DEPARTMENT COMPREHENSIVE EXAMINATION #79

January 6 and 7, 1997

Comprehensive Examination for Winter 1997

PART I

General Instructions

This Comprehensive Examination for Winter 1997 (#79) consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, January 6, and lasts three hours. The second part (Problems 3-4) will be handed out on the same day, at 1:30 pm, and also lasts three hours. The third and fourth parts (Problems 5-6 and Problems 7-8) will be administered in the same way on Tuesday, January 7, 1997.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit - especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulae and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

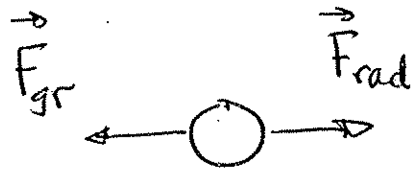
Use the last pages of your bluebooks for "scratch" work separated by at least one page from your solutions. "Scratch" work will not be graded.

An "artificial planet" has the form of a large spherical balloon made of a very thin foil - the mass of one square meter of this material being σ (there are no other supporting elements since the balloon retains its shape under weightless conditions). The balloon is launched into orbit around the Sun, such that it ends up moving in the Earth's orbit (assumed circular). The balloon has a perfectly black surface, and the integrated energy flux of the Sun's radiation at the location of the balloon (I.e., the total energy of all photons passing through one square meter in one second) is Σ. Show that the balloon's period around the Sun, T_b, is related to the Earth's period around the Sun, T_e, by

$$T_b^2 = T_e^2 \left(1 - \frac{r^2 \Sigma}{4GM_S \sigma c}\right)^{-1}$$

where r is the radius of the balloon's (or Earth's) orbit, M_S, is the mass of the Sun, and G is the constant of proportionality in Newton's Universal Law of Gravitation.

#1



Photons: $p = \frac{E}{c}$ or $P = \frac{\Sigma}{c}$

$$F_{rad} = P \pi a^2$$

$$F_{gr} = \frac{m G M_\odot}{r^2}$$

$$\therefore \frac{G m M_\odot}{r^2} - \frac{\Sigma}{c} \pi a^2 = m r \omega^2$$

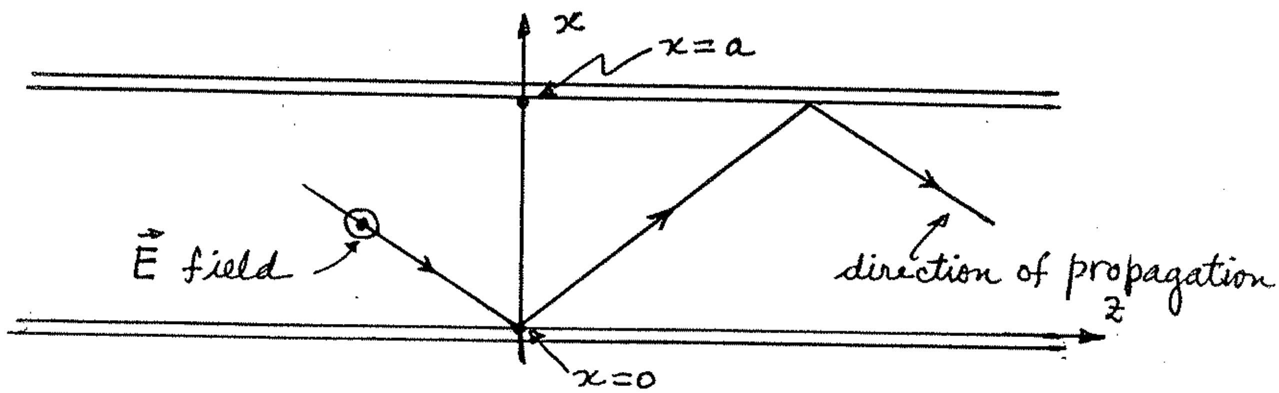
But $\omega = \frac{2\pi}{T}$

$$\therefore \quad T^2 = \frac{16\pi^2 \sigma c r^3}{4\sigma c G M_\odot - r^2 \Sigma} = \frac{4\pi^2 r^3}{G M_\odot - \frac{r^2 \Sigma}{4\sigma c}}$$

Now if the balloon orbits at Earth's distance from Sun, then

$$T^2 = \frac{T_\oplus^2}{1 - \frac{r^2 \Sigma}{4 G M_\odot \sigma c}}.$$

An electromagnetic plane wave is propagating between two infinite, parallel, perfectly conducting plates as shown in the sketch. For simplicity, assume that the conductors are located at x = 0 and x = a. Assume that the electric field is in the y-direction. The direction of propagation of the wave can have x and z components by no y component.



(a) Use Maxwell's equations to decide what boundary conditions the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields satisfy at the surface of the conductor.

(b) Use the boundary conditions to decide what frequencies can propagate in this way between the plates.

(c) Calculate the Poynting vector. In which direction does it point?

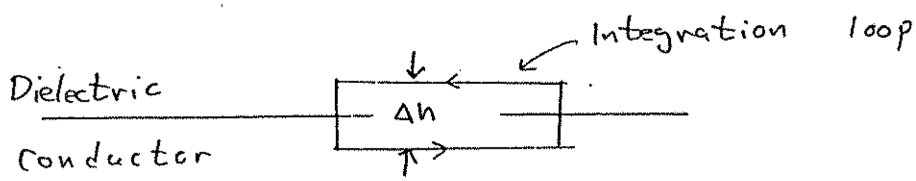
#2

1 (a) For time-harmonic fields:

$$\vec{\nabla} \times \vec{E}_\omega = i K \vec{B}_\omega$$

$$\vec{\nabla} \times \vec{H}_\omega = -i K \vec{E}_\omega + \frac{4\pi}{c} \vec{J}_\omega$$

Inside a perfect conductor $\vec{E} = 0$. at the surface

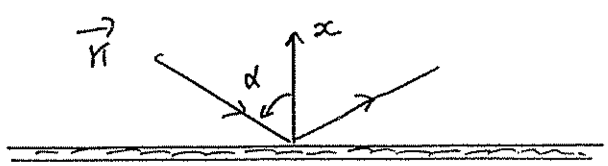


$$\oint \vec{E}_\omega \cdot d\vec{l} = i K \int_S \vec{B} \cdot \hat{n} \, da \Rightarrow 0 \quad \text{as } \Delta h \Rightarrow 0$$

so the tangential component of $\vec{E}$ vanishes at surface.

