OSU PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #74

March 27 and 28, 1995

Comprehensive examination for Spring 1995

PART I

General Instructions

This Comprehensive Examination for Spring 1995 consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, March 27, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:30 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, March 28.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit – especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

1. A quantum mechanical system in contained in an infinitely long cylinder of radius a. The potential is

$$V(\rho) = V \quad \rho < a$$

$$V(\rho) = \infty \quad \rho > a.$$

(a) Solve Schrodinger's equation in cylindrical coordinates to find the wave functions for this system. You will need the Laplacian,

$$\nabla^2 \psi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}.$$

You might also find Bessel's equation interesting. In its usual form it is

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2} \right) R = 0.$$

(b) For what energies are bound-state solutions possible? Write an equation for these energies.

1.

$$-\frac{\hbar^2}{2m}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2}+\frac{\partial^2\psi}{\partial z^2}\right]+V\psi=E\psi$$

assume $\psi = f(\rho)\, e^{im\varphi}\, e^{i\kappa z}$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{\rho}\frac{d}{d\rho}(\rho \dot{f})-\frac{m^2}{\rho^2}f-\kappa^2 f\right]=(E-V)f$$

let $\varepsilon = -\frac{2m}{\hbar^2}(E-V)$

$$f'' + \frac{1}{\rho}f' + \left(-\frac{m^2}{\rho^2}-\kappa^2+\varepsilon\right)f = 0$$

let $-\kappa^2+\varepsilon = \alpha^2 > 0$ and $x = \alpha\rho$

$$\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}+\left(1-\frac{m^2}{x^2}\right)f=0$$

Since $f(\rho=a)=0$ we must choose the oscillatory fn. that are regular at $\rho=0$.

so $f = c\, J_m(x_{mn}\rho/a)$ $\qquad \alpha = \frac{x_{mn}}{a}$

the oscillatory condition also implies $\alpha^2>0$, so

$$\frac{2m}{\hbar^2}(V-E) > \kappa^2$$

or $V-E > \frac{\hbar^2\kappa^2}{2m}$ = K.E. associated with motion in z-direction

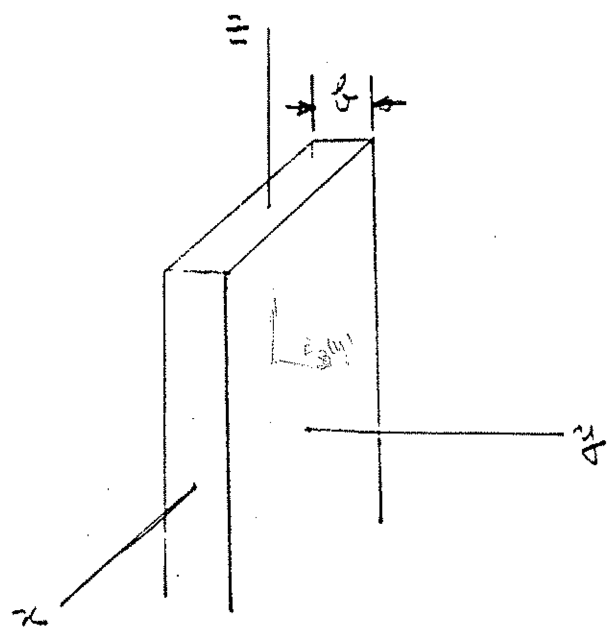
Since KE > 0, $E < V$

2. Consider the dielectric slab shown below, having a width b in the y-direction, but unlimited extent in the x- and z-directions. It has non-zero permittivity ε and permeability μ, but no conductivity (*i.e.* $\sigma = 0$). Consider a transverse magnetic (TM) wave, for which

$$E_z(y, z, t) = E_z(y) e^{i(kz - \omega t)}$$

to be propagating in the $+z$-direction. There is no x-dependence to the electric and magnetic fields in this arrangement.

Total internal reflection occurs at the dielectric-air interfaces at $y = \pm b/2$, and so the system acts as a type of dielectric waveguide. Determine the dispersion relation (ω versus k) for this type of wave traveling in the dielectric. [You won't be able to give an analytic form for $\omega = \omega(k)$.]



