Department of Physics Comprehensive Examination No. 81

30 September - 1 October 1997

This Comprehensive Examination for Fall 1997 consists of eight problems each worth 20 points. The problems are grouped into four sessions, each of which lasts for three hours. Session One (problems 1 and 2) begins at 9:00 AM Tuesday 30 September. Session Two (problems 3 and 4) begins at 1:30PM Tuesday 30 September. Session Three (problems 5 and 6) begins at 9:00 AM Wednesday 1 October. Session Four (problems 7 and 8) begins at 1:30PM Wednesday 1 October.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it is possible to obtain partial credit, especially if you demonstrate conceptual understanding. Use no scratch paper. Do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter, but not your name, is on the inside of the back cover of every bluebook. Be sure to remember your student letter for use in the remaining sessions of the examination.

If something is omitted from the statement of the problem or you feel there is an ambiguity, please ask your question quietly and privately, so as not to disturb the others. Only your bluebooks and the examination should be on the table before you. Any other items should be stored on the floor. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

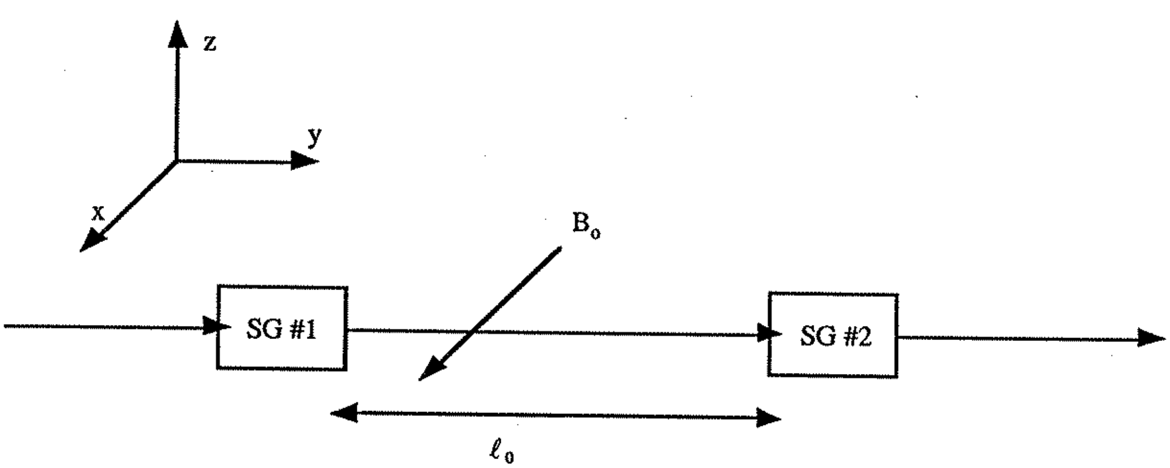
Use the last pages of your notebooks for scratch work separated by at least one page from your solutions. Scratch work will not be graded.

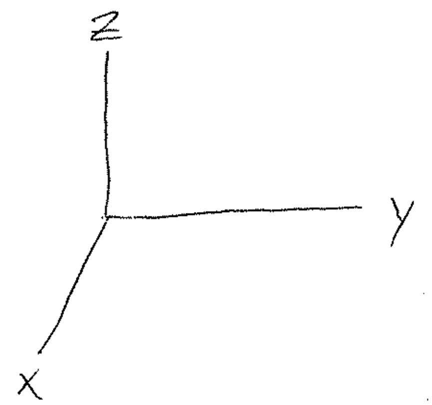
A beam of identical neutral particles with spin 1/2 (but no other angular momentum) travels along the y-axis. The beam passes through a series of two Stern-Gerlach (SG) spin analyzing magnets, each of which is designed to analyze the spin projection along the z-axis. Each Stern-Gerlach analyzer only allows particles with spin **up** (along the z-axis) to pass through. The particles travel at speed v_0 between the two analyzers, which are separated by a region of length ℓ_0 in which there is a uniform magnetic field B_0 pointing in the x-direction, as shown in the figure below. The particles have a gyromagnetic ratio γ (ratio of magnetic dipole moment to angular momentum).

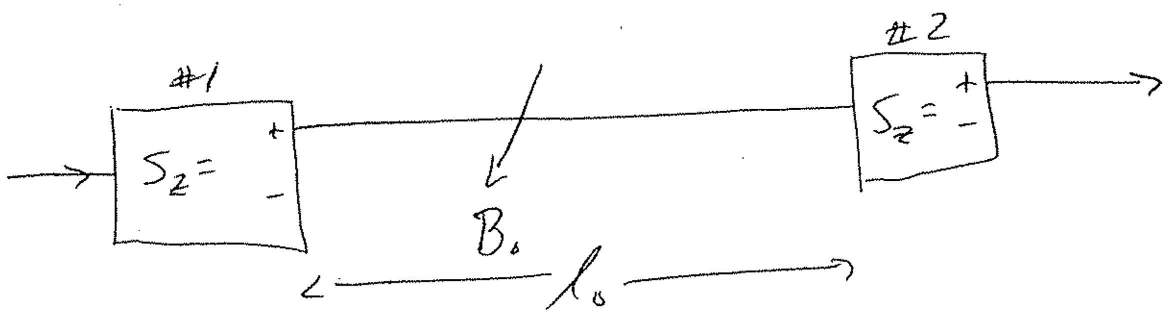
Determine the smallest value of ℓ_0 such that only 25% of the particles transmitted by the first analyzer are transmitted by the second analyzer.