If $\vec{E} = 0$ then $\vec{B} = 0$. Since $\vec{\nabla} \cdot \vec{B} = 0$, the normal component of $\vec{B}$ must vanish. Because of the possibility of surface charge and current, we can say nothing about $B_\parallel$ & $E_\perp$.

(b.) Look what happens at $x = 0$



$$\vec{E}\,(in) = \hat{y} E_0^+ e^{i \vec{K} \cdot \vec{r}} = \hat{y} E_0^+ e^{i(-x \cos\alpha + z \sin\alpha) K}$$

$$\vec{E}\,(out) = \hat{y} E_0^- e^{i(x \cos\alpha + z \sin\alpha) K}$$

at $x=0$ $\quad \vec{E}(in) + \vec{E}(out) = 0$ $\quad$ so $E_0^+ = -E_0^-$.

Then when $x>0$

$$\vec{E}(in) + \vec{E}(out) = \hat{y} E_0 e^{ikz \sin\alpha} \left\{ e^{-ikx\cos\alpha} - e^{+ikx\cos\alpha} \right\}$$

$$= -2i\hat{y} E_0 e^{ikz\sin\alpha} \sin(kx\cos\alpha)$$

this must also vanish when $x = a$, so

$$\sin(ka\cos\alpha) = 0$$

$$ka\cos\alpha = m\pi \qquad m = 1, 2, 3 \ldots$$

$$\text{so } k > \pi/a \text{ or } \omega > \pi c/a$$

this is the minimum cut-off frequency. all higher frequencies propagate.

(c) $S_x \neq 0$, but if we average over the distance $0 \leq x \leq a$ then $\langle S_x \rangle = 0$, since there can be no net flow of power in this direction.

$$S_z = -\frac{c}{8\pi} E_y B_x^*$$

$$B_x = \frac{2ik_2}{k} E_0 \sin\left(\frac{m\pi x}{a}\right) e^{ik_2 z}$$

$$E_y = -2i E_0 \sin\left(\frac{m\pi x}{a}\right) e^{ik_2 z}$$

where $\quad K_x \equiv K \sin\alpha$

Finally $\quad S_z = \left(\frac{c}{2\pi}\right)\left(\frac{K_x}{K}\right)|E_0|^2 \sin^2\left(\frac{m\pi x}{a}\right)$

Consider a system of N_a non-interacting atoms, represented by harmonic oscillators in equilibrium with a heat reservoir at absolute temperature, T. Let the possible energy levels of these harmonic oscillators be given by

$$\varepsilon_n = \left(n+\tfrac{1}{2}\right)\hbar\,\omega_0,$$

where ω_0 is the classical frequency at which all of the harmonic oscillators are considered to oscillate.

(a) Derive an expression for the specific heat at constant volume of the oscillators in the high temperature limit.

(b) Derive an expression for the specific heat at constant volume of the oscillators in the low temperature limit.

#3

$$Z = \left(\sum_{n=0}^{\infty} e^{-\beta E_n}\right)^{3N_a} = \left[e^{-\frac{1}{2}\beta\hbar\omega_0} \sum_{n=0}^{\infty} e^{-n\beta\hbar\omega_0}\right]^{3N_a}$$

$$= \left(\frac{e^{-\frac{1}{2}\beta\hbar\omega_0}}{1-e^{-\beta\hbar\omega_0}}\right)^{3N_a}$$

$$\log Z = -\frac{3N_a}{2}\beta\hbar\omega_0 - 3N_a \ln\left(1-e^{-\beta\hbar\omega_0}\right)$$

$$\bar{E} = -\frac{\partial}{\partial\beta}\log Z = -\left[-\frac{1}{2}\hbar\omega_0 - \frac{\hbar\omega_0 e^{-\beta\hbar\omega_0}}{1-e^{-\beta\hbar\omega_0}}\right]3N_a$$

$$\therefore \bar{E} = 3N_a\hbar\omega_0\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega_0}-1}\right)$$

$$C_v = \left(\frac{\partial \bar{E}}{\partial T}\right)_V = -\frac{1}{kT^2}\left(\frac{\partial \bar{E}}{\partial\beta}\right)_V = -\frac{3N_a\hbar\omega_0}{kT^2}\left[\frac{-\hbar\omega_0 e^{\beta\hbar\omega_0}}{(e^{\beta\hbar\omega_0}-1)^2}\right]$$

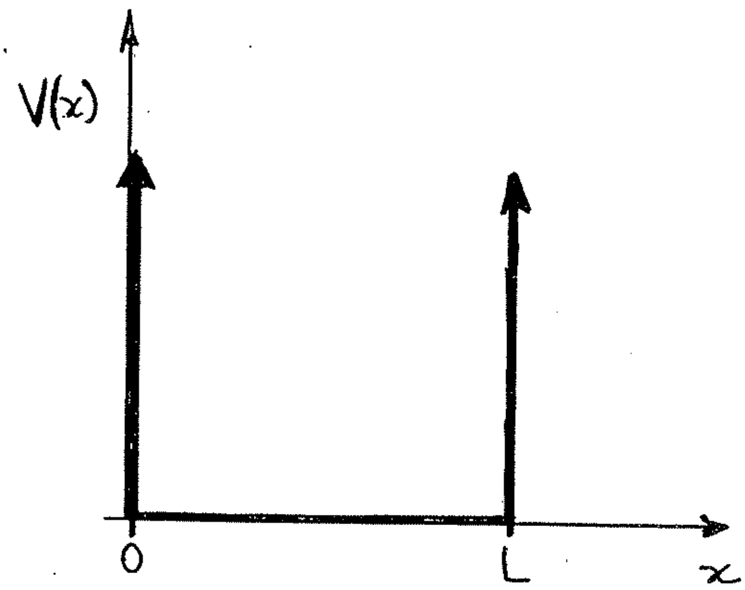
$$\therefore \quad C_v = 3R\left(\frac{\Theta_0}{T}\right)^2 \frac{e^{\Theta_0/T}}{(e^{\Theta_0/T}-1)^2}$$

where $R \equiv N_a k$, and $\Theta_0 = \hbar\omega_0/k$.

