OSU PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #75

September 21 and 22, 1995

P&D 80
mean 75
MS 10

Comprehensive Examination for Fall 1995

PART I

General Instructions

This Comprehensive Examination for Fall 1995 (#75) consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Thursday, September 21, and lasts three hours. The second part (Problems 3-4) will be handed out on the same day, at 1:30 pm, and also lasts three hours. The third and fourth parts will be administrated in the same way on Friday, September 22.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit - especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulae and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one page from your solutions. "Scratch" work will not be graded.

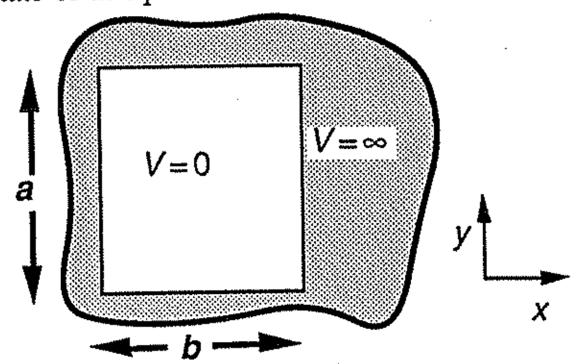
A spinless particle of mass m moves nonrelativistically in two dimensions in a rectangular potential box of dimensions a and b.

$$V(x,y) = 0 \text{ inside the box.}$$

$$V(x,y) = +\infty \text{ outside the box.}$$

For all early times $t < 0$ the particle is in the ground state of this potential.

(a) Find the energy of the particle at an early time $t < 0$.



(b) Suppose that the x-component of the particle's momentum is measured at time $t < 0$. What values may be found, and with what probabilities?

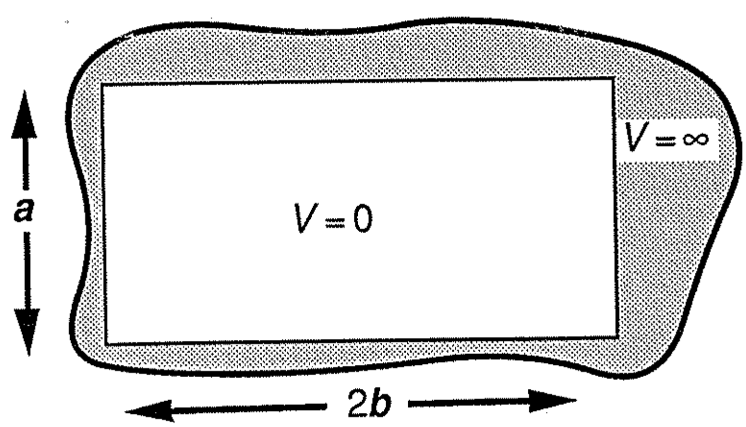
Suppose now that the momentum of the particle has not been measured.

At time $t = 0$, the length of the box is suddenly doubled, so that its dimensions are a and $2b$.

At time $t > 0$, the energy of the particle is measured.

(c) What values of the energy may now be found?

(d) What is the probability of finding an energy less than the answer to part (a)?



Solution to Problem QMU.

(a) Separable potential, constant $\Rightarrow$ wave function $\phi_{nm}(x,y) = C_{nm} \sin k_n x \quad \sin k_m y$

boundary conditions ϕ (boundary) $= 0 \Rightarrow k_n b = n\pi$, $k_m a = m\pi$

normalize $1 = \int_0^a dy \int_0^b dx\, |\phi_{nm}(x,y)|^2 = |C_{nm}|^2 \int_0^a dy \int_0^b dx\, \sin^2 k_n x \quad \sin^2 k_m y = |C_{nm}|^2 \frac{a}{2}\frac{b}{2}$

$$\phi_{nm} = \frac{2}{\sqrt{ab}} \sin\frac{n\pi x}{b} \sin\frac{m\pi y}{a}\ , E_{nm} = \frac{\hbar^2}{2m}(k_n^2 + k_m^2) = \frac{\hbar^2\pi^2}{2m}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

ground state $n = m = 1$

(b) all values are possible, $-\infty < p_x < +\infty$.

