OSU PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #73

Jan. 5 and 6, 1995

Comprehensive examination for Winter 1995

PART ~~III~~ all 4

General Instructions

This Comprehensive Examination for Winter 1995 (#73) consists of eight problems of equal weight (20 points each). It has four parts. The first two parts (Problems 1-4) were administered in two three-hour periods on Thursday, Jan. 5. The third part (Problems 5-6) is handed out at 9:00 am on Friday, Jan. 6, and will also last three hours. The last part (Problems 7-8) will be handed out at 1:30 pm on the same day and will also last three hours.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit – especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, <u>work each problem in its own numbered bluebook</u>, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are <u>not</u> allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

1. Two **identical**, non-interacting particles are contained within a one-dimensional box of length L, *i.e.* $0 \leq x \leq L$.

a) The most general two-particle wave functions for this system that are allowed by the laws of quantum mechanics are called $|m, n>_S$ and $|m, n>_A$, where S and A refer to states that are symmetric or antisymmetric under the permutation of particle coordinates, and m and n are quantum numbers. Write out these wave functions in detail.

b) Calculate the difference in the expected value of the square of the interparticle distance in the antisymmetric and symmetric states, *i.e.* find

$$\Delta = < d^2 >_A - < d^2 >_S$$

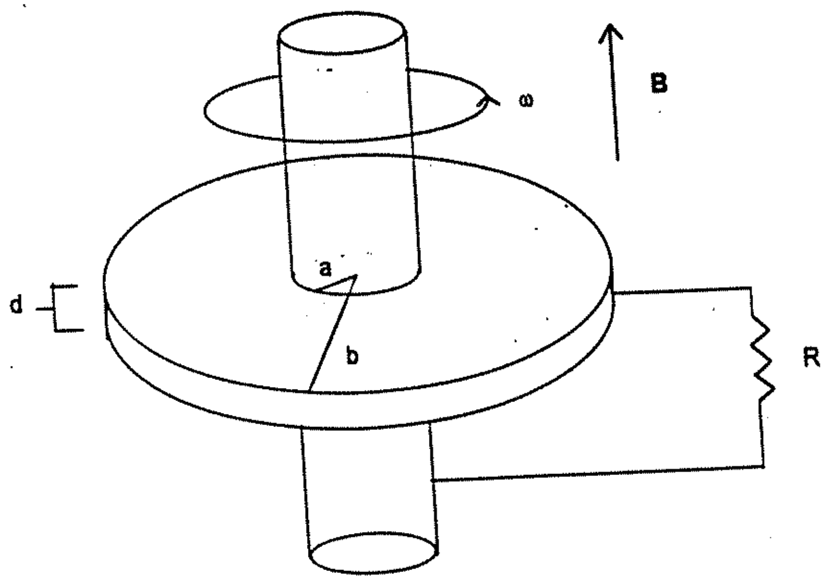
where

$$d^2 = (x_1 - x_2)^2.$$

You may leave your answer in the form of an integral. Show that if $m \neq n$, $\Delta \geq 0$.

c) What is the physical interpretation of the fact that $\Delta \geq 0$?

2. A Faraday disk generator consisting of a metal disk of conductivity σ is rotated with constant angular velocity ω in a uniform $\boldsymbol{B}$ field. Power is delivered to a load resistor. This type of generator delivers high current at a low potential. Assume that the contacts are frictionless and that the



connecting wires are perfect conductors. The inner radius of the disk is a, the outer radius is b, and the thickness is d. The shaft does not contribute to the generator.

a) Find the potential across R as a function of ω, σ, a, b, and R.

b) In the open circuit case, *i.e.* $R = \infty$, calculate the volume charge density ρ and the total volume charge.

c) In the open circuit case, calculate the surface charge density everywhere.

d) In the open circuit case, calculate the surface current density and find the magnetic field just inside the conductor.

e) With a load resistor present, find the torque required to generate a steady current I.