2

$$E_z(y,z) = E_z(y)\, e^{i(kz-\omega t)}$$

① Inside slab: $\nabla^2 E_z = -\mu\varepsilon \frac{\omega^2}{c^2} E_z$

or $\frac{\partial^2 E_z(y)}{\partial y^2} + \left(\mu\varepsilon\frac{\omega^2}{c^2} - k^2\right) E_z(y) = 0.$

Try as solution, $E_z(y) = E_0 \cos \gamma y$ (transversely symmetric TM modes — we could have chose the antisymmetric ones just as well).

This tells us that $-\gamma^2 + \left(\mu\varepsilon\frac{\omega^2}{c^2} - k^2\right) = 0.$

② Outside the slab: $\frac{\partial^2 E_z(y)}{\partial y^2} + \left(\frac{\omega^2}{c^2} - k^2\right) E_z(y) = 0$

Solutions of form $E_z(y) = A e^{-\beta y}$; $y \geq b/2$

$= A e^{\beta y}$; $y \leq -b/2$.

and so $\beta^2 + \left(\frac{\omega^2}{c^2} - k^2\right) = 0.$

③ Boundary conditions: (a) Continuity of E_z across face at $y = b/2$

$$E_0 \cos \gamma b/2 = A e^{-\beta b/2} \quad \text{——— (I)}$$

(b) Normal component of $\vec{D}$ across $y = b/2$ has to be continuous.

However, we need to get it from Maxwell:

$\vec{\nabla} \times \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \Rightarrow ikB_x - \frac{\partial B_z}{\partial x} = -i\mu\varepsilon \frac{\omega}{c} E_y$

and we can get B_x in terms of the electric field via

2 continued

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \implies B_x = -\frac{ic}{\omega}\frac{\partial E_z}{\partial y} - \frac{kc}{\omega}E_y$$

Putting this in for B_x on previous page yields

$$E_y = \frac{1}{\left(\mu\varepsilon\frac{\omega^2}{c^2} - k^2\right)}\left[\frac{\partial^2 E_z}{\partial y\,\partial z} - \cancel{\frac{i\omega}{c}\frac{\partial B_z}{\partial x}}\right]$$

0

$\therefore$ D_y (inside) at $y = b/2$ is $\dfrac{\varepsilon i k \gamma E_0 \sin \gamma b/2}{\left(\mu\varepsilon\frac{\omega^2}{c^2} - k^2\right)}$

and this must equal D_y (outside) at $y = b/2$ which is

$$\frac{A}{\frac{\omega^2}{c^2} - k^2}\left(-\beta i k e^{-\beta b/2}\right)$$

$\therefore$

$$\frac{\varepsilon E_0}{\mu\varepsilon\frac{\omega^2}{c^2} - k^2}\left[-ik\gamma\sin\left(\frac{\gamma b}{2}\right)\right] = \frac{A}{\frac{\omega^2}{c^2} - k^2}\left(-ik\beta e^{-\beta b/2}\right)$$

II

Divide LHS and RHS of I by II:

$$\frac{\varepsilon\gamma}{\mu\varepsilon\frac{\omega^2}{c^2} - k^2}\tan\left(\gamma b/2\right) = \frac{\beta}{\omega^2/c^2 - k^2}.$$

You know γ and β as functions of ω and k, so given a value of k you can solve (numerically) for all permitted ω's, and so get the dispersion relations for the mode.

4. An electron is at rest in a uniform magnetic field in the z direction. (The statement "at rest" means that you should calculate the precession of the electron's spin in the magnetic field and ignore the electron's orbit in the field.) Assume that the interaction between the electron spin and the field is given by the Hamiltonian,

$$V = -\boldsymbol{\mu} \cdot \boldsymbol{B}$$

where

$$\boldsymbol{\mu} = \frac{e\hbar}{2mc}\boldsymbol{\sigma}$$

(a) Write Schrodinger's equation for the **spin** wave function.

(b) Assume that at $t = 0$ the electron spin is pointing in the $+x$ direction. Find the spin wave function for the electron.