(a) when $T \gg \Theta_0$; $\quad C_v \simeq 3R$

(b) when $T \ll \Theta_0$; $\quad C_v \propto e^{-\Theta_0/T}$; $\left[3R\frac{\Theta_0^2}{T^2}e^{-\Theta_0/T}\right]$

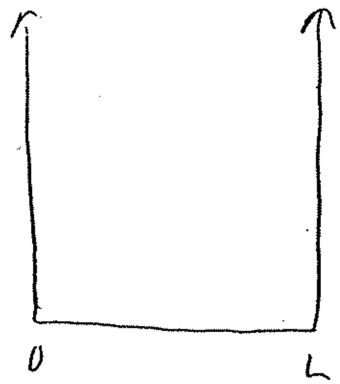
Consider a particle of mass m confined to an infinite square well potential as shown below.



(a) At $t = 0$, the particle's wave function is given by $\Psi(x,0) = \phi_n(x)$, where $\phi_n(x)$ is the eigenstate corresponding to the nth energy level above the ground state ($n = 1$). Find $\Psi(x,t)$ and $\langle E \rangle$, $\langle x \rangle$, and $\langle p \rangle$, the expectation values of the particle's energy, position, and momentum, respectively.

(b) Instead, assume that at $t = 0$, the particle has a probability of 4/5 to be in the state E_1 and a probability of 1/5 to be in the state E_2. Find $\Psi(x,0)$ and $\Psi(x,t)$, and the probability that a measurement of the energy at time $t = \frac{\hbar}{E_1}$ yields the value E_2. If, in fact, the energy is measured to be E_2, find $\Psi(x,t)$ for $t > \frac{\hbar}{E_1}$.

#4



eigenstates: $\phi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$

eigenenergies $E_n = n^2 \frac{\hbar^2 \pi^2}{2mL^2} = n^2 E_1$

a) $\Psi(x,0) = \phi_n(x)$

$$\Psi(x,t) = \exp\left(\frac{-iE_n t}{\hbar}\right) \phi_n(x)$$

$$\Psi(x,t) = \exp\left(\frac{-in^2 E_1 t}{\hbar}\right) \phi_n(x)$$

$$\langle E \rangle = \langle \Psi(x,t) | H | \Psi(x,t) \rangle$$

if $\Psi(x,t) = \sum a_{n'}(t)\, \phi_{n'}(x)$, then

$$\langle E \rangle = \sum_{n'} |a_{n'}(t)|^2 E_{n'}$$

and $P(E_{n'}) = |a_{n'}(t)|^2$ gives probability of measuring $E_{n'}$

For our case of $\Psi(x,t)$; $a_{n'}(t) = \delta_{nn'} \exp\left(\frac{-in^2 E_1 t}{\hbar}\right)$

$$\Rightarrow |a_{n'}(t)|^2 = \delta_{nn'}$$

$$\Rightarrow \boxed{\langle E \rangle = E_n}$$

i.e. since system is in energy eigenstate, measured energy must be E_n.

$$\langle x\rangle = \langle \psi | x | \psi\rangle$$

$$= \int_0^L \phi_n^*(x)\, e^{\frac{+in^2E_1t}{\hbar}}\, x\, e^{\frac{-in^2E_1t}{\hbar}}\, \phi_n(x)\, dx$$

$$= \int_0^L x\, |\phi_n^*(x)|^2\, dx$$

All $|\phi_n(x)|^2$ are symmetric w.r.t. $x = \frac{L}{2}$

$$\Rightarrow \boxed{\langle x\rangle = \frac{L}{2}}$$

$$\langle p\rangle = \langle \psi | p | \psi\rangle$$

$$= \int_0^L \phi_n^*(x)\, e^{\frac{+in^2E_1t}{\hbar}} \left(-i\hbar \frac{\partial}{\partial x}\right) e^{\frac{-in^2E_1t}{\hbar}}\, \phi_n(x)\, dx$$

$$= -i\hbar \int_0^L \phi_n^*(x)\, \frac{\partial}{\partial x}\, \phi_n(x)\, dx$$

$$= -i\hbar\, \frac{2}{L}\, \frac{n\pi}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx$$

$\boxed{\langle p\rangle = 0}$ since sin & cos orthogonal

b) Now at $t=0$, $P(E_1) = 4/5$
$P(E_2) = 1/5$

$\Rightarrow a_1(0) = 2/\sqrt{5}$
$a_2(0) = 1/\sqrt{5}$

$$\Rightarrow \psi(x,0) = \frac{2}{\sqrt{5}}\phi_1(x) + \frac{1}{\sqrt{5}}\phi_2(x)$$

as above $\psi(x,t) = \frac{2}{\sqrt{5}} \exp\left(-i\frac{E_1t}{\hbar}\right)\phi_1(x) + \frac{1}{\sqrt{5}} \exp\left(\frac{-i4E_1t}{\hbar}\right)\phi_2(x)$

$P(E_{n'})$ is independent of time, so

$P(E_2) = 1/5$ at all times ~~and~~ $\Rightarrow$ at $t = \frac{E_1}{\hbar}$

if E_2 measured, then state projects onto ϕ_2

$$\Rightarrow \Psi(x,t) = \exp\left(\frac{-i4E_1t}{\hbar}\right)\phi_2(x) \qquad t > \frac{E_1}{\hbar}$$

A rigid cylinder of radius R is attached to a solid foundation with its axis horizontally oriented. A solid uniform bar of rectangular cross-section, length $2l$, thickness $2d$, and mass M, is placed on the top of the cylinder (Fig. 1). The $2l$ edge of the bar is perpendicular to the cylinder axis. When the bar is horizontal, its center of gravity (point C in Fig. 1) is located exactly above the axis of the cylinder.

(a) For what values of d/R is this position a stable equilibrium, and for what values is it an unstable equilibrium?

(b) Find the frequency of small "rocking" oscillations of the bar about the stable equilibrium position (see Fig. 2 — assume that within the entire range of relevant rocking angles θ the static friction between the cylinder and the board materials is sufficiently large to prevent slippages between the moving surfaces).