3. A plastic rod initially of length $L = L_0$, is at temperature $T = T_0$ and under no compression ($f = 0$).

An approximate equation of state for the rod near T_0 and L_0 is

$$f = A\left\{\frac{L}{L_0}[1 - \alpha(T - T_0)] - 1\right\},$$

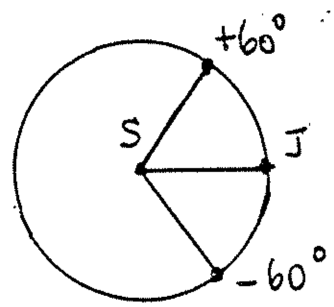
where f is the compression force, with α and A constants assumed to be independent of f, L and T.

If an external compressive force is suddenly (adiabatically) applied along the length of the rod compressing it by a small but finite amount ΔL, show that the final temperature of the rod T_f is given by

$$T_f = T_0\, exp\left[\frac{\alpha A}{c_L}\Delta L\right]$$

where c_L is the specific heat at constant length, which is assumed independent of L and T.

4. Consider Jupiter to revolve around the Sun in a circular orbit. It turns out that there is a condition of unstable equilibrium at points in Jupiter's orbit, 60° ahead of and 60° behind the planet, see the figure. Consider the gravitational potential to be the sum of the following two terms; the first representing that due to the Sun and the second representing that due to Jupiter; $-(1-\mu)/r_S$, and $-\mu/r_J$. This is equivalent to choosing units such that $G(M_S + M_J) = 1$. r_S and r_J are the distances measured from the Sun and Jupiter, respectively. Let the unit of distance be taken to be the separation between the Sun and Jupiter, and let the unit of time be taken such that the angular velocity of the mutual orbit of the Sun and Jupiter in an inertial frame is unity.



a) Find the equations of motion in the rotating frame.

b) Consider the situation of no motion in the rotating frame, and show that the "effective" potential is a maximum at the positions in Jupiter's orbit. indicated above.

c) Describe the action of velocity dependent forces ever present in the rotating frame when motion occurs there.

5. A quantum mechanical particle of mass m and charge e moves under the influence of a constant, uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. The appropriate Hamiltonian is

$$H = \frac{1}{2m}(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})^2$$

where $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$.

a) Choose a gauge such that $\boldsymbol{A}$ points in the (negative) $\hat{x}$ direction. Prove that

$$[p_x, H] = [p_z, H] = 0.$$

b) Prove that with the above assumptions the wave functions have the form

$$\varphi = e^{i(k_x x + k_z z)} f(y).$$

c) Derive a formula for the eigenenergies.

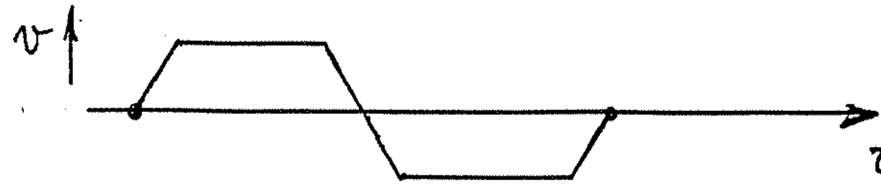
d) What effect will a gauge transformation have on these solutions?

6. For an extremely relativistic ($\epsilon_k = \hbar ck$), highly degenerate ($\mu >> k_B T$) electron gas confined to a volume V, find the energy per particle E/N, where N is the number of electrons.

