OSU PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #71

March 28 and 29, 1994

Comprehensive examination for Spring 1994

PART I

General Instructions

This Comprehensive Examination for Winter 1994 (#71) consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, March 28, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:30 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, March 29.

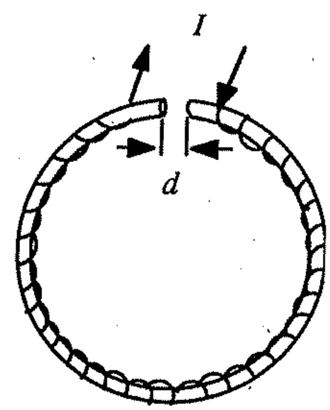
Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit – especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Problem 1 A ferromagnetic rod of length L and negligible diameter is bent into a circular shape leaving a small gap of width d between the ends of the rod. A coil consisting of N turns of wire is wrapped around the rod as shown in the sketch. The coil carries a current I. Assume that the permeability μ is large and that $d << L$. Make any other assumptions necessary to get a simple answer without any complications due to the geometry.

(a) Find the magnetic field $\boldsymbol{H}$ and induction $\boldsymbol{B}$ in the gap and in the material.



(b) Calculate all the above fields in the limit $\mu \to \infty$. Prove that $\boldsymbol{H} \to 0$ in the rod.

(c) There is an apparent paradox here. We are accustomed to thinking that $\boldsymbol{H}$ is produced by the true currents since the curl of $\boldsymbol{H}$ is proportional to the true current density, but in the above case $\boldsymbol{H}$ only exists in the gap. How does this situation come about?

Problem 2 A spinless particle of mass m moving in three dimensions is scattered by the potential

$$V(r) = V_0 \quad \text{for } |r| < a$$
$$= 0 \quad \text{for } |r| \geq a$$

Its kinetic energy is $E = \frac{\hbar^2 k^2}{2m} << mc^2$, and the potential is repulsive, $V_0 > 0$.

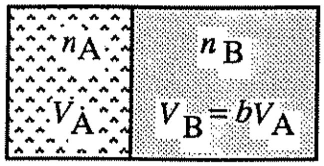
(a) Find a suitable approximation for the differential cross section $d\sigma/d\Omega$ valid in the case when the energy E is very large.
How does your answer depend on the scattering angle θ ?
How large does E have to be for your expression to be a good approximation?

(b) Find a suitable approximation for the differential cross section $d\sigma/d\Omega$ valid in the case when the energy E is very small.
How does your answer depend on the scattering angle θ ?
How small does E have to be for your expression to be a good approximation?

(c) How would your answers to parts (a) and (b) change if the potential were attractive, $V_0 < 0$?

END OF PART I

Problem 3 A box is maintained at constant temperature T and is divided into two sections as shown below.



The volume V_B of the right-hand compartment is b times as large as the volume V_A of the left-hand one, $V_B = b\,V_A$. The left-hand compartment contains n_A moles of ideal gas A and the right-hand compartment contains n_B moles of ideal gas B. The temperature T is the same in both sections and is sufficiently high that classical (Boltzmann) statistics may be used. The partition is now removed and the system is allowed to come to equilibrium. Both gases are monoatomic.

(a) Show that the entropy of n moles of an ideal gas of volume V at temperature T is

$$S = nR\left\{ \frac{5}{2} + \ln\left[(2\pi m k_B T/\hbar^2)^{3/2}\frac{V}{nN_0}\right]\right\}$$

where N_0 is Avogadro's number, k_B is the Boltzmann constant, $R = k_BT$ is the ideal gas constant, and m is the mass of each atom.

(b) Derive an expression for the total change of entropy $S_{final} - S_{initial}$.

(c) Repeat the calculation of part (b) for the case when gasses A and B are identical, and use your result to obtain an expression for the entropy of mixing ΔS_{mix}.

(d) The entropy of mixing appears to be paradoxical in the sense that its value does not depend on "how different" gases A and B are, *i.e.* ΔS_{mix} does not vanish if A and B are infinitesimally different. Explain how this dilemma is resolved.