probabilities $P(p_x, p_y)\, dp_x dp_y = |\langle p_x p_y|\phi\rangle|^2 dp_x dp_y$

$$= \left| \int_0^a dy \int_0^b dx\ \phi(x,y)\ \frac{\exp(-ip_x x/\hbar)}{\sqrt{2\pi\hbar}}\ \frac{\exp(-ip_y y/\hbar)}{\sqrt{2\pi\hbar}} \right|^2 =$$

$$= \frac{2}{a}\left| \int_0^a dy\ \sin \pi y/a\ \frac{\exp(-ip_y y/\hbar)}{\sqrt{2\pi\hbar}} \right|^2 \times \frac{2}{b}\left| \int_0^b dx\ \sin \pi x/a\ \frac{\exp(-ip_x x/\hbar)}{\sqrt{2\pi\hbar}} \right|^2$$

$$= P(p_y)dp_y \times P(p_x)\, dp_x$$

each distribution is normalized, so need only consider $P(p_x) = \int dp_y\, P(p_x, p_y)$

$$P(p_x) = \frac{2}{b}\left| \int_0^b dx\ \sin \pi x/a\ \frac{\exp(-ip_x x/\hbar)}{\sqrt{2\pi\hbar}} \right|^2 = \frac{2}{b}\left| \int_0^b dx\ \frac{\exp(i\pi x/b) - \exp(-i\pi x/b)}{2i}\ \frac{\exp(-ip_x x/\hbar)}{\sqrt{2\pi\hbar}} \right|^2$$

$$= \frac{1}{4\pi\hbar b}\left[\frac{-\exp(-ip_x b/\hbar) - 1}{-\pi/b + p_x/\hbar} - \frac{-\exp(-ip_x b/\hbar) - 1}{\pi/b + p_x/\hbar} \right]^2$$

(c) $b \rightarrow 2b$, $E_{nm}{}^{\text{new}} = \frac{\hbar^2\pi^2}{2m}(\frac{m^2}{a^2} + \frac{n^2}{4b^2})$

however not all these energies will be represented in the wave packet:

$$|\Psi(t>0)\rangle = \sum_{nm} |\phi_{nm}{}^{\text{new}}\rangle \exp(-iE_{nm}{}^{\text{new}}t) \langle\phi_{nm}{}^{\text{new}}|\Psi(t{=}0)\rangle$$

the matrix element is zero unless $m = 1$, because of orthogonal y- dependence.

the m.e. is also zero when $n/2$ is an odd integer greater than one, because then the new wave function in the left half of the box is one of the old ones and thus orthogonal!

result: $E_n{}^{\text{new}} = \frac{\hbar^2\pi^2}{2m}(\frac{1}{a^2} + \frac{n^2}{4b^2})$, $n = 1, 2, 3, 4, 5, 7, 8, 9, 11, \ldots$

(d) $E_n{}^{\text{new}} < E_{gs}{}^{\text{old}} \Rightarrow n = 1$

Probability $= |\langle\phi_{nm}{}^{\text{new}}|\Psi(t{=}0)\rangle|^2 = \frac{2}{2b}\frac{2}{b}\left|\int_0^b dx \sin\frac{\pi x}{2b} \sin\frac{\pi x}{b}\right|^2$

OSU Physics Depart. Comp. Exam #75, Sept. 21-22, 1995. **PROB 2**

Nuclei of a particular isotopic species contained in a crystal have spin, $I = 1$, and, thus, $m = +1, 0, -1$. The interaction between the nuclear quadrupole moment and the gradient of the crystalline electric field in the crystal produces a situation where the nucleus has the same energy, $E = \varepsilon$, in the state $m = +1$ and the state $m = -1$, compared with an energy $E = 0$ in the state $m = 0$.

(a) Find an expression, as a function of temperature, for the nuclear contribution to the entropy, S, of the solid.

(b) Indicate what your result predicts for this contribution to the entropy, S, at the extremes of very high temperature and very low temperature.

(c) Show that the nuclear contribution to the heat capacity of the solid, C_V, goes to zero at both high and low temperature extremes.

$m = \pm 1$ ———— ϵ

$m = 0$ ———— 0

(a) $\bar{E} - TS = F$ But $F = -kT \ln Z$.