You may want to know that

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$







let $|\psi(t=0)\rangle = |+\rangle$ be spin wave function of particle leaving analyzer #1

We nece to find $|\psi(t)\rangle$ at analyzer #2

such that

$$P_+ = |\langle +|\psi(t)\rangle|^2 = 0.25$$

Since particles travel at speed v_0 for distance l_0 between analyzers:

$$t = \frac{l_0}{v_0}$$

Hamiltonian of spin in magnetic field is

$$H = -\vec{\mu}\cdot\vec{B}$$

$\vec{\mu} = \gamma \vec{S}$ w/ γ = gyromagnetic ratio

$\Rightarrow H = -\gamma \vec{S} \cdot \vec{B}$

$= -\gamma S_x B_0$ since $\vec{B} = B_0 \hat{x}$

$\Rightarrow H = -\frac{\gamma \hbar B_0}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

so eigenstates of H are eigenstates of S_x

$\Rightarrow |\psi_1\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle + |-\rangle \right)$ w/ $E_1 = -\frac{\gamma \hbar B_0}{2}$

$|\psi_2\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle - |-\rangle \right)$ w/ $E_2 = +\frac{\gamma \hbar B_0}{2}$

$|\psi(t=0)\rangle = |+\rangle = \frac{1}{\sqrt{2}} \left(|\psi_1\rangle + |\psi_2\rangle \right)$

Now put in time evolution

$\Rightarrow |\psi(t)\rangle = \frac{1}{\sqrt{2}} \left(e^{-iE_1 t/\hbar} |\psi_1\rangle + e^{-iE_2 t/\hbar} |\psi_2\rangle \right)$

$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left(e^{i \frac{\gamma B_0}{2} t} |\psi_1\rangle + e^{-i \frac{\gamma B_0}{2} t} |\psi_2\rangle \right)$

Now take projection along ~~the~~ $\langle +|$ at 2^{nd} analyzer

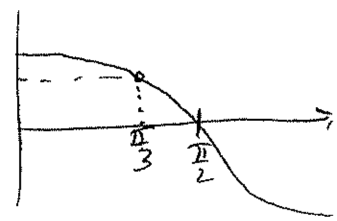
$\Rightarrow \langle + | \psi(t) \rangle = \frac{1}{2} e^{i \frac{\gamma B_0}{2} t} + \frac{1}{2} e^{-i \frac{\gamma B_0}{2} t}$

$= \cos \left(\frac{\gamma B_0}{2} t \right)$

$$\Rightarrow P_+ = \cos^2\left(\frac{\gamma B_0}{2} t\right)$$

to get 25% through after distance l_0, need:

$$\cos^2\left(\frac{\gamma B_0}{2} \frac{l_0}{v_0}\right) = \frac{1}{4}$$



$$\Rightarrow \frac{\gamma B_0 l_0}{2 v_0} = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow \frac{\gamma B_0 l_0}{2 v_0} = \frac{\pi}{3}$$

$$\boxed{l_0 = \frac{2\pi v_0}{3 \gamma B_0}}$$

A white dwarf star can be considered to be made up of carbon nuclei and free electrons. Such stars typically have a mass about that of the Sun, while occupying a volume about that of the Earth.

(8) (a) Establish a criterion, relating the electron number density and the temperature, which when satisfied, guarantees that the collection of electrons in the white dwarf star can be treated as a degenerate Fermi gas.

(12) (b) Given that the energy of an individual electron in such a degenerate Fermi gas can be written as $\varepsilon = \sqrt{(pc)^2 + (mc^2)^2}$, derive an expression for the pressure, P, exerted against gravity by this gas in (i) the non-relativistic limit, $mc^2 >> pc$, and (ii) the highly relativistic limit, $pc >> mc^2$.

Hint: Differentiation under the integral sign obeys the following general rule,

$$\frac{d}{d\alpha}\int_{u_0(\alpha)}^{u_1(\alpha)} f(x,\alpha)dx = f(u_1,\alpha)\frac{du_1}{d\alpha} - f(u_0,\alpha)\frac{du_0}{d\alpha} + \int_{u_0(\alpha)}^{u_1(\alpha)} \frac{\partial}{\partial\alpha}f(x,\alpha)dx$$

(a) The criterion is that the Fermi Energy, E_F, be much greater than $\frac{3}{2}kT$.

d^3p

$$N = 2\frac{V}{h^3}\int_0^{p_F} 4\pi p^2\,dp = \frac{V}{3\hbar^3\pi^2}\,p_F^3$$

or $$\frac{N}{V} = \frac{(2mE_F)^{3/2}}{3\pi^2\hbar^3} \quad \text{and so} \quad E_F = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}$$

where $n = N/V$, the number density of electrons.

Thus the condition that $E_F \gg \frac{3}{2}kT$ becomes

$$n \gg \frac{3^{1/2}(mkT)^{3/2}}{\pi^2\hbar^3}.$$

When this is met, the electron gas is a degenerate Fermi gas.