Hints: The criterion for "small oscillations" in this particular problem should not be taken in the usual $\theta \ll 1$ form, but rather as $\theta^2 \ll d^2/R^2$. The moment of inertia of a rigid uniform bar of length $2l$ and thickness $2d$ is $I = \frac{M}{12}(l^2 + d^2)$.

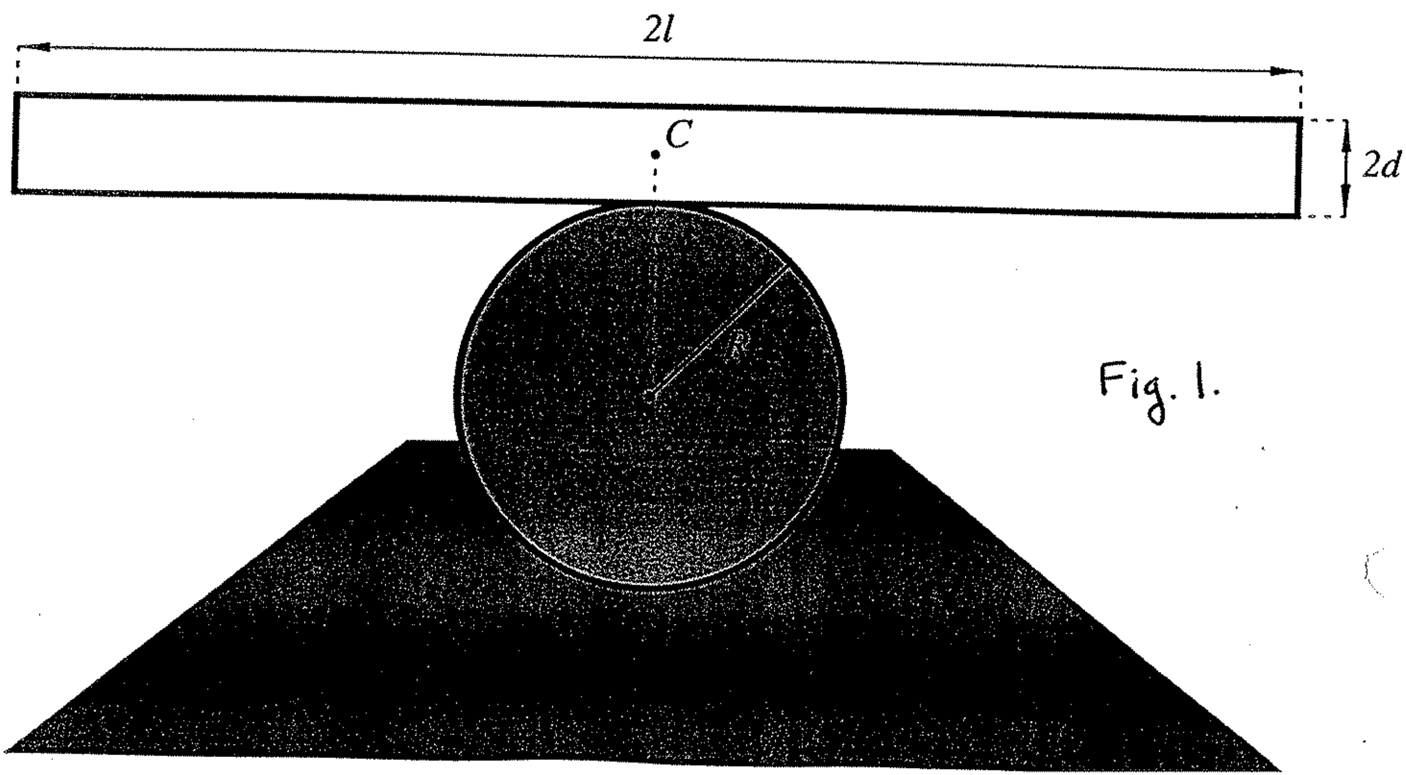


Fig. 1.