$\therefore \; S = \dfrac{\bar{E}}{T} + k \ln Z.$

$$Z = S^N = (1 + e^{-\beta\epsilon} + e^{-\beta\epsilon})^N = (1 + 2e^{-\beta\epsilon})^N$$

$$\bar{E} = \left(\frac{0(1) + \epsilon e^{-\beta\epsilon} + \epsilon e^{-\beta\epsilon}}{1 + 2e^{-\beta\epsilon}} \right) N = \frac{N \, 2\epsilon \, e^{-\beta\epsilon}}{1 + 2e^{-\beta\epsilon}}$$

$$\therefore \; S = Nk \ln (1 + 2e^{-\beta\epsilon}) + \frac{2N\epsilon \, e^{-\beta\epsilon}}{T(1 + 2e^{-\beta\epsilon})}.$$

(b) $T \to 0$, means $\beta \to \infty$

$S \to 0$. Nernst: 3rd Law.

$T \to \infty$, means $\beta \to 0$

$S \to Nk \ln 3$. All 3 states equally pop-

(c) $$C_v = \frac{\partial \bar{E}}{\partial T} = \frac{2N\epsilon^2 e^{-\epsilon/kT}}{kT^2 (1 + 2e^{-\epsilon/kT})^2}$$

(i) as $T \to 0$, the exponential term in numerator dominates and $C_v \to 0$

(ii) as $T \to \infty$, $C_v \propto \dfrac{1}{T^2}$, which also $\to 0$.

Consider an infinitely long straight wire of radius, a, to carry a current, I, having a uniform current density distributed over the cross-section of the wire.

(a) Find the energy/length of the wire stored in its magnetic field.

(b) Find the self-inductance per unit length of the wire.

(c) Now, consider a transmission line consisting of two parallel wires of radii, a and b, separated by a distance, $d > a + b$. The current flows down one wire and back the other. Find the total inductance per unit length of the transmission line.

Calculate B – 4 pts

" W – 4

$L \propto W/I^2$ – 4

2 wires – 8

Problem # 3

(a) $W = \frac{1}{2c} \int \vec{J} \cdot \vec{A} \, dv$ magnetic energy

For a long straight wire $B_\varphi = \frac{-2I\rho}{ca^2}$ (Ampere's law

and $\vec{J} = \frac{I}{\pi a^2} \hat{z}$

To find $\vec{A}$ use $B_\varphi = \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}$

$\partial A_\rho / \partial z$ should $= 0$ by symmetry, so

$$A_z = - \int_0^\rho B_\varphi \, d\rho' = \frac{\rho^2 I}{ca^2}$$

$$W = \frac{1}{2c} \int \vec{J} \cdot \vec{A} \, dv = \frac{1}{2c} \left(\frac{I}{\pi a^2} \right) (2\pi l) \int_0^a \rho \, d\rho \left(\frac{\rho^2 I}{ca^2} \right)$$

$$= \frac{I^2 l}{4c^2}$$

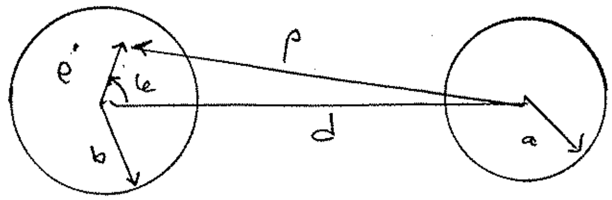
so $\frac{W}{l} = I^2 / 4c^2$

(b) Inductance can be defined in terms of energy as

$$L = 2W / I^2$$

so $\frac{L}{l} = \frac{1}{2c^2}$ (Self inductance)

(c) In addition to the above, we need the the interaction between the field of wire a and the current in wire b. (and vice versa



The field outside of a wire is $B_\varphi = -2I/\rho c$

so $$A_z = \int_a^\rho \frac{2I}{c}\frac{d\rho'}{\rho'} = \frac{2I}{c}\ln(\rho/a)$$

Integrate this over wire b

$$\rho^2 = d^2 + \rho'^2 - 2d\rho'\cos\varphi$$

$$W = \frac{1}{2c}\left(\frac{I}{\pi b^2}\right) \ell \int_0^b \rho' d\rho' \int_0^{2\pi} d\varphi \, \frac{2I}{c}\ln(\rho/a)$$

$$= \frac{I^2 \ell}{2\pi c^2 b^2}\int_0^b \rho' d\rho' \int_0^{2\pi} d\varphi \left\{2\ln(d/a) + \ln\left[1 + \left(\frac{\rho'}{d}\right)^2 - 2\left(\frac{\rho'}{d}\right)\cos\varphi\right]\right.$$