HINT: The free energy of an ideal gass is $F = -nRT\left\{ 1 + \ln\left[(2\pi m k_B T/\hbar^2)^{3/2}\frac{V}{nN_0}\right]\right\}$

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Problem 4 A tree has a very large main trunk, which is vertical, and a branch which branches out at an angle of $90^o = \pi/2$ radians. A child likes to climb out on the limb until she is 2 m from the trunk, and then bounce up and down on the branch, which oscillates up and down with a period of $T_{||}$= 1.6 s. However sometimes a gust of wind causes the child and branch to swing horizontally, in which direction the branch and child swing sideways with a period of $T_\perp = 1.8$ s. When the child is not on the branch, the wind makes it sway horizontally with a period $T_\perp^0 = 1.2$ s.

(a) What is the period of the branch's vertical oscillation when the child is not on the branch? $T_{||}^0 = ?$ You may assume that the shape of the branch is unaffected by the presence of the child and is the same for all types of motion.

To avoid having to climb the tree, the child cuts it down, after which the main trunk lies horizontally on the ground with the branch pointing into the air at an angle of 45^o from the vertical. She jumps onto it, seating herself at the top of her jump, and the branch starts swinging in the vertical direction, dipping 0.3 m before it starts to move up again.

To the child's surprise, her vertical motion soon turns into a horizontal motion, unseating her.

(b) Why does the branch start to oscillate horizontally?

(c) How long does it take before the motion is completely horizontal?

(d) What is the maximum acceleration in the horizontal plane?

HINT: The trunk does not move. See model on display in examination room.

END OF PART II

Problem 5 An He^+ ion consists of a nucleus with charge $+2e$ bound to a single electron with charge $-e$. In this problem, neglect the spins of the electron and nucleus, and assume that the mass of the nucleus is so large that its motion may be neglected (it is fixed at the origin r=0), while its size is so small that it can be considered a point charge.

(a) Give a complete set of commuting observables (CSCO) which includes the Hamiltonian. What bound-state energies are possible? Express your answer in terms of the Rydberg energy $E_0 = \frac{m_e e^4}{2\hbar^2} = 13.6\ eV$.

(b) What is the degeneracy of each energy level? For each energy, what are the possible eigenvalues of the other observables in your CSCO?

(c) The He^+ ion is created at time t=0 by the sudden radioactive decay of an atom of tritium, which is a hydrogen atom with an unusually heavy nucleus of charge $+e$. Before and until t=0, the electron is in the ground state of the hydrogen atom. After the decay at time $t > 0$, which energies might the He^+ ion have? Which values can be found for the other observables?

(d) Find the probability that the electron is more tightly bound in the He^+ ion than it was in the original hydrogen atom.

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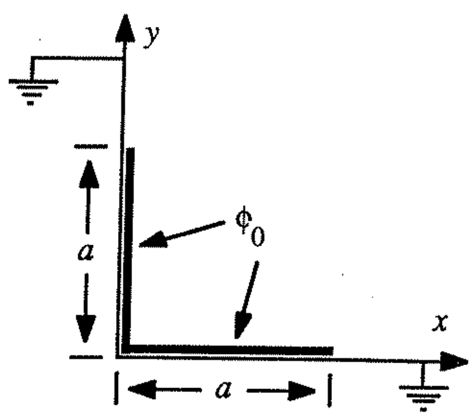
Problem 6 Two conducting planes with $x=0$ and $y=0$ isolate the quadrant $(x>0, y>0, z)$ from the rest of space. These conducting plates are grounded, *.i.e.* kept at an electrostatic potential $\phi=0$.

(a) What is the static, Dirichlet Green's function for the potential in this quadrant corresponding to the conducting-plane boundary conditions?

Now an infinitely long L-shaped conductor is placed in this quadrant, parallel to the z axis and very close to it. This conductor is bent at right angles, and each leg of the L has a length a. This new conductor is separated from the planes of part (a) by an infinitesimally thin layer of insulation, and is maintained at a constant potential ϕ_0.

(b) Find the electrostatic potential $\phi(r)$ everywhere in the quadrant $(x>0, y>0, z)$



END OF PART III

Problem 7 A spinless particle of mass m moves nonrelativistically in three dimensions in a harmonic potential $V(r) = \frac{1}{2}Kr^2$. Its wave function is given by $\langle r|\psi\rangle = \psi(r) = u_0(x)u_1(y)u_0(z)$ where $u_n(s)$ is the normalized wave function of the n'th eigenstate of the 1-dimensional oscillator ($u_0 \equiv$ ground state).