(b) $$E_0 = 2\sum_{|\vec{p}|<p_F}\sqrt{(pc)^2+(mc^2)^2} = 2\frac{V}{h^3}\int_0^{p_F} 4\pi p^2\sqrt{(pc)^2+(mc^2)^2}\,dp$$

$$E_0 = \frac{V mc^2}{\hbar^3\pi^2}\int_0^{p_F} p^2\left[1+\left(\frac{p}{mc}\right)^2\right]^{1/2} dp.$$

Put $x = \frac{p}{mc}$, then $$E_0 = \frac{V}{\hbar^3\pi^2}m^4c^5\int_0^{x_F} x^2(1+x^2)^{1/2}\,dx.$$

Now $P_0 = -\left(\frac{\partial E}{\partial V}\right)_T$. There will be a V dependence in x_F, via p_F's dependence on electron number density.

Making use of the hint allows

$$P_0 = \frac{m^4 c^5}{\hbar^3 \pi^2}\left[\frac{1}{3} x_F^3 \sqrt{1+x_F^2} - \int_0^{x_F} x^2 (1+x^2)^{1/2} dx\right]$$

(i) non-relativistic case ($x_F \ll 1$)

Integrand becomes $x^2(1+\frac{1}{2}x^2)$, and first term in bracket becomes $\frac{1}{3}x_F^3 + \frac{1}{6}x_F^5$.

$$P_0 = \frac{m^4 c^5}{\hbar^3 \pi^2}\left[\frac{1}{3}x_F^3 + \frac{1}{6}x_F^5 - \frac{x_F^3}{3} - \frac{x_F^5}{10}\right]$$

$$P_0 \simeq \frac{m^4 c^5}{15\pi^2 \hbar^3} x_F^5 .$$

(ii) relativistic case ($x_F \gg 1$)

Integrand becomes $x^3 + \frac{1}{2}x$, and first term in brackets becomes $x_F^4 + \frac{1}{2}x_F^2$.

$$P_0 \simeq \frac{m^4 c^5}{12\pi^2 \hbar^3}\left(x_F^4 - x_F^2\right)$$

where $x_F = \frac{p_F}{mc} = \frac{\hbar}{mc}\left(3\pi^2 n\right)^{1/3}$.

This problem is concerned with time-varying electric fields and conductors. Parts A and B are independent questions.

A. A plane wave of frequency ω is incident from air (n=1) upon a thick slab of metal of conductivity σ at $\theta = 0$ (normal incidence). Assuming that the dielectric function is

$$\varepsilon = 1 + i\frac{4\pi\sigma}{\omega},$$

that $\mu=1$ and that $\sigma/\omega >> 1$, find the power per unit area dissipated in the metal as a function of ω.

B. A time-varying electric field at a low frequency ω drives a current through a long, cylindrical, solid wire of radius d possessing a large conductivity σ. The wire is oriented along the z axis, and assume that $\mu=1$. The skin effect describes the fact that as ω increases, the time-dependent current density J is increasingly concentrated near the surface of the conductor. Show that the current density decreases exponentially (with skin depth $\delta = c / \sqrt{2\pi\sigma\omega}$) from the surface toward the center by following these two steps.

i. Develop a wave equation for the electric field inside the conductor.

ii. Find the electric field within the conductor by solving this equation in cylindrical coordinates.

Useful relations:

$$\sqrt{i} = (1+i)/\sqrt{2}$$

$$J_0(\sqrt{2i}x) \cong \sqrt{x}e^{(1-i)x}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}J_m(k\rho)\right) = \left(k^2 + \frac{m^2}{\rho^2}\right)J(k\rho)$$

A. $\epsilon = 1 + \frac{4\pi i\sigma}{\omega} \Rightarrow n = \sqrt{1+\frac{4\pi i\sigma}{\omega}} \sim \sqrt{\frac{4\pi\sigma}{\omega}}\sqrt{i} = \sqrt{\frac{2\pi\sigma}{\omega}}\,(1+i)$

Incident amplitude is $\mathcal{E}$, transmitted amplitude is $\mathcal{E}\frac{2}{1+n} = \mathcal{E}'$

Time-averaged power per unit area at the interface is $\langle \vec{S}' \rangle = \frac{c}{4\pi}\langle \vec{E}' \times \vec{H}' \rangle = \frac{c}{4\pi} n' \langle E'^2 \rangle$ into the metal

For large σ/ω, $\mathrm{Re}\langle S' \rangle = \frac{c}{8\pi}\sqrt{\frac{\omega}{2\pi\sigma}}\,\mathcal{E}^2$

B. In the wire $\vec{E}(t) = \vec{\mathcal{E}}e^{-i\omega t}$, $\vec{B}(t) = \vec{\mathcal{B}}e^{-i\omega t}$, $\vec{J}(t) = \vec{j}e^{-i\omega t}$

$$\vec{\nabla}\times\vec{\mathcal{B}} = \frac{4\pi}{c}\vec{j} - \frac{i\omega\epsilon}{c}\vec{\mathcal{E}} \simeq \frac{4\pi}{c}\vec{j} = \frac{4\pi\sigma}{c}\vec{\mathcal{E}}$$

$$\vec{\nabla}\times\vec{\mathcal{E}} = \frac{i\omega}{c}\vec{\mathcal{B}} \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{\mathcal{E}}) = \frac{i\omega}{c}\vec{\nabla}\times\vec{\mathcal{B}} = \frac{i\omega}{c}\frac{4\pi\sigma}{c}\vec{\mathcal{E}}$$

$$\vec{\nabla}\cdot\vec{\mathcal{E}} = 0 \Rightarrow -\nabla^2\vec{\mathcal{E}} = \frac{i\omega 4\pi\sigma}{c^2}\vec{\mathcal{E}} \quad \text{where } \vec{\mathcal{E}} = \mathcal{E}(\rho)\hat{z}$$

$$\vec{\mathcal{E}} = \mathcal{E}(\rho)\hat{z} \Rightarrow \nabla^2\mathcal{E} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\mathcal{E}}{\partial\rho}\right) \text{ only}$$

$$\nabla^2\mathcal{E} = -i\gamma^2\mathcal{E} \quad \text{where } \gamma^2 = 4\pi\omega\sigma/c^2 \Rightarrow \rho^2\frac{\partial^2}{\partial\rho^2}\mathcal{E} + \rho\frac{\partial}{\partial\rho}\mathcal{E} + i\gamma^2\rho^2\mathcal{E} = 0$$