one can show that

$$\ln\left[1 + \left(\frac{\rho'}{d}\right)^2 - 2\left(\frac{\rho'}{d}\right)\cos\varphi\right] = -2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\rho'}{d}\right)^n \cos n\varphi$$

which vanishes term by term when integrated over $\int_0^{2\pi} d\varphi$

Then $$W(a\to b) = \frac{I^2 \ell}{c^2}\ln(d/a)$$

of course $\quad W(b \to a) = \dfrac{I^2 l}{c^2} \ln\left(\dfrac{d}{b}\right)$

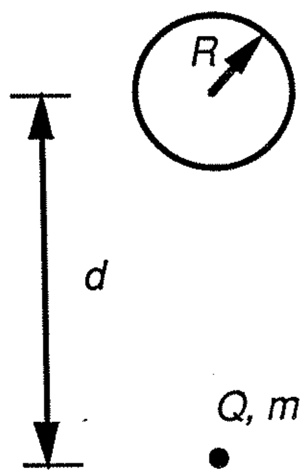
Finally $\quad \dfrac{L}{l} = \dfrac{2}{c^2} \ln\left(\dfrac{d^2}{ab}\right) + \dfrac{1}{c^2}$

OSU Physics Department Comprehensive Exam #75, September 21-22, 1995 **PROB 4**

A point charge Q of mass m is to be placed at rest in a uniform gravitational field g at its equilibrium position a distance d directly below the center of an uncharged conducting sphere of radius $R << d$.

(a) Find d .

(b) Find the frequency of small oscillations in the vertical direction.



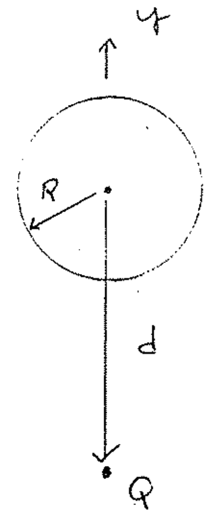
(a) 10 pts for F

4 pts for d

(b) 6 pts for equation of motion

Prob. 4

(a)



For the sphere to be an equipotential there must be a charge Q'

$$Q' = -\frac{R}{d} Q$$

located a distance y'

$$y' = -\frac{R^2}{d}$$

below the center of the sphere. In order to keep the ~~total~~ charge on the sphere $= 0$ there must be another image charge $Q'' = -Q'$ at the center of the sphere. Then the ~~total~~ force on Q is

$$F = \frac{QQ'}{(d-|y'|)^2} + \frac{QQ''}{d^2} - mg$$

$$= \frac{Q^2R}{d}\left[\frac{1}{(d-|y'|)^2} - \frac{1}{d^2}\right] - mg$$

$$= \frac{Q^2R}{d}\left[\frac{d^2}{(d^2-R^2)^2} - \frac{1}{d^2}\right] - mg$$

$$= \frac{Q^2R}{d}\left[\frac{+2d^2R^2 - R^4}{d^2(d^2-R^2)^2}\right] - mg$$

assume that $F=0$ at a point where $d \gg R$

$$F \cong \frac{2Q^2R^3 d_0^2}{d_0^7} - mg = 0$$

$$d \approx \left(2Q^2R^3 \right)^{1/5}$$

4b) Now if $|d| = |d_0 - \delta y|$ assume $d_0 >$

$$F = \frac{2Q^2R^3}{|d_0 - \delta y|^5} - mg = m\delta y''$$

if δy is small

$$F \approx \frac{2Q^2R^3}{d_0^5}\left(1 + \frac{5\,\delta y}{d_0}\right) - mg = m y''$$

$$m\delta y'' = \frac{10\,Q^2R^3}{d_0^6}\,\delta y$$

$$\omega = \left(\frac{10\,Q^2R^3}{m\,d_0^6}\right)^{1/2}$$

we do not get small oscillations in this case. the equilibrium is unstable.

Consider an insulating solid to be represented by a collection of 3N independent oscillators. (1) Develop the partition function, Z, for this system, and (2) show that the system has the following equation of state

$\left(\frac{\partial P}{\partial E}\right)_V = \frac{\Gamma}{V}$, where P is the pressure; E is the internal energy; V is the volume; and Γ is given by $\Gamma = \frac{\sum_{i=1}^{3N} \gamma_i c_i}{\sum_{i=1}^{3N} c_i}$, where $\gamma_i = -\frac{\partial \ln \omega_i}{\partial \ln V}$, indicating a dependence of the i^{th} oscillator's fundamental frequency, ω_i , on volume, and where c_i is the contribution to the specific heat at constant volume from the i^{th} harmonic oscillator.