(c) What happens to this wave function at subsequent times? Describe its behavior in detail. With what period (or periods) does the electron spin precess?

Pr 4-1

$$\vec{\mu} = \frac{e}{mc}\vec{s} = \frac{e\hbar}{2mc}\vec{\sigma} = -\mu_0\vec{\sigma}$$

$$V = -\vec{\mu}\cdot\vec{B}$$

$$H\psi_s = i\hbar\frac{\partial}{\partial t}\psi_s(\vec{r},t)$$

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \mu_z B \qquad \mu_z = \frac{g\mu}{2}\sigma_z$$

$$\psi_s = \psi(r,t)\xi(t) \qquad \xi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

$$-\frac{\hbar^2}{2m}\xi\nabla^2\psi - \frac{g\mu}{2}B_z\,\psi\sigma_z\xi = i\hbar\left(\xi\frac{\partial\psi}{\partial t} + \psi\frac{\partial\xi}{\partial t}\right)$$

Spin part is $\quad -\frac{g}{2}\mu B_z\,\sigma_z\xi = i\hbar\frac{\partial\xi}{\partial t}$

$$-\frac{g}{2}\mu B\begin{pmatrix} a \\ -b \end{pmatrix} = i\hbar\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix}$$

Specialize to electrons $\quad (-g/2) \simeq 1$

$$-i\frac{\mu B}{\hbar}\begin{pmatrix} a \\ -b \end{pmatrix} = \begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = -\frac{i}{2}\Omega\begin{pmatrix} a \\ -b \end{pmatrix}$$

where Ω is cyclotron freq. $= \frac{|e|B}{mc}$

$$a = a_0 e^{-i\Omega t/2}$$

$$b = b_0 e^{+i\Omega t/2}$$

~~Q2 Bonus?~~ 4-2

The spin operator $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$. The eigenfns are

$$\alpha_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad \beta_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\alpha_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \qquad \beta_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

so $S_x\,\alpha_x = +\frac{\hbar}{2}\alpha_x \qquad S_x\,\beta_x = -\frac{\hbar}{2}\beta_x$

$S_y\,\alpha_y = +\frac{\hbar}{2}\alpha_y \qquad S_y\,\beta_y = -\frac{\hbar}{2}\beta_y$

Then $\xi(t=0) = \alpha_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$$\xi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\Omega t/2}\\e^{+i\Omega t/2}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\pi t/T}\\e^{+i\pi t/T}\end{pmatrix}$$

where $T = \frac{2\pi}{\Omega} \qquad \frac{\Omega}{2} = \frac{\pi}{T}$

we see that $\xi(T) = -\alpha_x \qquad \xi(2T) = +\alpha_x$

so the spin must go around twice to get back where it started!

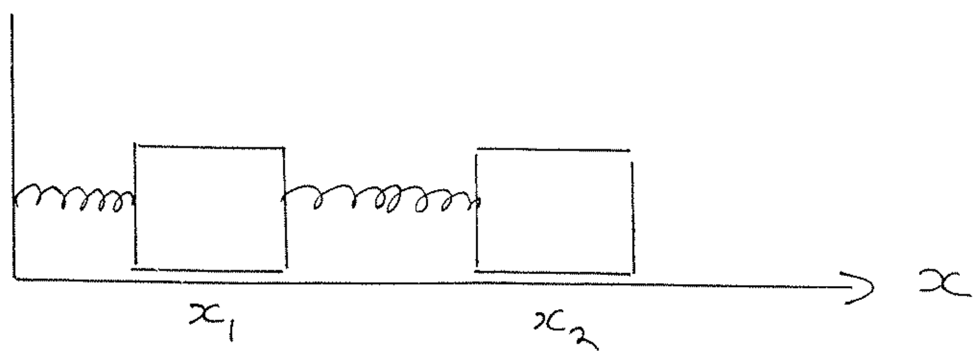
5. Two identical masses lie horizontally on a frictionless surface. One of the masses is attached by a spring to a fixed wall while the other mass, except for its spring coupled to the first mass, vibrates freely. Assuming the masses and spring constants to have unit values

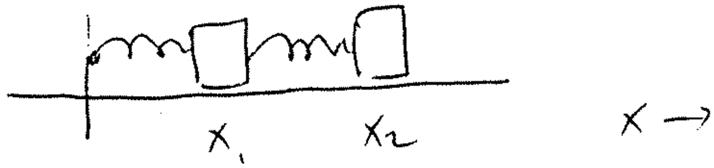
(a) Find the equations of motion for the masses.