The Solution is $J_0(\sqrt{i}\,\gamma\rho) = J_0\left(\sqrt{2i}\sqrt{\frac{2\pi\omega\sigma}{c^2}}\,\rho\right) \simeq \sqrt{\rho\sqrt{\frac{2\pi\omega\sigma}{c^2}}}\;e^{(1-i)\rho\sqrt{\frac{2\pi\omega\sigma}{c^2}}}$

Define skin depth $= \sqrt{c^2/2\pi\omega\sigma}$, then $J_0(\sqrt{i}\,\gamma\rho) \simeq \sqrt{\rho/\delta}\; e^{(1-i)\rho/\delta}$

So $\mathcal{E}(\rho) = A\sqrt{\frac{\rho}{\delta}}\,e^{\rho/\delta}\,e^{-i\rho/\delta}$ and $\mathcal{E}(\rho=a) = \mathcal{E}_0 \Rightarrow A = \mathcal{E}_0\sqrt{\frac{\delta}{a}}\,e^{-a/\delta}\,e^{ia/\delta}$

Then $\vec{E}(\rho,t) = \mathcal{E}_0\sqrt{\frac{\rho}{a}}\,e^{-(a-\rho)/\delta}\,e^{i(a-\rho)/\delta}\,e^{-i\omega t}\,\hat{z}$

And $\vec{J}(\rho,t) = \sigma\vec{E}(\rho,t)$

OSU Physics Depart. Comp. Exam #81. Sept. 30 - Oct. 1, 1997 **PROB 4**

Four self-guided missiles (*M1,...,M4*) are launched from the corners of an *ABCD* square, with side a (see Fig. 1). The target-seeking devices of the missiles are programmed in such a way that missile *M1* is homing in on missile *M2*, *M2* on *M3*, *M3* on *M4* and *M4* on *M1*. Assuming that all four missiles fly in the horizontal plane with the same constant speed V, and that the missile size is negligibly small compared to a, find:

(a) The total distance flown by each missile before the final crash at the square center O. You are supposed to solve this problem ***without*** using any calculus-based methods (such as, e.g., by solving a differential equation) — this can be done by applying an appropriate analysis **(10 pts).**

(b) the function describing the missile trajectory in the stationary frame— here you can use any method you want. After you find the function, check by appropriate integration whether the flight distance you have obtained using the non-calculus-based analysis is correct **(10 pts)**.

Hint to (b): it may be useful to remember that if a curve is described in polar coordinates: $r = r(\varphi)$, the angle α between the radius vector of a point on the curve, and a tangential line passing through this point is given by: $\cot\alpha = \frac{1}{r}\left(\frac{dr}{d\varphi}\right)$.

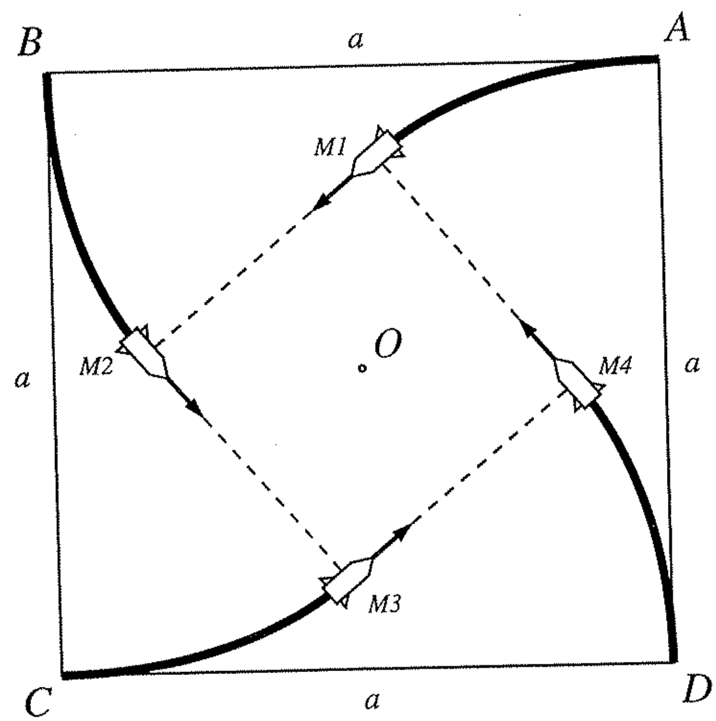


Figure 1. A schematic depicting the situation some time after the launch. The thick curves represent the missile trajectories. During the flight the missiles continuously adjust their courses, so that the speed vector of each missile (represented by the arrow) is always parallel to the line connecting the missile with its target (dashed lines in the figure).

Problem 4 - solution.