Hint: $c_i = k\sum_{i=1}^{3N} \frac{x_i^2 e^{x_i}}{(e^{x_i}-1)^2}$, where $x_i = \frac{\hbar\omega_i}{kT}$.

$$\epsilon_i = \left(n_j + \tfrac{1}{2}\right)\hbar\omega_i \quad ; \quad n_j = 0, 1, 2, \ldots$$

$$z_1 = \sum_{n_j} e^{-\beta\epsilon_1} = e^{-\frac{\beta\hbar}{2}\omega_1} \sum_{n_j=0}^{\infty} e^{-n_j\beta\hbar\omega_1}$$

$$= e^{-\frac{\beta\hbar}{2}\omega_1}\left(1 - e^{-\beta\hbar\omega_1}\right)^{-1}, \text{ ~~and~~}$$

similarly for Z_2, Z_3, ...

$$Z_{vib} = Z_1 Z_2 Z_3 \ldots$$

$$Z_{vib} = e^{-\frac{\hbar\beta}{2}\sum_{i=1}^{3N}\omega_i} \prod_{i=1}^{3N}\left(1 - e^{-\beta\hbar\omega_i}\right)^{-1}.$$

Since, as always,

$$F = -kT \ln Z$$

$$F = \tfrac{1}{2}\left(\hbar\omega_1 + \hbar\omega_2 + \ldots + \hbar\omega_{3N}\right) + kT \sum_{i=1}^{3N} \ln\left(1 - e^{-\beta\hbar\omega_i}\right.$$

Now $P = -\left(\frac{\partial F}{\partial V}\right)_T$, with the variation on V coming from dependence of $\{\omega_i\}$ on V.

$$P = -\tfrac{1}{2}\sum \frac{\partial}{\partial\omega_i}(\quad)\frac{\partial\omega_i}{\partial V} - kT\sum_{i=1}^{3N}\left[\frac{\partial}{\partial\omega_i}\ln\left(1 - e^{-\beta\hbar\omega_i}\right)\right]\frac{\partial\omega_i}{\partial V}$$

$$P = -\frac{1}{2}\left(\begin{matrix}\text{term with no}\\ T\text{ dependence}\end{matrix}\right) - \hbar \sum_{i=1}^{3N} \frac{1}{e^{\beta\hbar\omega_i} - 1}\left(\frac{\partial \omega_i}{\partial V}\right)$$

But $\dfrac{\partial \omega_i}{\partial V} = \dfrac{\omega_i}{V}\dfrac{\partial \ln \omega_i}{\partial \ln V}$, and so

$$-\gamma_i \frac{\omega_i}{V} = \frac{\partial \omega_i}{\partial V}$$

$$\therefore \quad P = -\frac{1}{2}\left(\qquad \right) + \frac{\hbar}{V} \sum_{i=1}^{3N} \frac{\gamma_i \omega_i}{e^{\beta\hbar\omega_i} - 1}$$

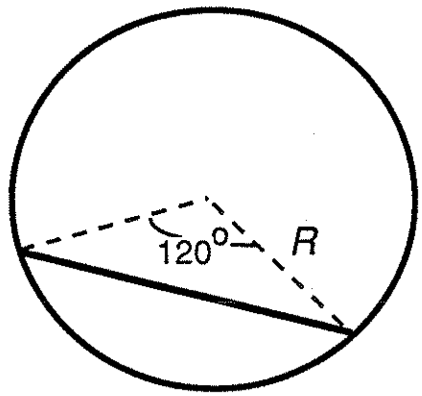
Now $\left(\dfrac{\partial P}{\partial E}\right)_V = \left(\dfrac{\partial P}{\partial T}\right)_V \left(\dfrac{\partial T}{\partial E}\right)_V = \dfrac{1}{C_V}\left(\dfrac{\partial P}{\partial T}\right)_V$

$$\therefore \left(\frac{\partial P}{\partial E}\right)_V = \frac{k}{V}\left\{\sum_{i=1}^{3N} \left(\frac{\hbar\omega_i}{kT}\right)^2 \frac{\gamma_i\, e^{\hbar\omega_i/kT}}{(e^{\hbar\omega_i/kT} - 1)^2}\right\} \times \frac{1}{C_V}$$