7. In order to travel to the stars, someone suggests building a rocket that will use photons as fuel. The proposal is to visit the neighborhood of a distant star and then return to Earth. The system is designed to achieve a time-dilation factor of γ during the constant cruising velocity phases of the trip. Show that the fraction (f) of the initial mass ($m_\circ$) that can constitute the payload is given by

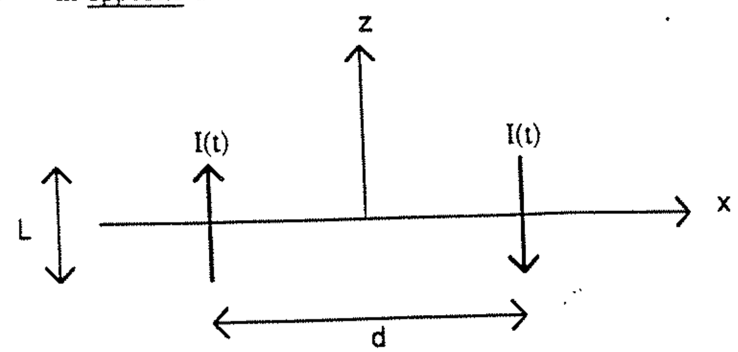
$$f = [\gamma - (\gamma^2 - 1)^{1/2}]^4$$

The graph below indicates the phases of the trip as measured by someone remaining on the Earth.



8. A small antenna located between two perfectly conducting surfaces can radiate a guided electromagnetic field.

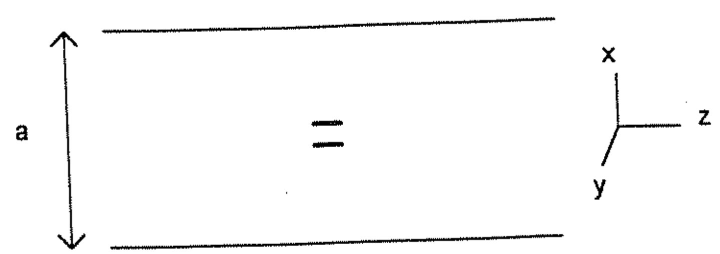
a) Consider an antenna which consists of two short and very thin linear conductors of length L separated by the distance d, as shown in the picture. These conductors carry time harmonic currents $I(t) = Ie^{-i\omega t}$ in opposite directions.



What is the current density representing this radiating system?

What is the first nonzero multipole moment for this radiating source?

b) Now consider positioning a similar antenna midway between two infinite, perfectly conducting planes as depicted below. The antenna is small compared to the gap between the surfaces of the waveguide, that is $a >> L, d$. However, the antenna is of infinite extent in the y direction.



Determine the electric or magnetic field propagating in the z direction for the first allowed mode of this waveguide. The first allowed mode is defined as that for which propagation commences at the lowest frequency.

1.

Single-particle wave fn. are

$$|n\rangle = \sqrt{\frac{2}{L}}\,\sin\left(\frac{n\pi x}{L}\right)$$

Two-particle wave functions must be symmetric or anti-symmetric under exchange

a) $$|n,m\rangle_{\substack{S\\A}} = \frac{1}{\sqrt{2}}\left\{ |n\rangle_1 |m\rangle_2 \pm |m\rangle_1 |n\rangle_2 \right\}$$

coordinate x_1 coordinate x_2

b) $$\langle d^2\rangle = \frac{1}{2}\Big\{ \langle n|x_1^2|n\rangle + \langle m|x_1^2|m\rangle \pm 2\langle n|x_1^2|m\rangle\delta_{nm}$$

$$- 2\langle n|x_1|n\rangle\langle m|x_2|m\rangle \pm 2\langle n|x_1|m\rangle\langle n|x_2|m\rangle + \langle m|x_1|m\rangle\langle n|x_2|n\rangle$$

$$+ \langle m|x_2^2|m\rangle \pm 2\langle n|x_2^2|m\rangle\delta_{nm} + \langle n|x_2^2|n\rangle \Big\}$$

$$= \langle n|x^2|n\rangle + \langle m|x^2|m\rangle \pm 2\langle n|x^2|m\rangle\delta_{nm}$$

$$- 2\langle n|x|n\rangle\langle m|x|m\rangle \mp 2\left(\langle n|x|m\rangle\right)^2$$

c) if $n=m$ then $|n,n\rangle_A = 0$. So assume $n \neq m$

$$\langle d^2\rangle_A - \langle d^2\rangle_S = 4\left(\langle n|x|m\rangle\right)^2 \geq 0$$

$$= 4\left\{\frac{2}{L}\int_0^L x\,\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx\right\}^2$$