Question (a): the answer can be found without using any calculus! One has to notice that at any time before the final collision the four missiles are always positioned in the corners of a square. This square rotates and shrinks, of course, but always remains a square. Let's choose the two diagonals of that rotating & shrinking square as the axes of a new coordinate system. For an observer placed in the center of this system, each missle is approaching him/her along the axis it belongs to.

Note that the projection of the missle velocity on the rotating square diagonal is always $V/\sqrt{2}$. The distance the missile has to go in the rotating frame is simply one-half of the initial diagonal length, i.e. $a/\sqrt{2}$. Accordingly, the time needed to cover that distance is $\Delta t = (a/\sqrt{2})/(V/\sqrt{2}) = a/V$. So, since the speed in the fixed frame is V, the total distance flown is simply $S = V \cdot \Delta t = a$.

Question (b): The velocity vector $\vec{V}$ is tangential to the missle trajectory, and it always makes a 45° angle with the rotating square diagonal, which is, of course, the radius-vector of the curve. Thus, using the formula given in the hint, one can write:

$$\frac{1}{R}\frac{dR}{d\varphi} = -1$$

i.e.,

$$\frac{dR}{R} = -d\varphi$$

so:

$$\ln R = -\varphi + C$$

or:

$$R = C' e^{-\varphi}$$

The initial value of R, for $\varphi = 0$, is $a\sqrt{2}$. Hence, we get the following trajectory equation:

$$R(\varphi) = \frac{a}{\sqrt{2}}\, e^{-\varphi}$$

This is a logarithmic spiral.

The length S of any curve given by a $R = R(\varphi)$ equation can be obtained using the formula:

$$S = \int_{\varphi_1}^{\varphi_2} \sqrt{R^2 + \left(\frac{dR}{d\varphi}\right)^2}\, d\varphi$$

In the present case $\varphi_1 = 0$ and $\varphi_2 = \infty$, and $\frac{dR}{d\varphi} = -\frac{a}{\sqrt{2}} e^{-\varphi} = -R$.

So, the integral is simple to find:

$$S = \int_0^\infty \sqrt{\left(\frac{a}{\sqrt{2}} e^{-\varphi}\right)^2 + \left(-\frac{a}{\sqrt{2}} e^{-\varphi}\right)^2}\, d\varphi = \int_0^\infty \sqrt{a^2 e^{-2\varphi}}\, d\varphi =$$

$$= \int_0^\infty a e^{-\varphi} d\varphi = -a e^{-\varphi}\Big|_0^\infty = -a \cdot \phi + a \cdot 1 = a.$$

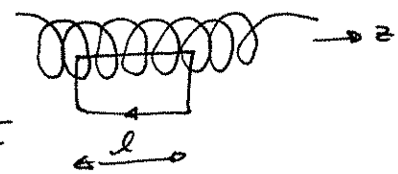
The result of the non-calculus solution in part (a) appears to be correct!

A very long solenoid of radius d with n turns of wire per unit length is subjected to a current $I(t) = I_o e^{-i\omega t}$. Find the time-dependent electric field inside the solenoid, and specify its direction and dependence upon the distance from the axis of the cylinder, ρ. μ=1 and ε=1 everywhere inside the solenoid.

For a slowly-varying current, $\vec{B}$ is approximately determined by the steady-current analysis

$$\vec{\nabla} \times \vec{B} = \frac{4\pi \vec{J}}{c} \Rightarrow$$ $\vec{B}$ is in the $\hat{z}$ direction and that

for the pictured loop



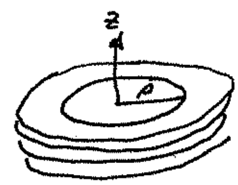
$$\int \vec{\nabla} \times \vec{B} \cdot d\vec{a} = \frac{4\pi}{c} \int \vec{J} \cdot d\vec{a} = \frac{4\pi}{c} n l I$$

So $\oint \vec{B} \cdot d\vec{l} = Bl = \frac{4\pi n l I}{c} \Rightarrow \vec{B} = \frac{4\pi n I}{c} \hat{z} e^{-i\omega t} = \mathcal{B} \hat{z} e^{-i\omega}$

Find $\vec{E}$ by using Faraday's Law with $\vec{E} = \vec{\mathcal{E}} e^{-i\omega t}$, $\vec{B} = \vec{\mathcal{B}} e^{-i\omega t}$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{\nabla} \times \vec{\mathcal{E}} = \frac{i\omega}{c} \mathcal{B}$$

Then $\int \vec{\nabla} \times \vec{\mathcal{E}} \cdot d\vec{a} = \frac{i\omega}{c} \int \vec{\mathcal{B}} \cdot d\vec{a}$ over ~~the~~ a surface $\perp \vec{z}$



So $\int \vec{\mathcal{E}} \cdot d\vec{l} = \mathcal{E}_\varphi 2\pi\rho = \frac{i\omega}{c} \mathcal{B} \pi \rho^2$

or $\mathcal{E}_\varphi = \frac{i\omega\rho\, 2\pi n I}{c^2}$

$$\vec{E} = \frac{i\omega\rho\, 2\pi n I}{c^2} \hat{\varphi} e^{-i\omega t}$$

Consider a 4-state system whose Hamiltonian has eigenstates labeled $|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle$, and $|\Psi_4\rangle$, with corresponding energies $E_1 = 3E_0$, $E_2 = 2E_0$, and $E_3 = E_4 = 0$. A perturbation W is applied to the system and is characterized by the following equations:

$$\begin{aligned}
W|\Psi_1\rangle &= a|\Psi_2\rangle + b|\Psi_3\rangle + b|\Psi_4\rangle \\
W|\Psi_2\rangle &= a|\Psi_1\rangle + b|\Psi_3\rangle + b|\Psi_4\rangle \\
W|\Psi_3\rangle &= b|\Psi_1\rangle + b|\Psi_2\rangle + c|\Psi_3\rangle + d|\Psi_4\rangle \\
W|\Psi_4\rangle &= b|\Psi_1\rangle + b|\Psi_2\rangle + d|\Psi_3\rangle + c|\Psi_4\rangle
\end{aligned}$$

where a ,b, c, and d are positive, real constants with dimensions of an energy.