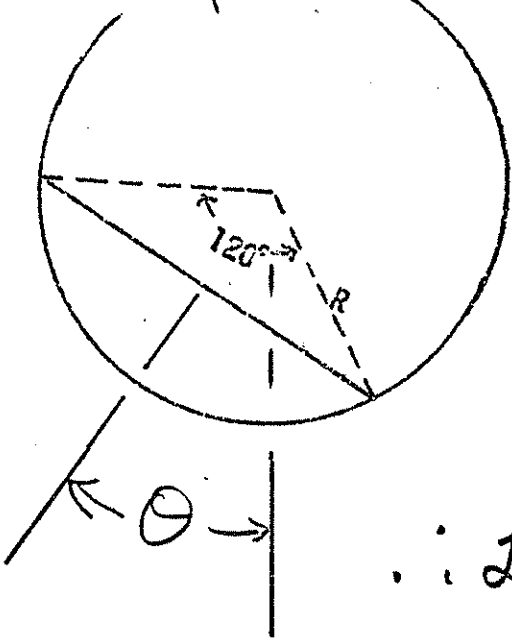
But from the definition of C_V, given in the statement of the problem

$$\left(\frac{\partial P}{\partial E}\right)_V = \frac{1}{V}\,\frac{\sum \gamma_i C_i}{\sum C_i} = \frac{\Gamma}{V}, \qquad \text{QED.}$$

A uniform rod of mass m slides with its ends on a smooth vertical circle in a uniform gravitational field g. The rod subtends an angle of 120^o at the center of the circle, as shown in the figure, and friction is negligible. Determine the Lagrangian and equation(s) of motion for this system, and solve for motion near equilibrium.



PROB 6.



$$T = \frac{1}{2}M\left(\frac{R}{2}\right)^2\dot{\theta}^2 + \frac{1}{2}I_c\dot{\theta}^2,$$

where $I_c = 2\rho\int_0^{\frac{\sqrt{3}}{2}R} r^2 dr = \frac{2\rho}{3}\left(\frac{\sqrt{3}}{2}R\right)$

$M = \sqrt{3}R\rho$; ∴, $I_c = \frac{2}{3}\left(\frac{\sqrt{3}}{2}R^3\right)\frac{M}{\sqrt{3}R}$

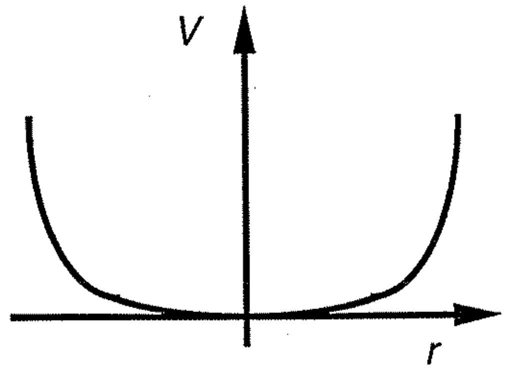
or, $I_c = \frac{MR^2}{4}$. $V = \frac{MgR}{2}(1-\cos\theta)$.

$$\therefore \mathcal{L} = \frac{MR^2\dot{\theta}^2}{8} + \frac{MR^2\dot{\theta}^2}{8} - \frac{MgR}{2}(1-\cos\theta).$$

$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{MR^2\dot{\theta}}{2}$, $\frac{\partial\mathcal{L}}{\partial\theta} = -\frac{MgR}{2}\sin\theta$; ∴ $\ddot{\theta} + \frac{g}{R}\sin\theta = 0$.

For $\theta \ll 1$ $\ddot{\theta} \simeq -\frac{g}{R}\sin\theta$, which yields S.H.M. with angular frequency $\omega = \sqrt{g/R}$.

A spinless particle of mass M moves nonrelativistically in 3 dimensions in a central potential $V(|r|) = \lambda\,|r|^4 > 0$.



(a) Find a complete set of commuting observables which includes the Hamiltonian.

(b) What is the degeneracy of the ground state?

(c) Estimate the energy of the ground state.

(d) Sketch the radial wave function of the ground state. Label your axes.

At approximately what radius is the particle most likely to be found, if it is in its ground state?