(a) What values are possible for the total angular momentum J^2? With what probabilities may they be found?

(b) What values are possible for the x-component of the angular momentum J_x? With what probabilities may they be found?

(c) What values are possible for the y-component of the angular momentum J_y? With what probabilities may they be found?

(d) Instead, the particle is a spin-1/2 particle with its spin fully polarized along the $+z$ -direction. Now, what values are possible for the total angular momentum J^2, and with what probabilities may they be found?

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Problem 8 The vibrational energy associated with a particular state of an N-atom crystal can be written

$$E = E_0 + \sum_{r=1}^{3N} n_r \hbar\omega_r \quad ,$$

where E_0 is the zero-point energy and $n_r = 0,1,2,...$ is the occupation number of the r^{th} normal mode, whose frequency is ω_r .

(a) Show that the partition function Z can be expressed in terms of the normal mode spectral density of states $\sigma(\omega)$ according to

$$\ln Z = -\beta E_0 - \int_0^\infty \ln(1-e^{-\beta\hbar\omega})\, \sigma(\omega)\, d\omega$$

where $\beta \equiv 1/kT$.

(b) Use the result of part (a) to evaluate $\ln Z$ for the Debye model using

$$\sigma(\omega) = A\omega^2 \text{ for } 0 \le \omega \le \omega_D$$
$$= 0 \text{ otherwise.}$$

Express your result in terms of $\Theta_D \equiv \hbar\omega_D/k$ and $D(y)$ where

$$D(y) \equiv \frac{3}{y^3} \int_0^y \frac{x^3 dx}{e^{+x} - 1}$$

The spectral density is normalized such that $\int_0^{\omega_D} \sigma(\omega)\, d\omega = 3N$.

(c) Again using the Debye model, evaluate the average energy of the solid at temperature T .

END OF EXAMINATION

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Problem 1

$$\oint \vec{H}\cdot d\ell = \frac{4\pi}{c} I \leftarrow \text{total real enclosed current}$$

Call the magnetic material Region 1 and the gap Region 2.

$$H_1 L + H_2 d = \frac{4\pi N I}{c}$$

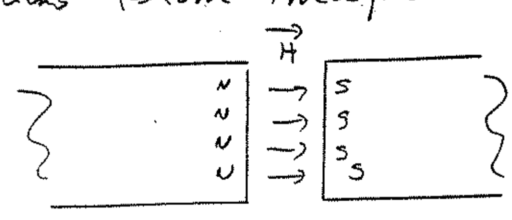
The normal comp. of $\vec{B}$ is continuous at the interface, so

$$B_1 = B_2 = H_2$$
$$B_1 = \mu H_1 = H_2$$

a)
$$H_1(L + \mu d) = 4\pi N I / c$$
$$H_1 = \frac{4\pi N I}{c\,(L+\mu d)}$$
$$B = H_2 = \frac{4\pi N I \mu}{c\,(L+\mu d)}$$

b) as $\mu \to \infty$ $\quad H_1 = 0$ & $H_2 = \dfrac{4\pi N I}{c d}$

c) A vector field is determined by its curl and divergence. $\nabla\cdot\vec{H} \neq 0$ but gets contributions from magnetic surface "charges"



Problem 2 (a) At very high energy, if $E >> V_0$, can use the first Born approximation,

$$d\sigma/d\Omega = \frac{m^2}{4\pi^2\hbar^2} \left| \int d^3r \; e^{i\mathbf{q}\cdot\mathbf{r}} \, V(r) \right|^2$$ where $\hbar \mathbf{q}$ is the momentum transfer

choose $\mathbf{q}$ as the axis of a spherical coordinate system, then

$$\int d^3r \; e^{i\mathbf{q}\cdot\mathbf{r}} \, V(r) = V_0 \int_0^a r^2 dr \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi \, \exp i\, qr \cos\theta$$

$$= 2\pi V_0 \int_0^a r^2 dr \, [\, \exp i\, qr \; - \; \exp -i\, qr \,]/(iqr) = \frac{4\pi}{q} V_0 \int_0^a r \, dr \sin qr \; = -\frac{4\pi}{q} V_0 \frac{\partial}{dq} \int_0^a dr \cos qr$$