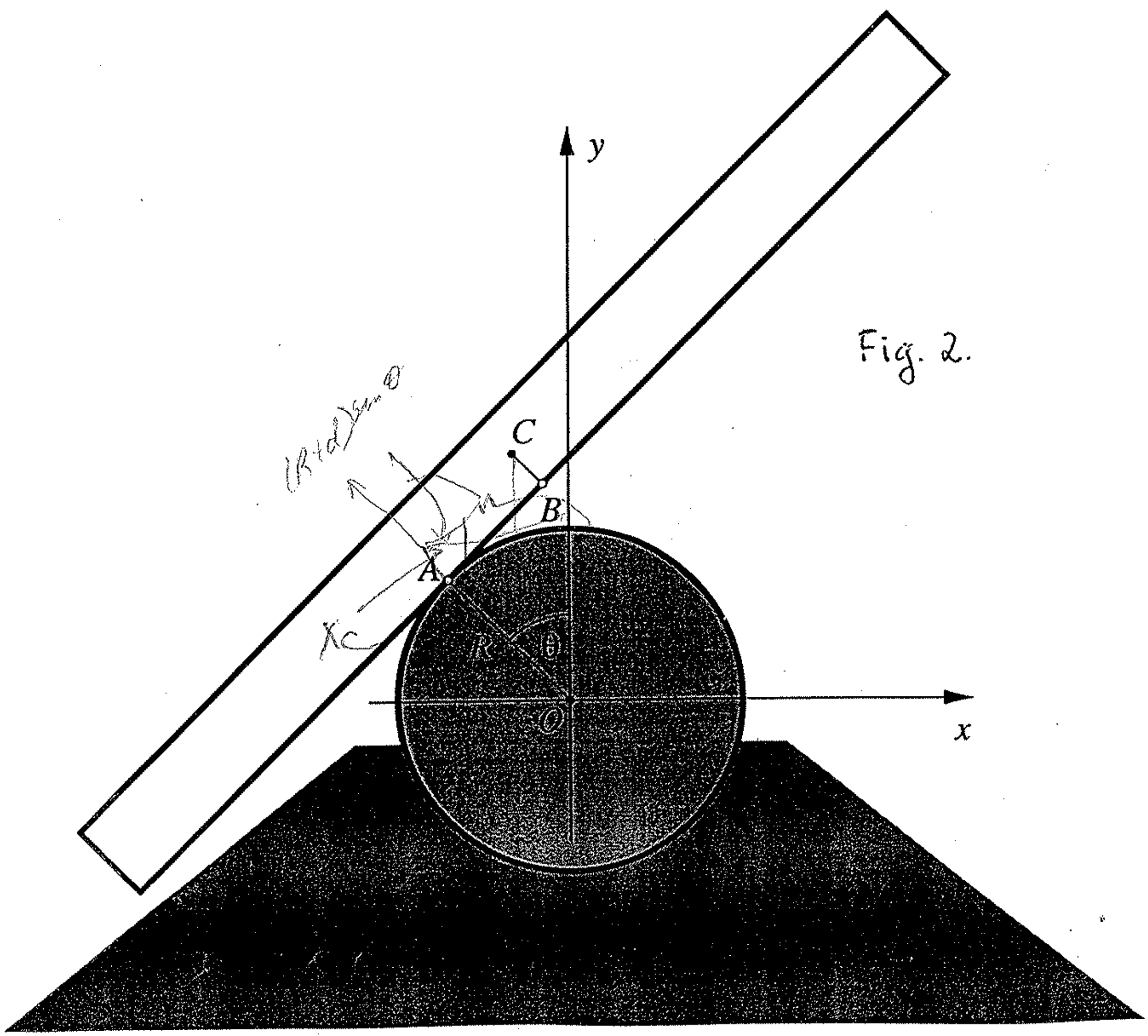
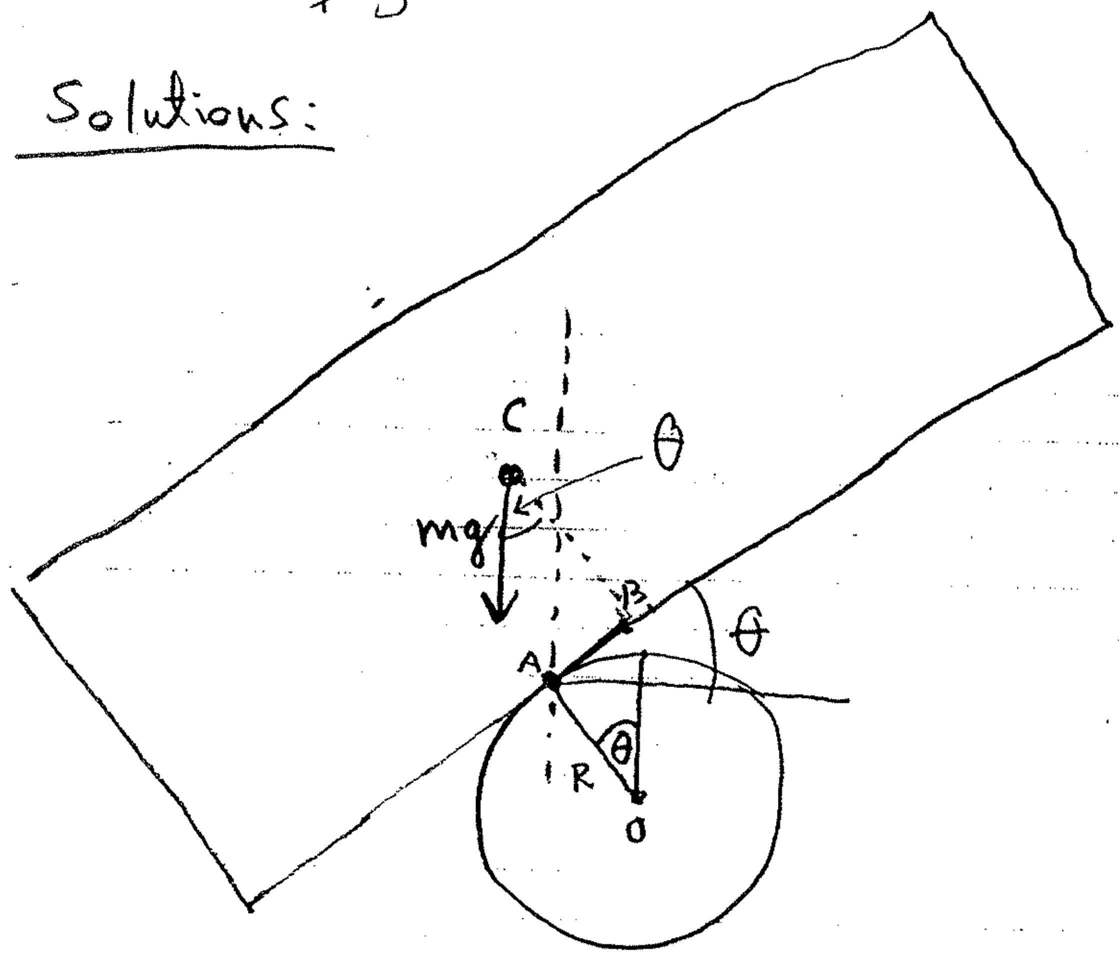


Fig. 2.

#5

Solutions:



(a) the equlibrium is unstable if the rocking moves the mass center C <u>outside</u> the vertical line drawn through point A.

Using the coordinate system xy from Fig. 2, it means that the x coordinate of point A is shorter then the x coordinate of C:

$$|x_A| < |x_C|$$

but $|x_A| = R\sin\varphi$; $|x_C| = |x_A| - AB\cos\theta + BC\sin\theta$

$BC = d$, $AB = R\cdot\theta$

Ergo: $R\theta\cos\theta < d\sin\theta$

If $\theta \ll 1$, $\sin\theta \cong \theta$ and $\cos\theta \cong 1$, so:

$$R\theta < d\theta \Rightarrow R < d$$

is the <u>unstable</u> equilibrium condition – i.e., <u>stable</u> equilibrium is when:

$$\boxed{R > d}$$

(b): Let's set up the Lagrangian:

x_c, y_c – coordinates of the mass center C:

$$\begin{cases} x_c = (R+d)\sin\theta - R\theta\cos\theta \\ y_c = (R+d)\cos\theta + R\theta\sin\theta \end{cases}$$

so:

$$\begin{cases} \dot{x}_c = (R+d)\dot{\theta}\cos\theta + R\theta\dot{\theta}\sin\theta - R\dot{\theta}\cos\theta \\ \dot{y}_c = -(R+d)\dot{\theta}\sin\theta + R\theta\dot{\theta}\cos\theta + R\dot{\theta}\sin\theta \end{cases}$$