Solution to Problem QMG

(a)Spherical symmetry $\Rightarrow$ angular momentum is conserved

C.S.C.O. is $H, \boldsymbol{L}^2, L_z$ where L_x, L_y, L_z are components of orbital angular momentum $\boldsymbol{L} = \boldsymbol{R} \times \boldsymbol{P}$, and $L^2 = L_x{}^2 + L_y{}^2 + L_z{}^2$; any component of $\boldsymbol{L}$ will do.

(b) ground state will be spherically symmetric, has $\boldsymbol{L} = 0$, so is non-degenerate (degeneracy = 1).

(c) variational estimate: expect wave function peaked at origin, falling off in all directions

try $\phi(r) = C(\alpha) \exp -\alpha r^2$, choose α to minimize mean energy

use formula from sheet $\int_0^\infty dx\, x^{2n} \exp -a^2x^2 = \dfrac{1\bullet 3\bullet 5\bullet\bullet\bullet(2n-1)\sqrt{\pi}}{2^{n+1}\, a^{2n\neq 1}}$

$$\text{Norm: } 1 = |C(\alpha)|^2\, 4\pi \int_0^\infty r^2 dr\ \exp -2\alpha r^2 = |C(\alpha)|^2\, 4\pi \frac{\sqrt{\pi}}{4}\, (2\alpha)^{-3/2} \quad \text{with } n = 1,\ a^2 = 2\alpha$$

$$.\ |C(\alpha)|^2 = \left(\frac{2\alpha}{\pi}\right)^{3/2}$$

$$\langle H \rangle \equiv \langle KE \rangle + \langle PE \rangle\,,$$

$$\langle PE \rangle = |C(\alpha)|^2\, 4\pi \int_0^\infty r^2 dr\ \lambda r^4 \exp -2\alpha r^2 = |C(\alpha)|^2\, 4\pi\lambda \frac{3\bullet 5\sqrt{\pi}}{16} (2\alpha)^{-7/2} = \lambda\, \alpha^{-2} \frac{15}{16} \quad \text{with } n = 3$$

$$\langle KE \rangle = 4\pi \int_0^\infty r^2 dr\ \frac{-\hbar^2}{2M}\, \nabla^2\phi = -|C(\alpha)|^2\, \frac{\hbar^2}{2M}\, 4\pi \int_0^\infty r^2 dr\ \exp -\alpha r^2 \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \exp -\alpha r^2\right)$$

$$= -|C(\alpha)|^2\, \frac{\hbar^2}{2M}\, 4\pi \int_0^\infty dr\ (-2\alpha)\{3r^2 - 2\alpha r^4\}\ \exp -2\alpha r^2\,, \text{ there are } n = 1 \text{ and } n = 2 \text{ terms}$$

$$= -\left(\frac{2\alpha}{\pi}\right)^{3/2} \frac{\hbar^2}{2M}\, 4\pi\, (-2\alpha)\{3 \frac{\sqrt{\pi}}{4}\, (2\alpha)^{-3/2} - 2\alpha\, \frac{3\sqrt{\pi}}{8} (2\alpha)^{-5/2}\,\} = \frac{\hbar^2}{2M}\, \alpha \frac{3}{8}$$

$$\langle H \rangle = \frac{\hbar^2}{2M}\, \alpha \frac{3}{8} + \lambda\, \alpha^{-2} \frac{15}{16}\,,$$

$$\text{minimize } 0 = \frac{\partial}{\partial\alpha}\langle H\rangle = \frac{\hbar^2}{2M}\,\frac{3}{8} - 2\,\lambda\,\alpha^{-3}\frac{15}{16} \Rightarrow \alpha = \left(\frac{\hbar^2}{2M}\,\frac{1}{5\lambda}\right)^{-1/3}$$

$$\Rightarrow \langle H\rangle = \frac{\hbar^2}{2M}\left(\frac{\hbar^2}{2M}\,\frac{1}{5\lambda}\right)^{-1/3}\frac{3}{8} + \lambda\left(\frac{\hbar^2}{2M}\,\frac{1}{5\lambda}\right)^{2/3}\frac{15}{16}$$