$$= -\frac{4\pi a}{q} V_0 \frac{\partial}{\partial q} q^{-1} \int_0^{qa} dx \cos x \; = -\frac{4\pi}{q} V_0 \frac{\partial}{\partial q} \frac{\sin qa}{q} = \frac{4\pi}{q^3} V_0 \, (\, \sin qa \; - qa \cos qa \,)$$

note $q = |\mathbf{k}_f - \mathbf{k}_i| = 2k \sin \theta/2$ where θ is the scattering angle $\Rightarrow$

cross section falls off rapidly for $\theta > (ka)^{-1}$

(b) If $ka << 1$, then only s-wave scattering occurs and the cross section is isotropic;

this is equivalent to $E << \hbar^2/2ma^2$. Then $d\sigma/d\Omega \approx k^{-2}\sin^2 \delta_0$.

If in addition $E << V_0$, then the barrier is essentially infinite so the l=0 radial wave function is just proportional to $\sin k\,(r-a) = \sin\,(kr - \delta_0)$, and $\sin^2 \delta_0 = \sin^2 ka \approx k^2a^2$,

and $d\sigma/d\Omega \approx a^2$.

(c) The answer to part (a) would not change since it doesn't depend on the sign of V_0, and neither does the test for validity of the Born approximation.

For part (b) the scattering would become isotropic when ka <<1 as before. However the expression for δ_0 would be different, and it might not go to a constant until the energy becomes very small if there is a resonance near E =0.

Problem 3

Free energy of ideal gas:

$$F = -kTN\left[\ln V - \tfrac{3}{2}\ln\beta + \tfrac{3}{2}\ln\left(\frac{2\pi m}{\hbar^2}\right) - \ln N + 1\right]$$

(a) $S = k\left[\ln Z + \beta \bar{E}\right]$

$$\ln Z = -\frac{F}{kT} = N\left[\ln V - \tfrac{3}{2}\ln\beta + \tfrac{3}{2}\ln\left(\frac{2\pi m}{\hbar^2}\right) - \ln N + 1\right]$$

$$\bar{E} = -\frac{\partial}{\partial\beta}\ln Z = \frac{3}{2}\frac{N}{\beta} = \frac{3}{2}NkT$$

so $$S = kN\left[\ln V - \tfrac{3}{2}\ln\beta + \tfrac{3}{2}\ln\left(\frac{2\pi m}{\hbar^2}\right) - \ln N + \tfrac{5}{2}\right]$$

$k = R/N_0 \qquad \frac{N}{N_0} = n$

$$S = nR\left[\ln\left[\left(\frac{2\pi m kT}{\hbar^2}\right)^{3/2}\frac{V}{nN_0}\right] + \tfrac{5}{2}\right]$$

Problem 3 (continued)

(b) For changes at constant T, need only consider

$$S' = nR \ln \frac{V}{nN_0}$$

$$S_{initial} = n_A R \ln \frac{V_A}{n_A N_0} + n_B R \ln \frac{bV_A}{n_B N_0}$$

$$S_{final} = n_A R \ln \frac{(1+b)V_A}{n_A N_0} + n_B R \frac{(1+b)V_A}{n_B N_0}$$

$$S_f - S_i = n_A R \ln \left[\frac{(1+b)V_A}{n_A N_0} \cdot \frac{n_A N_0}{V_A} \right]$$

$$+ n_B R \ln \left[\frac{(1+b)V_A}{n_B N_0} \cdot \frac{n_B N_0}{bV_A} \right]$$

$$= n_A R \ln (1+b) + n_B R \ln \left(\frac{1+b}{b} \right)$$

$$S_f - S_i = R \ln \left[\frac{(1+b)^{n_A + n_B}}{b^{n_B}} \right]$$

(c) If gases A + B are identical:

$$S_f = (n_A + n_B) R \ln \frac{(1+b)V_A}{(n_A + n_B) N_0}$$

$$S_f - S_i = n_A R \ln \left\{ \left(\frac{n_A}{n_A + n_B} \right) (1+b) \right\}$$

$$+ n_B R \ln \left\{ \left(\frac{n_B}{n_A + n_B} \right) \left(\frac{1+b}{b} \right) \right\}$$