a) (5 points) Write down the matrix representation of the full Hamiltonian $H = H_o + W$ in the basis defined by the states $|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle$, and $|\Psi_4\rangle$.

(b) (15 points) Find the first non-zero correction to each energy level caused by the perturbation.

$|\psi_1\rangle \qquad E_1 = 3E_0$

$|\psi_2\rangle \qquad E_2 = 2E_0$

$|\psi_3\rangle \qquad E_3 = 0$

$|\psi_4\rangle \qquad E_4 = 0$

$$\Rightarrow H_0 = \begin{pmatrix} 3E_0 & 0 & 0 & 0 \\ 0 & 2E_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$W|\psi_1\rangle = a|\psi_2\rangle + b|\psi_3\rangle + b|\psi_4\rangle$$

$$\Rightarrow \langle\psi_1|W|\psi_1\rangle = 0$$
$$\langle\psi_2|W|\psi_1\rangle = a$$
$$\langle\psi_3|W|\psi_1\rangle = b$$
$$\langle\psi_4|W|\psi_4\rangle = b$$

etc

$$\Rightarrow W = \begin{pmatrix} 0 & a & b & b \\ a & 0 & b & b \\ b & b & c & d \\ b & b & d & c \end{pmatrix}$$

$$\Rightarrow H = H_0 + W = \begin{pmatrix} 3E_0 & a & b & b \\ a & 2E_0 & b & b \\ b & b & c & d \\ b & b & d & c \end{pmatrix}$$

b) levels 1 & 2 are non-degenerate
levels 3 + 4 are degenerate

$\Rightarrow$ use corresponding perturbation theory.

1+2: use nondegenerate pert. th.

$$\Rightarrow \quad E_n^{(1)} = \langle \psi_n | W | \psi_n \rangle$$

but $\langle \psi_1 | W | \psi_1 \rangle = 0$
$\langle \psi_2 | W | \psi_2 \rangle = 0$

so need 2nd order pert. th.

$$E_n^{(2)} = \sum_{k \neq n} \frac{|\langle \psi_k | W | \psi_n \rangle|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$\Rightarrow E_1^{(2)} = \frac{a^2}{3E_0 - 2E_0} + \frac{b^2}{3E_0 - 0} + \frac{b^2}{3E_0 - 0}$$

$$E_1^{(2)} = \frac{a^2}{E_0} + \frac{2b^2}{3E_0}$$

$$E_2^{(2)} = \frac{a^2}{2E_0 - 3E_0} + \frac{b^2}{2E_0} + \frac{b^2}{2E_0}$$

$$E_2^{(2)} = -\frac{a^2}{E_0} + \frac{b^2}{E_0}$$

3+4: use degenerate pert. th.

So diagonalize W in subspace spanned by $|\psi_3\rangle$ & $|\psi_4\rangle$

call this W'

$$W' = \begin{pmatrix} c & d \\ d & c \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} (c-\lambda) & d \\ d & (c-\lambda) \end{vmatrix} = 0$$

$$(c-\lambda)^2 - d^2 = 0$$

$$c^2 - 2\lambda c + \lambda^2 - d^2 = 0$$

$$\lambda^2 - 2\lambda c + c^2 - d^2 = 0$$

$$\Rightarrow \lambda = c \pm \sqrt{c^2 - (c^2 - d^2)}$$

$$\lambda = c \pm d$$

$$\Rightarrow E_3^{(1)} = c + d$$

$$E_4^{(1)} = c - d$$

So perturbed energies are:

$$\boxed{\begin{aligned} E_1 &= 3E_0 + \frac{a^2}{E_0} + \frac{2b^2}{3E_0} \\ E_2 &= 2E_0 - \frac{a^2}{E_0} + \frac{b^2}{E_0} \\ E_3 &= c + d \\ E_4 &= c - d \end{aligned}}$$

OSU Physics Depart. Comp. Exam #81. Sept. 29-30, 1997 **PROB 7**

A double pendulum consists of mass m_1 attached to a weightless rod with length l_1, and mass m_2 attached to a weightless rod with length l_2. The upper end of rod l_1 is hinged at a fixed point, and the upper end of rod l_2 is hinged at mass m_1 (Fig. 1). Analyze small oscillations of this system in the xz plane.

Hint: choose the angles φ_1 and φ_2 (Fig. 1) as the generalized coordinates, and expand the trigonometric functions of φ_1 and φ_2 to the second-order terms. While solving the Lagrange equations, seek solutions in the form: $\varphi_1 = \alpha_1 e^{i\omega t}$, $\varphi_2 = \alpha_2 e^{i\omega t}$.