Problem 2

a) $\vec{E}' = \vec{E} + \frac{\vec{v}}{c}\times\vec{B}$ (3), so $\vec{J} = \sigma\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$ (2)

Easiest to assume that electric field is only radial, requiring that the inner and outer edges of the disc be perfect conductors (equipotential surfaces). Then $\vec{J}(\rho) = \frac{I}{2\pi\rho d}\hat{\rho}$. (2) Also, $\frac{\vec{v}\times\vec{B}}{c} = \frac{\omega\rho B}{c}\hat{\rho}$

Potential across resistor is $\Phi = \int_a^b \hat{\rho}\cdot\vec{E}(\rho)\,d\rho$ where

$$\vec{E}(\rho) = \frac{\vec{J}}{\sigma} - \frac{\vec{v}}{c}\times\vec{B} = \left[\frac{I}{2\pi d\rho\sigma} - \frac{\omega\rho B}{c}\right]\hat{\rho} \quad (2$$

So $\Phi = \frac{I}{2\pi\sigma d}\ln\frac{b}{a} - \frac{\omega B}{2c}(b^2-a^2)$ but $\frac{I}{2\pi\sigma d}\ln\frac{b}{a} = R_{disc}I$

b) With an open circuit, $\vec{J}=0 \Rightarrow \vec{E} = -\frac{\vec{v}\times\vec{B}}{c}$

So $\nabla\cdot\vec{E} = 4\pi\rho \Rightarrow \rho = -\frac{1}{4\pi c}\nabla\cdot(\vec{v}\times\vec{B}) = -\frac{1}{2\pi c}\omega B$

$Q_{TOTAL} = \int\rho\,dV = -\frac{\omega B}{2\cancel{\pi}c}d(b^2-a^2)$ (4)

c) Use $E(\rho<a)=0$ to get $\sigma(\rho=a)$ from $\left[\vec{E}(\rho=a_+)-\vec{E}(\rho=a_-)\right]\cdot\hat{\rho} = 4\pi\sigma(\rho=a)$

(3) So $\sigma(\rho=a) = -\frac{\omega a B}{4\pi c}$

(3) Also need $\sigma_{TOTAL} + Q_{TOTAL} = 0 \Rightarrow \int\sigma(\rho=b)\,da = -Q_{TOTAL} - \int\sigma(\rho=a)\,da$

or $2\pi b d\,\sigma_b = \frac{\omega B d}{2c}(b^2-a^2) + \frac{\omega a B}{4\pi c}2\pi a d \Rightarrow \sigma_b = +\frac{\omega b B}{4\pi c}$

d) current densities $K(\rho=a) = \sigma_a\omega a$ and $K(\rho=b) = \sigma_b\omega b$

(3) $\left[\vec{B}_{out} - \vec{B}_{in}\right]\times\hat{\rho} = \frac{4\pi}{c}\vec{K}$ is small so $\vec{B}_{in} \simeq \vec{B}_{out}$ at the edges

on inner + outer edges

e) Torque $= \vec{N} = \frac{1}{c}\int\vec{r}\times(\vec{J}\times\vec{B})\,dV = -\hat{z}\frac{1}{c}\int\rho J(\rho)B\,dV = -\hat{z}\frac{1}{c}\int\frac{I}{2\pi d}B\,dV$

$= -\hat{z}\frac{1}{2\pi dc}\pi(b^2-a^2)d = -\hat{z}\frac{I}{2c}(b^2-a^2) = -$ applied torque

3. Solution (I)

Process is isentropic, so with $T(S,L)$

$$dT = \left(\frac{\partial T}{\partial S}\right)_L dS + \left(\frac{\partial T}{\partial L}\right)_S dL$$

and $dS = 0$. Therefore

$$dT = \left(\frac{\partial T}{\partial L}\right)_S dL$$

But from "I/II Law" for elastic rods

$$T dS = du - f dL$$

So that, there is a Maxwell Relation found from

$$du = T dS + f dL$$

which is $\left(\frac{\partial T}{\partial L}\right)_S = \left(\frac{\partial f}{\partial S}\right)_L$.