$R\dot{\theta}\cos\theta$ in the first, and $R\dot{\theta}\sin\theta$ in the second cancel out:

$$\begin{cases} \dot{x}_c = \dot{\theta}(d\cos\theta + R\theta\sin\theta) \\ \dot{y}_c = \dot{\theta}(-d\sin\theta + R\theta\cos\theta) \end{cases}$$

Kinetic energy associated with <u>linear</u> motion of the mass center:

$$K_L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) =$$

$$= \frac{m}{2}\dot{\theta}^2\Big(d^2\cos^2\theta + R^2\theta^2\sin^2\theta + 2dR\theta\sin\theta\cos\theta +$$
$$+ d^2\sin^2\theta + R^2\theta^2\cos^2\theta - 2dR\theta\sin\theta\cos\theta\Big)$$

$$= \frac{m\dot{\theta}^2}{2}\left[d^2(\sin^2\theta + \cos^2\theta) + R^2\theta^2(\sin^2\theta + \cos^2\theta)\right] =$$

$$= \frac{m\dot{\theta}^2}{2}(d^2 + R^2\theta^2).$$

Kinetic energy of the rotary motion:

$$K_R = \frac{I}{2}\dot{\theta}^2 \quad (I - \text{moment of inertia of the bar}).$$

So, the total kinetic energy is:

$$T = \frac{m\dot{\theta}^2}{2}(d^2 + R^2\theta^2) + \frac{I}{2}\dot{\theta}^2$$

The potential energy is:

$$U = mg\,y_c = mg\left[(R+d)\cos\theta + R\theta\sin\theta\right]$$

The problem can be easily solved only in the $R^2\theta^2 \ll d^2$ region, where

$$T \cong \frac{m\dot{\theta}^2}{2} d^2 + \frac{I}{2}\dot{\theta}^2$$

Then:

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial T}{\partial \dot{\theta}} \cong \dot{\theta}(md^2 + I)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \ddot{\theta}(md^2 + I)$$

$$\frac{\partial L}{\partial \theta} \cong -\frac{\partial U}{\partial \theta} = -mg\left[-(R+d)\sin\theta + R\sin\theta + R\theta\cos\theta\right]$$

$$= -mg\left[R\theta\cos\theta - d\sin\theta\right]$$

For small θ, $\sin\theta \cong \theta$ and $\cos\theta \cong 1$

So:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \Rightarrow$$

$$\Rightarrow \ddot{\theta}(md^2 + I) = -\theta(R-d)$$

$$\ddot{\theta} = -\frac{(R-d)g}{md^2+I}\theta$$

So, the angular frequency ω is:

$$\boxed{\omega = \sqrt{\frac{(R-d)g}{md^2+I}}}$$

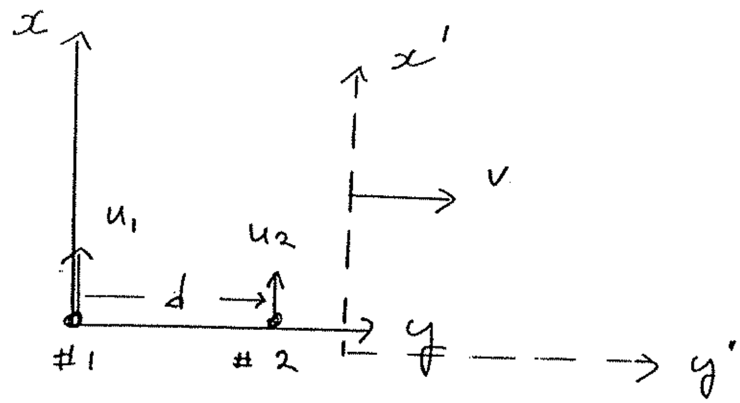
Again, one obtains the same $R > d$ condition as from the static considerations

In the reference frame K, two sprinters are lined up a distance d apart on the y-axis for a race parallel to the x-axis. Two starters, one beside each man, will fire their starting pistols at slightly different times, giving a handicap to the better of the two runners. The time difference in K is T.

(a) For what range of time differences will there be a reference frame K' in which there is no handicap, and for what range of time differences is there a frame K' in which there is a true (not apparent) handicap?

(b) Determine explicitly the Lorentz transformation to the frame K' appropriate for each of the two possibilities in (a), finding the velocity of K' relative to K and the space-time positions of each sprinter in K'.

Q. #6



a) $t_1' = \gamma(t_1 - y_1 v/c^2)$ event #1 – runner #1 starts

$t_2' = \gamma(t_2 - y_2 v/c^2)$ " #2 – " #2 "

$$\Delta t' = \gamma \cdot (T - dv/c^2) = t_2' - t_1' \geq 0$$

There will be no handicap when $\Delta t' = 0$ or

$$T = dv/c^2 \qquad \beta = \frac{cT}{d}$$

$\beta < 1$ implies $T < d/c$

b) assume $t_1 = t_1' = 0$ when the origins coincide.

$$t_1' = \gamma(0 - 0) = 0$$

$$t_2' = \gamma(T - dv/c^2)$$

In the first case there is no handicap, so

$t_2' = 0$ and $v = c^2 T/d$.

Since they both start at the same time in the primed frame we have ...

In K |

Runner # 1 $\begin{cases} x = u_1 t\, \Theta(t) \\ y = 0 \end{cases}$

Runner # 2 $\begin{cases} x = u_2 (t - T)\, \Theta(t - T) \\ y = d \end{cases}$

In K' |

Runner # 1 $\begin{cases} x' = u_1' t'\, \Theta(t') = u_1 t\, \Theta(t) \\ y' = \gamma(y - vt) = -\gamma c^2 T t / d^2 \end{cases}$

Runner # 2 $\begin{cases} x' = u_2' t'\, \Theta(t') = u_2 t\, \Theta(t) \\ y' = \gamma(d - vt) = \gamma(d - c^2 T t / d^2) \end{cases}$

If it should turn out that $c^2 T / d > c$

then they will never be simultaneous in any frame.