(b) Find the allowed frequencies of vibration, ω, for the system.

(c) Find the normal mode corresponding to each of the allowed frequencies.

(d) If the "free" mass at the right is initially ($t = 0$) displaced from its equilibrium position by a distance $\delta > 0$, describe the subsequent motion - in time - of each of the masses.





Trivializing the problem with the Lagrangian

(a) $\mathcal{L} = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 - \frac{1}{2}x_1^2 - \frac{1}{2}(x_2 - x_1)^2$

The equations of motion are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} = 0$$

$$\Rightarrow \quad \ddot{x}_1 + x_1 - (x_2 - x_1) = 0$$

$$\ddot{x}_2 + (x_2 - x_1) = 0$$

(b) Assume solutions $\left.\begin{matrix} x_1 = A_1 e^{i\omega t} \\ x_2 = A_2 e^{i\omega t} \end{matrix}\right\}$ $\Rightarrow$ substitute into Eq. of M.

$$\begin{vmatrix} -\omega^2 + 2 & -1 \\ -1 & -\omega^2 + 1 \end{vmatrix} = 0$$

gives normal frequencies

$$\omega^4 - 3\omega^2 + 1 = 0$$

$$\omega_1^2 = \frac{1}{2}(3 + \sqrt{5})$$

$$\omega_2^2 = \frac{1}{2}(3 - \sqrt{5})$$

$$\omega_1 = \frac{1}{2}(3 + \sqrt{5})^{1/2}$$

$$\omega_2 = \frac{1}{2}(3 - \sqrt{5})^{1/2}$$

(c) Normal modes are found from

$$(-\omega_{1,2}^2 + 2)A_1 - A_2 = 0$$

Thus $\tilde{A}_1 = A_1\left(1, \frac{1}{2}(1-\sqrt{5})\right)$

$\tilde{A}_2 = A_1\left(1, \frac{1}{2}(1+\sqrt{5})\right)$

(These are unnormalized)

(d) Since x_1, x_2 are real

$$x_1 = \mathrm{Re}\left(A_1 e^{i\omega_1 t} + A_2 e^{i\omega_2 t}\right)$$
$$x_2 = \mathrm{Re}\left(B_1 e^{i\omega_1 t} + B_2 e^{i\omega_2 t}\right)$$

$A_1 = A_{1r} + i A_{1I}$
$A_2 = A_{2r} + i A_{2I}$
$B_1 = B_{1r} + i B_{1I}$
$B_2 = B_{2r} + i B_{2I}$

Also At $t=0$ $\left.\begin{matrix} x_2 = a > 0 \\ x_1 = 0 \end{matrix}\right\}$ $\left.\begin{matrix} \dot{x}_2 = 0 \\ \dot{x}_1 = 0 \end{matrix}\right\}$

$\Downarrow$

we can see right off that

$$x_1 = A_1 \cos\omega_1 t + A_2 \cos\omega_2 t$$
$$x_2 = B_1 \cos\omega_1 t + B_2 \cos\omega_2 t$$