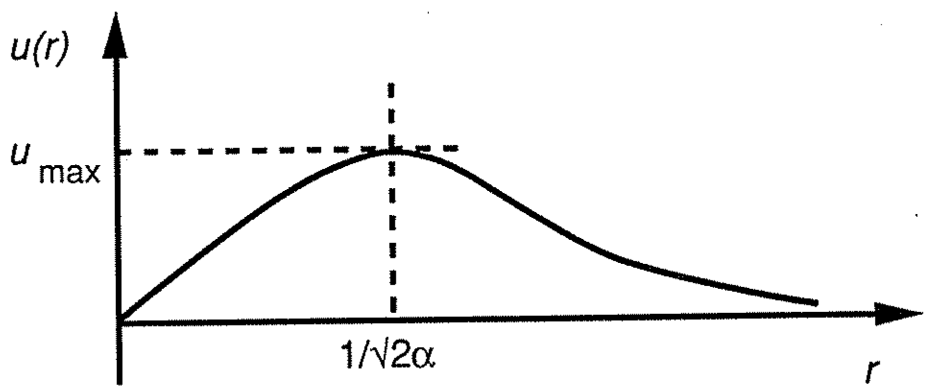
(d) $\phi(r) = Y_{00}(\Omega)\,R_{nl}(r) = \frac{1}{\sqrt{4\pi}}R_{nl}(r)$, radial wave function is $R_{nl}(r)$,

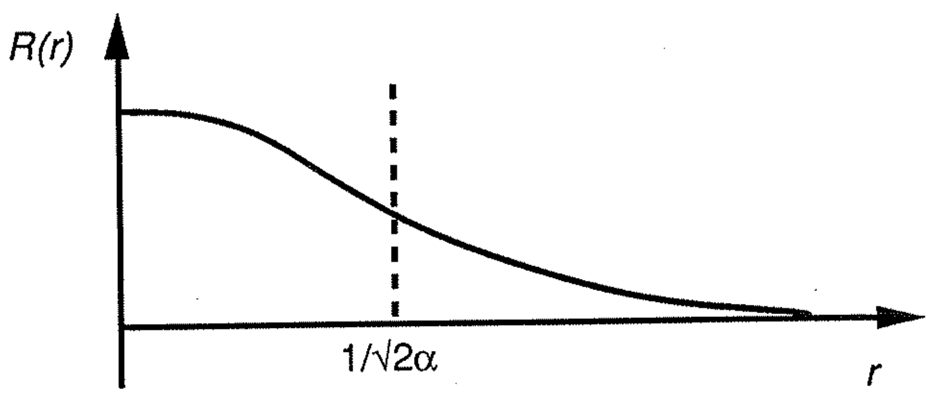
or $u_{nl}(r) = rR_{nl}(r) = r\sqrt{4\pi}\,\phi(r)$.

Probability density is $|\phi|^2$, d(probability of radius r) is $|\phi|^2r^2\mathrm{d}r = u_{nl}{}^2(r)$, maximum where u is max

$u(r) \sim r\exp -\alpha r^2$, max when $\frac{\partial u}{\partial r} = 0 \sim (1 - 2\alpha r^2)\exp -\alpha r^2$, *i.e.* at $r = \frac{1}{\sqrt{2\alpha}}$.

height at maximum is $u_{\max} = \sqrt{\left(\frac{4\pi}{2\alpha}\right)}\,C(\alpha)\,\exp(-1/2) = 2\left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-1/2}$





This problem may also be solved by WKB and/or by arguments based on the scaling and the uncertainty principle.

A uniform solid sphere of radius r and density ρ rolls, without slipping, down a plane inclined at an angle α to the horizontal. What is the velocity of its center after rolling a distance s, and how does this velocity compare with that which would be attained by the sphere in sliding without friction the same distance?

PROB 8

$$I = \frac{2}{5} M r^2 \quad ; \quad T_{rot} = \frac{1}{\cancel{2}\,5} M r^2 \dot{\phi}^2$$

$$T_{trans} = \frac{1}{2} M \left(\frac{ds}{dt}\right)^2$$

Rolling without slipping $\qquad ds = r\,d\phi$

Cons. of Energy

$$T = \left(\frac{1}{5} M + \frac{1}{2} M\right)\left(\frac{ds}{dt}\right)^2 = Mgs \sin\alpha$$

$$\left(\frac{ds}{dt}\right)^2 = \frac{10}{7} gs \sin\alpha$$

If sliding no rolling you can show

$$\left(\frac{ds}{dt}\right)^2_{slide} = 2\, gs \sin\alpha$$

$$\frac{v_{slide}}{v_{roll}} = \sqrt{\frac{2}{\frac{10}{7}}} = \sqrt{\frac{7}{5}}\,.$$