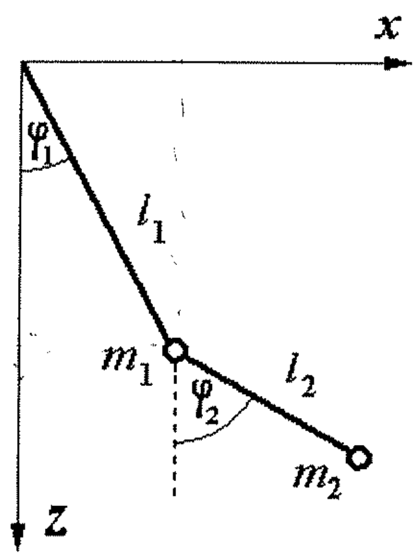


Figure 1. The double pendulum.

roblem 7 – solution:

The potential energy is: $V = -m_1 g z_1 - m_2 g z_2$

The x, y coordinates of the masses are:

$$x_1 = l_1 \sin\varphi_1$$
$$z_1 = l_1 \cos\varphi_1$$

$$x_2 = l_1 \sin\varphi_1 + l_2 \sin\varphi_2$$
$$z_2 = l_1 \cos\varphi_1 + l_2 \cos\varphi_2$$

Using expansions, and dropping the φ^n terms with $\varphi \geqslant 3$:

$$x_1 \cong l_1 \varphi_1$$
$$z_1 \cong l_1\left(1 - \frac{\varphi_1^2}{2}\right)$$
$$x_2 \cong l_1 \varphi_1 + l_2 \varphi_2$$
$$z_2 \cong l_1\left(1 - \frac{\varphi_1^2}{2}\right) + l_2\left(1 - \frac{\varphi_2^2}{2}\right)$$

The Lagrangian in (x, z) coordinates is:

$$L = T - V = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{z}_1^2\right) + \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{z}_2^2\right) + m_1 g z_1 + m_2 g z_2$$

Dropping all terms of the third order and higher in φ_1, $\dot{\varphi}_1$, φ_2 and $\dot{\varphi}_2$, we obtain:

$$L \cong \frac{m_1}{2} l_1^2 \dot{\varphi}_1^2 + \frac{m_2}{2}\left(l_1 \dot{\varphi}_1 + l_2 \dot{\varphi}_2\right)^2 - \frac{(m_1 + m_2) g l_1}{2} \varphi_1^2 -$$
$$- \frac{m_2 g l_2}{2} \varphi_2^2 + \text{const.}$$

The Lagrange Eqs. are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}_j} - \frac{\partial L}{\partial \varphi_j} = 0 \qquad (j = 1, 2)$$

Inserting the Lagrangian, we obtain:

$$\begin{cases} (m_1+m_2)\, l_1 \ddot{\varphi}_1 + m_2 l_2 \ddot{\varphi}_2 + (m_1+m_2) g \varphi_1 = 0 \\ l_1 \ddot{\varphi}_1 + l_2 \ddot{\varphi}_2 + g \varphi_2 = 0 \end{cases}$$

Now, we seek solutions in the form:

$$\varphi_1 = \alpha_1 e^{i\omega t} \quad ; \quad \varphi_2 = \alpha_2 e^{i\omega t}$$

Inserting into the eq. system, one gets:

$$\begin{cases} (m_1+m_2)(g - l_1\omega^2)\alpha_1 - m_2 l_2 \omega^2 \alpha_2 = 0 \\ -l_1\omega^2\alpha_1 + (g - l_2\omega^2)\alpha_2 = 0 \end{cases} \qquad (*)$$

This system has non-zero solutions for α_1, α_2 only if its determinant is zero:

$$\begin{vmatrix} (m_1+m_2)(g - l_1\omega^2) & m_2 l_2 \omega^2 \\ -l_1\omega^2 & (g - l_2\omega^2) \end{vmatrix} = 0$$

From which we get:

$$m_1 l_1 l_2 \omega^4 - g(m_1+m_2)(l_1+l_2)\,\omega^2 + (m_1+m_2) g^2 = 0$$

This equation has two solutions for ω^2, call them ω_1^2 and ω_2^2; after some tedious but straightforward algebra one obtains:

$$\left.\begin{matrix}\omega_1^2\\ \omega_2^2\end{matrix}\right\} = \frac{(m_1+m_2)(l_1+l_2)}{2m_1 l_1 l_2} g \pm g\frac{\sqrt{m_1+m_2}}{2m_1 l_1 l_2} \times \sqrt{m_1(l_1-l_2)^2 + m_2(l_1+l_2)^2}$$

The sum in the rightmost square root is always >0, so ω_1^2 and ω_2^2 are different ($\omega_1^2 \neq \omega_2^2$) and are real numbers.

Furthermore, the second right-hand term is smaller then the first one — one can readily see that by replacing (l_1-l_2) in the second term by (l_1+l_2); then the ~~se~~ second terms becomes larger and equal to the first one. So, ω_1^2 and ω_2^2 may take only <u>positive</u> values. Hence, there are four <u>real</u> solutions:

$$+\omega_1, -\omega_1, +\omega_2, -\omega_2$$

We still have to determine the constants α_1, α_2 in the exponential functions for φ_1, φ_2. After inserting ω_1^2 and ω_2^2 into the equations marked with (*) on the preceding page, we get two sets of equations which yield two different pairs of α_1, α_2 coefficients. Let's denote them as $\alpha_1^{(1)}, \alpha_2^{(1)}$, and $\alpha_1^{(2)}, \alpha_2^{(2)}$.