These must satisfy the Eq of motion

$$\begin{cases} B_1(-\omega_1^2) + B_2(-\omega_2^2) + B_1 + B_2 - A_1 - A_2 = 0 \\ A_1(-\omega_1^2) + A_2(-\omega_2^2) + A_1 + A_2 - B_1 - B_2 + A_1 + A_2 = 0 \end{cases}$$

$$\begin{cases} B_1(1-\omega_1^2) + B_2(1-\omega_2^2) - A_1 - A_2 = 0 \\ A_1(2-\omega_1^2) + A_2(2-\omega_2^2) - B_1 - B_2 = 0 \end{cases}$$

and
$$\begin{cases} A_1 + A_2 = 0 \\ B_1 + B_2 = a \end{cases}$$

These can be solved to give

$$A_1 = -\frac{a}{\sqrt{5}} \qquad A_2 = \frac{a}{\sqrt{5}}$$

$$B_1 = \frac{1}{10}(5-\sqrt{5})\,a \qquad B_2 = \frac{1}{10}(5+\sqrt{5})\,a$$

6. The earth is an oblate spheroid (its moment of inertia about the body axis of symmetry I_z is slightly larger than the moment of inertial about the remaining principal body axes, $I_x = I_y = I$. (The degree of asymmetry is given by $\frac{I_z - I}{I} \approx 0.003$.)

Show that in the absence of external torques the earth's angular velocity vector

$$\boldsymbol{\Omega} = \omega_x \hat{e}_x + \omega_y \hat{e}_y + \omega_z \hat{e}_z$$

where $\hat{e}_x, \hat{e}_y, \hat{e}_z$ are the body-fixed axes, precesses about the body symmetry axis $\hat{e}_z$ with the frequency

$$\omega_p = \frac{I_z - I}{I} \omega_o$$

where ω_o is the earth's rotational velocity about the e_z axis.

Euler's Eq with $I_x = I_y = I$

$$I \dot{\omega}_x + (I_z - I)\, \omega_y \omega_z = 0$$
$$I \dot{\omega}_y + (I - I_z)\, \omega_z \omega_x = 0$$
$$I_z \dot{\omega}_z = 0$$

$$\omega_z = \text{constant} = \omega_0$$

$$I \dot{\omega}_x + (I_z - I)\, \omega_y \omega_0 = 0$$
$$I \dot{\omega}_y + (I - I_z)\, \omega_x \omega_0 = 0$$

Assume $\vec{\omega} = \omega_x \hat{e}_x + \omega_y \hat{e}_y + \omega_z \hat{e}_z$

$$\vec{\omega} = \mathrm{Re}\left(A_x e^{i\omega t} \hat{e}_x + A_y e^{i\omega t} \hat{e}_y\right) + \omega_0 \hat{e}_z$$

i.e. $\omega_x = A_{xr} \cos\omega t - A_{xI} \sin\omega t$

$\omega_y = A_{yr} \cos\omega t - A_{yI} \sin\omega t$

$\omega_z = \omega_0$

Substitute (then take real part)

$$I A_x i\omega + \omega_0 (I_z - I) A_y = 0$$
$$I A_y i\omega + \omega_0 (I - I_z) A_x = 0$$

$$\begin{vmatrix} I i\omega & \omega_0 (I_z - I) \\ \omega_0 (I - I_z) & I i\omega \end{vmatrix} = 0$$

$$\Rightarrow \omega = \left| \frac{I - I_z}{I} \right| \omega_0$$

Therefore

$$A_y = A_x \frac{I \, i \omega}{\omega_0 (I_z - I)}$$

$$= - i A_x$$

and $\vec{\omega} = \omega_x \hat{e}_x + \omega_y \hat{e}_y + \omega_0 \hat{e}_z$

$$\vec{\omega} = (A_{xr} \cos \omega t - A_{xI} \sin \omega t) \hat{e}_x$$
$$+ (A_{xI} \cos \omega t + A_{xr} \sin \omega t) \hat{e}_y$$
$$+ \omega_0 \hat{e}_z$$

Pick the time origin so that

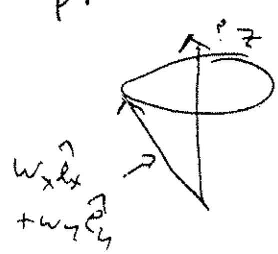
at $t = 0$ $\omega_x = 0$

therefore $A_{xr} = 0$

and

$$\vec{\omega} = \underbrace{- A_{xI} \sin \omega t \, \hat{e}_x + A_{xI} \cos \omega t \, \hat{e}_y}_{\downarrow} + \omega_0 \hat{e}_z$$

