pts.)**, and

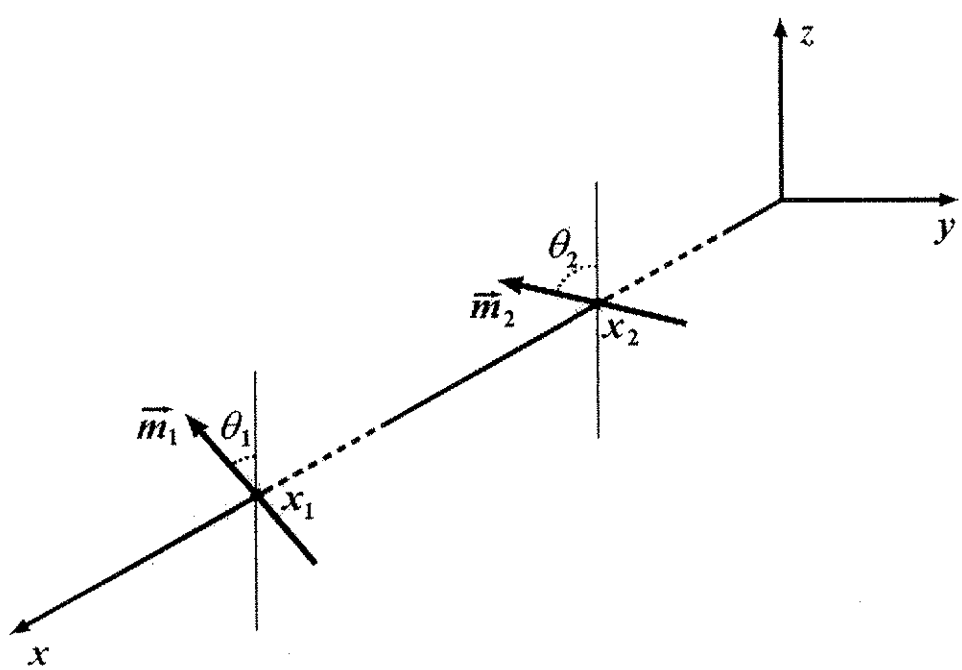
(b) at high temperatures (i.e., for $k_B T >> m_1 m_2/(r_{12})^3$). **(17 pts.)**.

Hints: **(i)** The energy of interaction between two dipoles is:

$$E_{\text{dip}} = \frac{\vec{m}_1 \bullet \vec{m}_2 - 3(\vec{m}_1 \bullet \hat{r}_{12})(\vec{m}_2 \bullet \hat{r}_{12})}{r_{12}^3} ,$$

where $\hat{r}_{12}$ denotes a unit vector parallel to $\vec{r}_{12}$.

(ii) It's a good idea to use angular coordinates; **(iii)** in **(b)**, expand the Boltzmann factor in the partition function up to the *second-order* term ($e^x = 1 + x + x^2 + \ldots..$).



Problem # 7 – Solution:

(a) the $3(\vec{m}_1 \cdot \hat{r}_{12})(\vec{m}_2 \cdot \hat{r}_{12})$ term vanishes, so there is only $\vec{m}_1 \vec{m}_2$ in the interaction energy. In angular coordinates, it's $m_1 m_2 \cos(\theta_1 - \theta_2)$. For $T \to 0$, the dipoles choose an energy-minimizing configuration $\theta_1 - \theta_2 = \pi$ (antiparallel), so that $E = -(m_1 m_2)/r_{12}^3$, and $F = -dE/dr_{12} = -3 m_1 m_2 / r_{12}^4$; the force is attractive (–), as expected.

(b) To investigate the finite T case, we need the partition f-tion (classical):

$$Z = \int_0^{2\pi}\int_0^{2\pi} d\theta_1 d\theta_2\, e^{-E(\theta_1,\theta_2)/kT} = \iint d\theta_1 d\theta_2\, e^{m_1 m_2 \cos(\theta_1-\theta_2)/kT r_{12}^3}$$

For $m_1 m_2 / kT r_{12}^3 \ll 1$ one can expand the exp function: $e^x \cong 1 + x$. But since $\cos(\theta_1 - \theta_2) = \sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2$, the linear term disappears in integration. So, we have to take one more term, $e^x \cong 1 + x + x^2/2$. Thus the exponential function is:

$$1 + (\text{the linear term}) + \tfrac{1}{2}(m_1 m_2 / kT r_{12}^3)^2 \cos^2(\theta_1 - \theta_2) =$$

$$= 1 + (\text{linear term}) + \tfrac{1}{2}(m_1 m_2 / kT r_{12}^3)(\cos^2\theta_1 \cos^2\theta_2 + \sin^2\theta_1 \sin^2\theta_2 + \tfrac{1}{2}\sin 2\theta_1 \sin 2\theta_2)$$

The term $\frac{1}{2}\sin 2\theta_1 \cdot \sin 2\theta_2$ vanishes on integration, and the other two yield π^2 each. So, the partition function becomes:

$$Z = 4\pi^2 + \tfrac{1}{2}(m_1 m_2 kT r_{12}^3)^2 \cdot 2\pi^2 = 4\pi^2\left[1 + (m_1 m_2)^2/4(kT)^2 r_{12}^6\right]$$

In the canonical formalism, the thermal average of the force $\langle F \rangle$ is:

$$\langle F \rangle = \frac{1}{Z}\iint d\theta_1 d\theta_2 \left(F(\theta_1,\theta_2)\, e^{-\frac{E(\theta_1,\theta_2)}{kT}}\right) = \frac{1}{Z}\iint d\theta_1 d\theta \left(-\frac{dE}{dr_{12}}\right) e^{-\frac{E}{kT}}$$

But $\left(-\frac{dE}{dr_{12}}\right) e^{-E/kT}$ is the same as $kT\left(\frac{d}{dr_{12}} e^{-E/kT}\right)$, so the integral in the above formula can be expressed as $\frac{d}{dr_{12}} kT \iint d\theta_1 d\theta_2\, e^{-E/kT} = kT \frac{d}{dr_{12}} Z$

So, $\langle F \rangle = \frac{kT}{Z}\frac{d}{dr_{12}} Z = kT \frac{d}{dr_{12}} \ln Z$. Using $\ln(1+\delta) \cong \delta$ for $\delta \ll 1$ (totally legitimate here), we get:

$$\langle F \rangle = kT \frac{d}{dr_{12}}\left(\frac{m_1 m_2^2}{4(kT)^2 r_{12}^6}\right) = -\frac{2}{3}\left(\frac{m_1 m_2}{kT}\right)^2 \cdot \frac{1}{r_{12}^7}$$

A very long cylinder oriented along the z axis carries the surface current density $\vec{K} = \hat{z}K\cos\varphi$. The radius of the cylinder is R, and $\mu = \mu_o$ inside and outside the surface.

a. In which direction does the vector potential $\vec{A}$ point?

b. What are the relevant boundary conditions?

c. Find the vector potential $\vec{A}$ and the magnetic field $\vec{B}$ for $\rho < \mathrm{R}$ and $\rho > \mathrm{R}$.