Therefore

$$dT = \left(\frac{\partial f}{\partial S}\right)_L dL$$

$$= \frac{\left(\frac{\partial f}{\partial T}\right)_L}{\left(\frac{\partial S}{\partial T}\right)_L} dL$$

$$= \frac{T}{C_L}\left(\frac{\partial f}{\partial T}\right)_L dL$$

From Eq of State:

$$\left(\frac{\partial f}{\partial T}\right)_L = - A\frac{L}{L_0}\alpha$$

so $\frac{dT}{T} = -\frac{A\alpha L\, dL}{C_L L_0}$ which is easily integrated

(II) Most students proceeded along the lines:

1st LAW: $đQ = du - f\,dL$

with $đQ = 0$,

so that: $du = f\,dL$. (*)

They then made the mistake of saying

$$du = \left(\frac{\partial u}{\partial T}\right)_L dT$$

$$= c_L\,dT$$

which would be true for an "ideal" system.

However, the equation of state is not that of an ideal system, so that $\left(\frac{\partial u}{\partial L}\right)_T \neq 0$ and

$$du = \left(\frac{\partial u}{\partial T}\right)_L dT + \underline{\underline{\left(\frac{\partial u}{\partial L}\right)_T}}\, dL \quad ; \left[\left(\frac{\partial u}{\partial T}\right)_L = c_L\right]$$

This gives from (*)

$$c_L\,dT = \left[f - \left(\frac{\partial u}{\partial L}\right)_T\right] dL$$

and $\left(\frac{\partial u}{\partial L}\right)_T$ <u>must</u> <u>be</u> <u>found</u>.

It turns out (see next page) that

$$\left(\frac{\partial u}{\partial L}\right)_T = -T\left(\frac{\partial f}{\partial T}\right)_L + f$$

so $c_L\,dT = T\left(\frac{\partial f}{\partial T}\right)_L dL$ as in method (I).

1st Law for elastic rod:

$$dS = \frac{1}{T}\,dU - \frac{f}{T}\,dL$$

$$= \frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_L dT - \frac{1}{T}\left[f - \left(\frac{\partial U}{\partial L}\right)_T\right] dL$$

Since dS is exact there is a Maxwell relation

$$\frac{\partial}{\partial L}\left[\frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_L\right]_T = -\frac{\partial}{\partial T}\left\{\frac{1}{T}\left[f - \left(\frac{\partial U}{\partial L}\right)_T\right]\right\}_L$$

So that

$$\frac{1}{T}\,\frac{\partial^2 U}{\partial L\,\partial T} = +\left\{\left(+\frac{1}{T^2}\right)\left(f - \left(\frac{\partial U}{\partial L}\right)_T\right)\right\} - \frac{1}{T}\left(\frac{\partial f}{\partial T}\right)_L + \frac{1}{T}\left(\frac{\partial^2 U}{\partial L\,\partial T}\right)$$

and

$$\left(\frac{\partial U}{\partial L}\right)_T = f - T\left(\frac{\partial f}{\partial T}\right)_L$$

4. $$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{\omega}\times \quad .$$

$$\frac{\partial^2 \vec{r}}{\partial t^2} = \underbrace{\frac{d^2\vec{r}}{dt^2} - \vec{\omega}\times(\vec{\omega}\times\vec{r})}_{} - 2\vec{\omega}\times\frac{\partial \vec{r}}{\partial t}$$

$$= \quad -\vec{\nabla} U \qquad + 2\omega\frac{\partial y}{\partial t}\vec{i} - 2\omega\frac{\partial x}{\partial t}\vec{j}$$