ω processes about $\hat{e}_z$ axis



7. A gas of molecules, each with a permanent dipole moment α, can be cooled by reducing an applied uniform electric field under adiabatic conditions. The number of molecules N, and the pressure are constant. A static uniform electric field E_a is applied when the system is in contact with a thermal reservoir at some temperature. The system is then disconected from the thermal reservoir and cooled adaibatially by reducing the field. The dielectric susceptibility of the gas is

$$\chi = \chi_\circ + \frac{\alpha}{3kT}\frac{N}{V}$$

, with $\chi_\circ \approx 0$ for this case. C_{PE} is the heat capacity at constant pressure and applied electric field. Its dependence upon E_a can be determined from the susceptibility.

(a) What is dU, the differential internal energy of the system?

(b) Find the thermodynamic potential A such that $dA = TdS = dQ$ at constant P, E_a and N.

(c) Find another thermodynamic potential A', which is a function of T, P, E_a and N.

(d) What are the twelve Maxwell relations derivable from these two potentials?

(e) Show that, when the external field is changed by dE_a adiabatically, the change in temperature is

$$dT = \frac{N\alpha E_a}{3kT}\frac{dE_a}{C_{PE}(T, E_a = 0) + \frac{N\alpha E_a^2}{3kT^2}}$$

a) $dU' = TdS - PdV + Ed\rho + \mu dN$ where ρ = total dipole moment
and $U' = U - \frac{1}{8\pi}\int |\vec{E}|^2 dV$ (2)

b) $A = U' + PV - E\rho \Rightarrow dA = TdS + VdP - \rho dE + \mu dN$
So, for constant P, E and N, $dA = TdS = dQ$ (2)

c) $A' = U' - TS + PV - E\rho \Rightarrow dA' = -SdT + VdP - \rho dE + \mu dN$ (2)

d) From A: $\left(\frac{\partial V}{\partial S}\right)_{PEN} = \left(\frac{\partial T}{\partial P}\right)_{SEN}$, $-\left(\frac{\partial \rho}{\partial S}\right)_{PEN} = \left(\frac{\partial T}{\partial E}\right)_{SPN}$, $\frac{\partial \mu}{\partial S} = \frac{\partial T}{\partial N}$

$-\frac{\partial \rho}{\partial P} = \frac{\partial V}{\partial E}$, $\frac{\partial \mu}{\partial P} = \frac{\partial V}{\partial N}$, $\frac{\partial \mu}{\partial E} = -\frac{\partial \rho}{\partial N}$ (2)

From A': $\left(\frac{\partial V}{\partial T}\right)_{PEN} = -\left(\frac{\partial S}{\partial P}\right)_{TEN}$, $\frac{\partial \rho}{\partial T} = \frac{\partial S}{\partial E}$, $\frac{\partial \mu}{\partial T} = \frac{\partial S}{\partial N}$ (2)

$-\frac{\partial \rho}{\partial P} = \frac{\partial V}{\partial E}$, $\frac{\partial \mu}{\partial P} = \frac{\partial V}{\partial N}$, $\frac{\partial \mu}{\partial E} = -\frac{\partial \rho}{\partial N}$

e) $dT = \left(\frac{\partial T}{\partial E}\right)_{SPN} dE$ (2) $= -\frac{\partial \rho}{\partial S} dE = -\frac{\partial \rho}{\partial T}\frac{\partial T}{\partial S} dE = -\frac{\partial \rho}{\partial T}\frac{T}{C_{PE}} dE$

since $C_{PE} = T\left(\frac{\partial S}{\partial T}\right)_{PE}$ (2)

Use $\mathcal{P}$ = polarization
Then $dT = -V\left(\frac{\partial \mathcal{P}}{\partial T}\right)_{PE}\frac{T}{C_{PE}} dE$

Use $\mathcal{P} = \chi E$. (2) Then $dT \simeq -\frac{VET}{C_{PE}}\left(\frac{\partial \chi}{\partial T}\right) dE = \frac{NE\alpha}{3kTC_{PE}} dE$ (2)