There is now no relationship between T, d, and v,

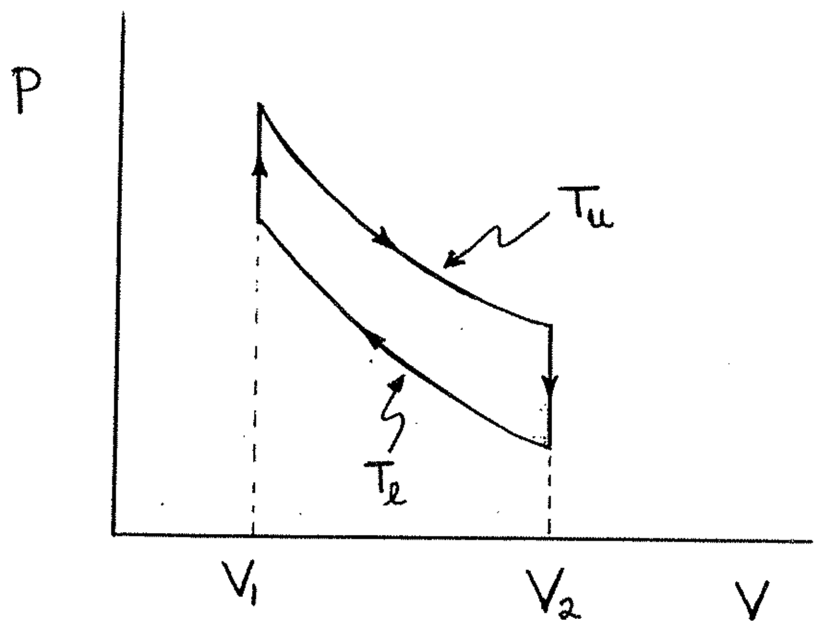
so in K' |

Runner # 1 $\begin{cases} x' = u_1' t'\, \Theta(t') = u_1 t\, \Theta(t) \\ y' = -\gamma v t \end{cases}$

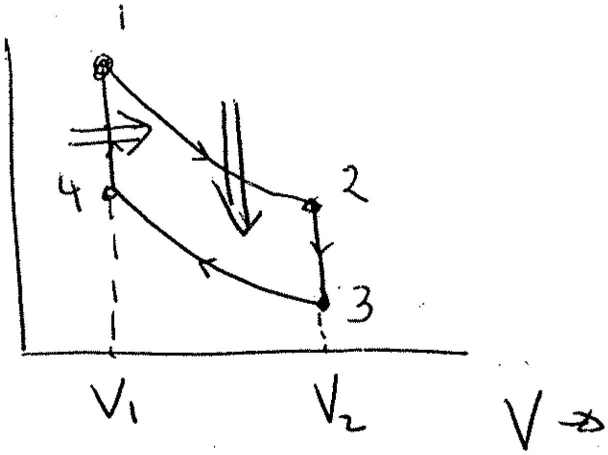
Runner # 2 $\begin{cases} x' = u_2' (t' - T') = u_2 (t - T)\, \Theta(t - T) \\ y' = \gamma (d - vt) \end{cases}$

The figure shown below represents one cycle of operation of the ideal Stirling (heat) engine. Consider the working substence to be n moles of an ideal monatomic gas. The upper and lower curves are isotherms, at temperatures of T_u and T_l, respectively. The right and left hand sides of the cycle are isochors (paths at constant volume). Show that for very small changes in the volume, $\Delta V \ll V$, the efficiency of this engine, η, can be expressed as

$$\eta = \frac{2}{3}\left(\frac{\Delta V}{V}\right), \quad \text{where } \frac{\Delta V}{V} = \frac{V_2 - V_1}{V_1}.$$



#7



$1 \to 2$: $Q_{1\to 2} = \int \frac{nRT_2}{V} dV = nRT_2 \ln\left(\frac{V_2}{V_1}\right) > 0$

$W_{1\to 2} = Q_{1\to 2}$, since $\Delta U = 0$

$2 \to 3$ $W_{2\to 3} = 0$

$Q_{2\to 3} = \frac{3}{2} nR(T_1 - T_2) < 0$

$3 \to 4$ $Q_{3\to 4} = nRT_1 \ln\left(\frac{V_1}{V_2}\right) < 0$

$W_{3\to 4} = Q_{3\to 4}$

$4 \to 1$ $W_{4\to 1} = 0$

$Q_{4\to 1} = \frac{3}{2} nR(T_2 - T_1) > 0$

$$\eta = \frac{W_{1\to 2\to 3\to 4\to 1}}{Q_{\text{taken in}}} = \frac{nR(T_2 - T_1)\ln(V_2/V_1)}{nRT_2 \ln\left(\frac{V_2}{V_1}\right) + \frac{3}{2} nR(T_2 - T_1)}$$

for $V_2 - V_1 \ll V_1$; $\ln \frac{V_2}{V_1} \simeq \frac{V_2 - V_1}{V_1} \equiv x$

$$\therefore \eta \simeq \frac{x(T_2 - T_1)}{(x + \frac{3}{2})(T_2) - \frac{3}{2} T_1}$$

$$\eta \simeq \frac{x(T_2 - T_1)}{\frac{3}{2}(T_2 - T_1)} \simeq \frac{2}{3} x$$

$$\eta = \frac{2}{3}\left(\frac{V_2 - V_1}{V_1}\right).$$

Consider two distinguishable spin 1/2 particles whose interaction with each other is described by a potential $V = a\,\vec{S}_1 \bullet \vec{S}_2$, where a is a real constant.

(a) Find the eigenenergies and eigenstates of this system, assuming that the interaction potential is the full Hamiltonian.

(b) Now, the two particles are placed in a uniform external magnetic field, $\vec{B}$. Their magnetic moments can be written as $\vec{M}_1 = \alpha\,\vec{S}_1$, and $\vec{M}_2 = \beta\,\vec{S}_2$. Find the new eigenenergies of the system.