Here, $\vec{\omega}$ has been taken to be in z-direction, and so

$$U = -\left(\frac{1-\mu}{r_1}\right) - \frac{\mu}{r_2} - \frac{1}{2}\left(\omega^2 x^2 + \omega^2 y^2\right).$$

(1) Yielding Equs. of Motion,

$$\ddot{x} = -\frac{\partial}{\partial x}U + 2\dot{y}\omega$$

$$\ddot{y} = -\frac{\partial}{\partial y}U - 2\dot{x}\omega$$

$$\ddot{z} = -\frac{\partial U}{\partial z}$$

with $\omega = R = 1$, we get when operating on U, the following expressions

(2)

$$-\frac{\partial U}{\partial x} = -\frac{(1-\mu)}{2}\frac{2x}{(x^2+y^2)^{3/2}} + \frac{\mu\, 2(x-1)}{2\left[(x-1)^2+y^2\right]^{3/2}} + x = 0$$

and

$$-\frac{\partial U}{\partial y} = \frac{-(1-\mu)\,y}{(x^2+y^2)^{3/2}} - \frac{\mu y}{\left[(x-1)^2+y^2\right]^{3/2}} + y = 0,$$

satisfied for $x = \frac{1}{2}$ and $y = \pm\frac{\sqrt{3}}{2}$, thus specifying the points at 60° ahead and behind Jupiter in its orbit.

To see that the extremum is a max., check $\frac{\partial^2 U}{\partial x^2}$ and $\frac{\partial^2 U}{\partial y^2}$ at these points. The second derivatives are negative. Thus the points are maxima in the potential.

(3) "Equilibrium" is produced by action of the Coriolis force when particle has some non-zero velocity in x-y. ~~Potential force alone~~ $\frac{\partial U}{\partial z}$ returns in z-direction.

5. Let $\vec{A} = (-yB, 0, 0)$ so that

$$\vec{B} = \vec{\nabla} \times \vec{A} = B\hat{z}$$

a) Then $H = \frac{1}{2m}\left(p_x + \frac{eyB}{c}\right)^2 + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}$

since $[y, \hat{p}_y] \neq 0$ but $[p_x, \hat{y}] = [p_z, y] = 0$

b) p_x, p_z, and H have simultaneous eigenstates. thus we can choose $\psi = e^{i(k_x x + k_z z)} f(y)$

c) Let $H\psi = E\psi$ and try to find f.

$$\left[\frac{p_y^2}{2m} + \frac{m\Omega^2}{2}(y - y_0)^2\right] f = \left(E - \frac{\hbar^2 k_z^2}{2m}\right) f$$

where $\Omega = \frac{eB}{mc}$, the cyclotron frequency, and $y_0 \equiv -\frac{c\hbar k_x}{eB}$. The quantity in [ ] looks like the hamiltonian for a harmonic oscillator oscillating along the y-axis around y_0 with spring constant

$$K = m\Omega^2.$$

The eigenenergies must be $E_n = \hbar\Omega(n + 1/2) + \frac{\hbar^2 k_z^2}{2m}$.

d) A gauge transformation would change the axis of oscillation.

6. (a)

$$E = \frac{V\, 2 \times 4\pi}{(2\pi)^3} \int_0^{k_0} (\hbar c k) \; k^2 dk$$

$$= V \frac{\hbar c}{\pi^2} \frac{k_0^4}{4}$$

where $\hbar c k_0 = \mu_0$

$$N = V \times \frac{2 \times 4\pi}{(2\pi)^3} \int_0^{k_0} k^2 dk$$

$$= \frac{1}{\pi^2} \frac{k_0^3}{3}$$

$$\frac{E}{N} = \frac{3}{4} \hbar c k_0 = \frac{3}{4} \mu_0$$

(b) Can proceed with $\quad p = \frac{1}{3}\left(\frac{\partial E}{\partial V}\right)_T$

or $\quad \Omega = -pV$

$$\Omega = -\frac{1}{\beta} \log Z_{gr} = -\frac{1}{\beta} \times 2 \times \frac{V}{(2\pi)^3} \times 4\pi \int_0^{\infty} k^2 dk \log(1 + e^{-\beta(\cdot}$$