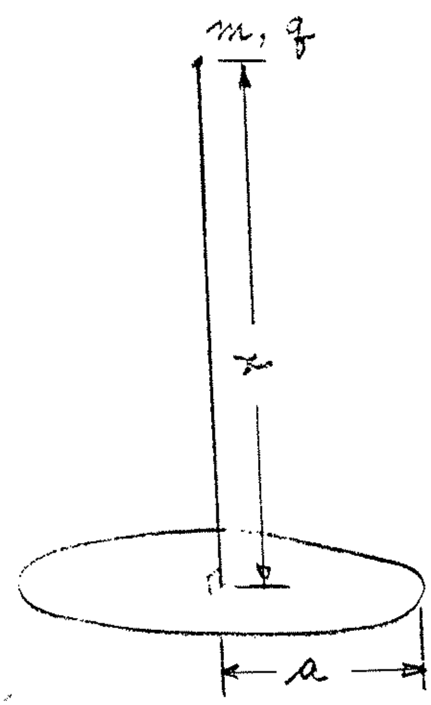
Finally, $C_{PE}(T,E) = \int_0^E \frac{\partial C_{PE}}{\partial E'}(T,E')dE'$

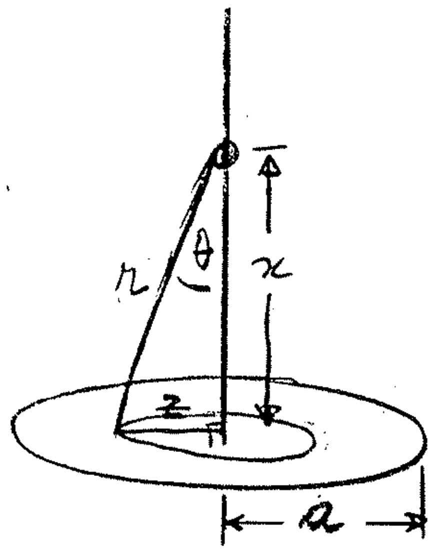
But $\frac{\partial C_{PE}}{\partial E} = T\frac{\partial}{\partial E}\left(\frac{\partial S}{\partial T}\right) = T\frac{\partial}{\partial T}\frac{\partial S}{\partial E} = T\frac{\partial}{\partial T}\frac{\partial \rho}{\partial T} = VT\frac{\partial^2 \mathcal{P}}{\partial T^2}$

$\simeq VT\frac{\partial^2}{\partial T^2}(\chi E) = \frac{VTEN\alpha}{3kV}\frac{2}{T^3} = \frac{2NE\alpha}{3kT^2}$

So $C_{PE}(T,E) = C_{PE}(T, E=0) + \frac{2}{3}\frac{N\alpha}{kT^2}\int_0^E E'dE' = C_{PE}(T,0) + \frac{N\alpha E^2}{3kT^2}$

8. We are told that a new type of charge detector has been developed in the following manner. A circular disk of radius a has a uniform charge density of σ on its surface. A "point" mass m having a radius much smaller than x, has a total charge q distributed on its surface. The height x above the center of the disk that this small sphere "sits at" is obviously determined by the amount of charge on it. Derive an expression that tells us what q is as a function of x. Check the figure below for the configuration.





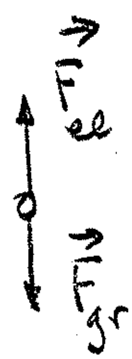
$$E_x = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\, 2\pi z\, dz}{r^2}\cos\theta$$

$$= \frac{\sigma x}{2\varepsilon_0}\int_0^a \frac{z\,dz}{(z^2+x^2)^{3/2}}$$

$$E_x = \frac{\sigma}{2\varepsilon_0}\left[1-\frac{x}{(a^2+x^2)^{1/2}}\right]$$

$$|F_{el}| = q E_x$$

$$|F_{gr}| = mg$$



$$\sum \vec{F} = 0$$

$$F_{el} - mg = 0 \quad \therefore \quad q = \frac{2\pi m g a^2 \varepsilon_0}{Q\left[1-\frac{x}{(a^2+x^2)^{1/2}}\right]}$$

$$Q = \sigma\pi a^2$$