From the second of the (*) equations we can find that for both pairs of α:

$$\frac{\alpha_1^{(n)}}{\alpha_2^{(n)}} = \frac{g}{l_1\omega_n^2} - \frac{l_2}{l_1}; \quad n=1,2 \qquad (**)$$

The ratio value is the same for both $+\omega_n$ and $-\omega_n$. Hence, we obtain four linearly independent solutions so of the original Lagrange equations:

$$\varphi_1 = \alpha_1^{(1)} e^{i\omega_1 t}, \quad \varphi_2 = \alpha_2^{(1)} e^{i\omega_1 t}; \quad \varphi_1 = \alpha_1^{(1)} e^{-i\omega_1 t}, \quad \varphi_2 = \alpha_2^{(1)} e^{-i\omega_1 t}$$

$$\varphi_1 = \alpha_1^{(2)} e^{i\omega_2 t}, \quad \varphi_2 = \alpha_2^{(2)} e^{i\omega_2 t}; \quad \varphi_1 = \alpha_1^{(2)} e^{-i\omega_2 t}, \quad \varphi_2 = \alpha_2^{(2)} e^{-i\omega_2 t}$$

Taking advantage of:

$$\frac{e^{ix}+e^{-ix}}{2} = \cos x \,, \quad \frac{e^{ix}-e^{-ix}}{2} = \sin x \,,$$

we can convert the exponential complex functions into real trigonometric functions. Accordingly, we obtain four linearily independent solutions for φ_1, φ_2 in terms of real functions:

$$\varphi_1 = \alpha_1^{(1)} \cos \omega_1 t \,, \quad \varphi_2 = \alpha_2^{(1)} \cos \omega_1 t$$

$$\varphi_1 = \alpha_1^{(1)} \sin \omega_1 t \,, \quad \varphi_2 = \alpha_2^{(1)} \sin \omega_1 t$$

$$\varphi_1 = \alpha_1^{(2)} \cos \omega_2 t \,, \quad \varphi_2 = \alpha_2^{(2)} \cos \omega_2 t$$

$$\varphi_1 = \alpha_1^{(2)} \sin \omega_1 t \,, \quad \varphi_2 = \alpha_2^{(2)} \sin \omega_2 t$$

The general solution can be constructed from these special solutions by taking their linear combination with arbitrary coefficients $a_{1 \dots 4}$:

$$\varphi_i = a_1 \alpha_i^{(1)} \cos \omega_1 t + a_2 \alpha_i^{(1)} \sin \omega_1 t + a_3 \alpha_i^{(2)} \cos \omega_2 t \cdot$$
$$+ a_4 \alpha_i^{(2)} \sin \omega_2 t \quad \text{with, } (i = 1, 2)$$

This can be transformed into equivalent form:

$$\varphi_i = h_1 \alpha_i^{(1)} \cos(\omega_1 t + \gamma_1) + h_2 \alpha_i^{(2)} \cos(\omega_2 t + \gamma_2)$$
$$\text{with } (i = 1, 2).$$

The $h_1, h_2, \gamma_1, \gamma_2$ are arbitrary constants which have to be determined from the boundary conditions.

The $\alpha_1^{(i)}, \alpha_2^{(i)}$ coefficients (the "amplitudes"), let's recall, are constants that fulfill the (**) ~~equat~~ relationship.

One mole of a van der Waals gas is characterized by the following equation of state,

$$\left(P+\frac{a}{V^2}\right)(V-b)=RT$$

(7) (a) Show that the heat capacity at constant volume, C_V, is independent of V.

(7) (b) Under the assumption that C_V is also independent of T, show that the internal energy, E, of the gas is given by

$$E=C_V T-\frac{a}{V}+const$$

(6) (c) Also under the assumption that C_V is independent of T, show that the entropy, S, of the gas is given by

$$S=C_V \ln T+R\ln(V-b)+const'.$$

$$dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV \quad ; \quad T\,dS = C_V\,dT + T\left(\frac{\partial P}{\partial T}\right)_V dV$$

But also, $T\,dS = dE + P\,dV$ and so

$$dE = C_V\,dT + \left[T\left(\frac{\partial P}{\partial T}\right)_V - P\right] dV$$

which means that $\displaystyle\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$

(a) Now $C_V = \left(\frac{\partial E}{\partial T}\right)_V$, and we want to check for any V dependence, so let's examine

$$\left(\frac{\partial C_V}{\partial V}\right)_T = \frac{\partial^2 E}{\partial V\,\partial T} = \frac{\partial}{\partial T}\left(\frac{\partial E}{\partial V}\right)_T = \frac{\partial}{\partial T}\left[T\left(\frac{\partial P}{\partial T}\right)_V - P\right]$$

$$= T\frac{\partial^2 P}{\partial T^2} = T\frac{\partial}{\partial T}\left[\frac{R}{V-b}\right] = 0 .$$

$\therefore$ C_V is independent of V.

(b) $dE = C_V\,dT + \left[T\left(\frac{\partial P}{\partial T}\right)_V - P\right] dV$, from above.

$$\therefore \quad dE = C_V\,dT + \frac{a}{V^2}\,dV$$

$$E = C_V T - \frac{a}{V} + \text{const.}$$

(c) $dS = \dfrac{C_V}{T}\,dT + \left(\dfrac{\partial P}{\partial T}\right)_V dV$, from above.

$$S = \int C_V \frac{dT}{T} + \int \frac{R}{V-b}\,dV$$

$$S = C_V \ln T + R \ln(V-b) + \text{const}'.$$