Integrate by parts with $\mu \gg kT$

$$\Omega = -2 \times \frac{\hbar c}{2\pi^2} \frac{V}{3} \int_0^{k_0} k^3 dk$$

$$= -\frac{\hbar c}{\pi^2} \frac{1}{3} V \frac{k_0^4}{4} = -\frac{V}{(\hbar c)^3} \frac{1}{3} \frac{\mu_0^4}{4}$$

7.

$$\gamma = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}} \quad ; \quad \tau = \gamma \tau_0 .$$

Conservation of mass-energy:

$$m_0 c^2 = (f_1 m_0) \gamma c^2 + E_r$$

Conservation of momentum:

$$0 = (f_1 m_0) \gamma v - \frac{E_r}{c}$$

$$\therefore \quad 1 = \gamma f_1 + \gamma f_1 \frac{v}{c} \text{ , or since } \frac{v}{c} = \frac{1}{\gamma}\left(\gamma^2 - 1\right)^{1/2}$$

$$f_1^2 - 2\gamma f_1 + 1 = 0 .$$

$$\therefore \; f_1 = \gamma \pm \left(\gamma^2 - 1\right)^{1/2} \text{ , and since } f_1 < 1$$

$$f_1 = \gamma - \left(\gamma^2 - 1\right)^{1/2}$$

There are such stages of acceleration on the total return trip,

$$\therefore \; f = f_1^4 = \left[\gamma - \left(\gamma^2 - 1\right)^{1/2}\right]^4 .$$

Problem 8

1. $\vec{J}(\vec{r}) \simeq I L\, \delta(z)\delta(y) \left[\delta(x+\frac{d}{2}) - \delta(x-\frac{d}{2})\right] e^{-i\omega t}$ (4)

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \frac{\vec{J}(\vec{r}')\, e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\, dv' \simeq \frac{e^{ikr}}{cr}\int \vec{J}(\vec{r}')\, e^{-ik\hat{n}\cdot\vec{r}'}\, dv'$$

$$= \frac{e^{ikr}}{cr}\sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}\int \vec{J}(\vec{r}')\,(\hat{n}\cdot\vec{r}')^n\, dv' \quad (2)$$

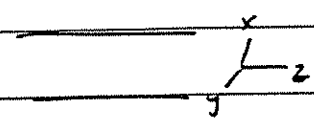
for elec. dipole term, $\int \vec{J}(\vec{r}')\,dv' = 0$ (2)

So elec. quadrupole and magnetic dipole terms are the first nonzero terms

For mag. dip., $\vec{A}(\vec{r}) \simeq ik(\hat{n}\times\vec{m})\dfrac{e^{ikr}}{r}$ where $\vec{m} = \dfrac{1}{2c}\int \vec{r}\times\vec{J}\, dv$ (2)

$\vec{m} = \dfrac{IL}{2c} d\,\hat{y}$ and $\vec{E} = -k^2 \dfrac{e^{ikr}}{r}\,\hat{n}\times\vec{m}$ and $\vec{B} = \hat{n}\times\vec{E}$

2. Elec. quad ~~or~~ or mag. dip. source both can radiate only TM waves into this structure. (1)



B.C.: $E_{\parallel} = E_z = 0$ at conducting surfaces. (2)

$k^2 = \beta^2 + \gamma^2$

(2) $\left[\nabla^2 + \gamma^2\right] E_z = 0 \Rightarrow E_z(x,z) = E_0 \cos\gamma x\; e^{+i\beta z} e^{-i\omega t}$ (2)

$\gamma = m\pi/a$ (1)

$E_x = \dfrac{i\beta}{\gamma^2}\vec{\nabla}_t E_z$ (2)