#8

2 Spin $\frac{1}{2}$ particles

$V = a\,\vec{S}_1 \cdot \vec{S}_2$

a) Rewrite V using: $\vec{S} = \vec{S}_1 + \vec{S}_2$ total spin

$\vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1 \cdot \vec{S}_2$

$\vec{S}_1^2 = \vec{S}_2^2 = \frac{1}{2}(\frac{1}{2}+1)\hbar^2 = \frac{3}{4}\hbar^2$

$\Rightarrow \vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}(\vec{S}^2 - \frac{3}{2}\hbar^2)$

$\Rightarrow V = \frac{a}{2}(\vec{S}^2 - \frac{3}{2}\hbar^2)$

So it will be most useful to use basis of eigenvectors of $\vec{S}_1^2, \vec{S}_2^2, \vec{S}^2, S_z$ (i.e. $|S, M\rangle$) rather than basis of eigenvectors of $\vec{S}_1^2, \vec{S}_2^2, S_{1z}, S_{2z}$ (i.e. $|\varepsilon_1, \varepsilon_2\rangle$)

Recall the relations between the two basis sets.

$|0,0\rangle = \frac{1}{\sqrt{2}}\left[|+,-\rangle - |-,+\rangle\right]$

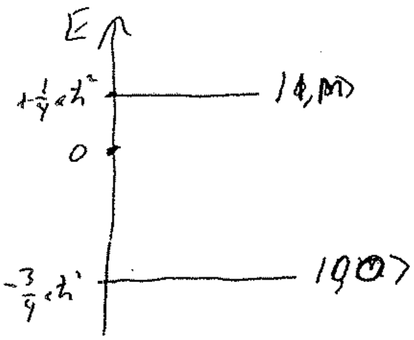
$|1,1\rangle = |+,+\rangle$
$|1,0\rangle = 1/\sqrt{2}\left[|+,-\rangle + |-,+\rangle\right]$
$|1,-1\rangle = |-,-\rangle$

Clearly $V|S,M\rangle = \frac{a}{2}\left[S(S+1)\hbar^2 - \frac{3}{2}\hbar^2\right]|S,M\rangle$

so V is diagonal in $|S,M\rangle$ basis. Since V is the full Hamiltonian, $H=V$, energy eigenstates are clearly $|S,M\rangle$ states

$$\Rightarrow H|S,M\rangle = \frac{a\hbar^2}{2}\left[S(S+1) - \tfrac{3}{2}\right]|S,M\rangle$$

Energies	Eigenstates	
$-\frac{3}{4}a\hbar^2$	$\|0,0\rangle$	
$+\frac{1}{4}a\hbar^2$	$\left\{\begin{array}{l}\|1,1\rangle \\ \|1,0\rangle \\ \|1,-1\rangle\end{array}\right.$	3-fold degenerate



b) In external $\vec{B}$ field, each spin has extra potential energy:

$$V_i = -\vec{M}_i \cdot \vec{B}$$

or if we let $\vec{B} = B\hat{z}$

$$V_i = -M_{iz} B$$

$$\Rightarrow \boxed{H = a\,\vec{S}_1 \cdot \vec{S}_2 - \alpha B S_{1z} - \beta B S_{2z}}$$

Now write full Hamiltonian in $|S,M\rangle$ basis.
Note that

$$S_{1z}|0,0\rangle = \frac{\hbar}{2}|1,0\rangle$$
$$S_{2z}|0,0\rangle = -\frac{\hbar}{2}|1,0\rangle$$

$$S_{1z}|1,0\rangle = \frac{\hbar}{2}|0,0\rangle$$
$$S_{2z}|1,0\rangle = -\frac{\hbar}{2}|0,0\rangle$$

$S_{1z}|1,1\rangle = \frac{\hbar}{2}|1,1\rangle$ etc (i.e. $|1,1\rangle, |1,-1\rangle$ are eigenstates of S_{1z}, S_{2z}, but $|1,0\rangle, |0,0\rangle$ are <u>not</u>)

Now write in matrix form w/ labeling as shown:

$$\text{e.g.}\quad S_{1z} = \frac{\hbar}{2}\begin{array}{cccc} |1,1\rangle & |1,-1\rangle & |1,0\rangle & |0,0\rangle \end{array}$$
$$S_{1z} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad S_{2z} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\Rightarrow H = \begin{pmatrix} \frac{a}{4}\hbar^2 - B\frac{\hbar}{2}(\alpha+\beta) & 0 & 0 & 0 \\ 0 & \frac{a}{4}\hbar^2 + B\frac{\hbar}{2}(\alpha+\beta) & 0 & 0 \\ 0 & 0 & \frac{a}{4}\hbar^2 & -B\frac{\hbar}{2}(\alpha-\beta) \\ 0 & 0 & -B\frac{\hbar}{2}(\alpha-\beta) & -\frac{3a}{4}\hbar^2 \end{pmatrix}$$

Two eigenvalues are obvious:

$$E_1 = \frac{a}{4}\hbar^2 - B\frac{\hbar}{2}(\alpha+\beta)$$

$$E_2 = \frac{a}{4}\hbar^2 + B\frac{\hbar}{2}(\alpha+\beta)$$

Diagonalize this 2×2 matrix to find 2 eigenvalues E_3 & E_4

$$\begin{vmatrix} \frac{a}{4}\hbar^2 - \lambda & -B\frac{\hbar}{2}(\alpha-\beta) \\ -B\frac{\hbar}{2}(\alpha-\beta) & -\frac{3}{4}a\hbar^2 - \lambda \end{vmatrix} = \lambda^2 + \frac{a}{2}\hbar^2\lambda - \frac{3a^2}{16}\hbar^4 - B^2\left(\frac{\hbar}{2}\right)^2(\alpha-\beta$$

$$= 0$$

$$\Rightarrow \lambda = -\frac{a\hbar^2}{4} \pm \sqrt{\left(\frac{a\hbar^2}{2}\right)^2 + B^2\left(\frac{\hbar}{2}\right)^2(\alpha-\beta)^2}$$

$$\Rightarrow E_3 = -\frac{a\hbar^2}{4} + \sqrt{\left(\frac{a\hbar^2}{2}\right)^2 + B^2\left(\frac{\hbar}{2}\right)^2(\alpha-\beta)^2}$$

$$E_4 = -\frac{a\hbar^2}{4} - \sqrt{\left(\frac{a\hbar^2}{2}\right)^2 + B^2\left(\frac{\hbar}{2}\right)^2(\alpha-\beta)^2}$$