Problem 3 (continued)

$$\Delta S_{mix} = S^{\text{different}}_{\text{final}} - S^{\text{identical}}_{\text{final}}$$

$$= n_A R \ln \frac{(1+b)V_A}{n_A N_0} + n_B R \frac{(1+b)V_A}{n_B N_0}$$

$$- (n_A + n_B) R \ln \frac{(1+b)V_A}{(n_A+n_B)N_0}$$

$$= n_A R \ln \left\{ \frac{(1+b)V_A}{n_A N_0} \cdot \frac{(n_A+n_B)N_0}{(1+b)V_A} \right\}$$

$$+ n_B R \ln \left\{ \frac{(1+b)V_A}{n_B N_0} \cdot \frac{(n_A+n_B)N_0}{(1+b)V_A} \right\}$$

$$\Delta S_{mix} = n_A R \ln \left(\frac{n_A+n_B}{n_A} \right) + n_B R \ln \left(\frac{n_A+n_B}{n_B} \right)$$

if $n = n_A + n_B$

$$\Delta S_{mix} = R \left[- n_A \ln n_A - (n-n_A) \ln (n-n_A) + n \ln n \right]$$

$$= -R \left[n_A \ln n_A + (n-n_A) \ln (n-n_A) - n \ln n \right]$$

(d) This is an example of the "Gibbs paradox". The point is that at the level of individual particles - atoms, molecules, nucleons - quantum mechanics does not permit "infinitesimal differences". The classical limit of the statistics refers to the temperature being sufficiently high that both F-D and B-E distributions ⇒ Boltzmann. One is still concerned with quantum particles.

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Problem 4

(a) In each case we assume the motion is harmonic, so it is characterized by an inertia I and a restoring force constant $k = I\omega^2$. The inertia is the sum of a contribution from the child and one from the branch, $I = I_{\text{child}} + I_{\text{branch}}$. The restoring force comes from the branch and is therefore the same whether the child is there or not. Note that gravity adds a constant force and thus doesn't affect the oscillation's period, but merely displaces its equilibrium position. Evidently the restoring force constant is different for the horizontal and vertical motion.

Thus the ratio $T/T^0 = \omega_0/\omega = (I/I^0)^{1/2}$ is the same in both cases, $T/T^0 = (1.8)/(1.5) = 6/5$, so $T^0_{\|} = \frac{5}{6} T_{\|} = 1.33$ s.

Due to an unfortunate error in transcribing this problem, the geometry of the fallen tree in parts (b) through (d) is described incorrectly. For the given geometry, the vertical oscillation remains vertical, but it has the frequency of the previously horizontal oscillation of the upright tree.

In the intended geometry, after the tree is felled, the branch is horizontal and the trunk is at 45° from the vertical. The solution for this geometry follows:

(b) The normal modes are parallel to and perpendicular to the trunk, so the vertical motion is now a combination of the normal modes. Since they have different frequencies, the relative phase of the normal-mode oscillations will change with time until they are 90° out of phase and then the motion will be horizontal. Since the angle of the branch is 45° from the vertical, the initial vertical combination contains an equal amplitude of both normal modes, as does the combination corresponding to pure horizontal motion.

Problem 4 (continued)

(c) Let x be the horizontal coordinate and y the vertical coordinate, with their origin chosen at the equilibrium (with the child on!). The normal modes are:

parallel: $x_{||}(t) = a \cos(\omega_{||}t + \phi_{||})$, $y_{||}(t) = a \cos(\omega_{||}t + \phi_{||})$

transverse: $x_{\perp}(t) = b \cos(\omega_{\perp}t + \phi_{\perp})$, $y_{\perp}(t) = -b \cos(\omega_{\perp}t + \phi_{\perp})$

From the initial condition $x = x_{||} + x_{\perp} = 0$, $y = y_{||} + y_{\perp} = 0.15$ m, $dx/dt = dy/dt = 0$ we have $b = -a = 0.075$ m, $\phi_{||} = \phi_{\perp} = 0$,

$$y(t) = a\,[\cos(\omega_{||}t) + \cos(\omega_{\perp}t)] = 2a \cos[(\omega_{||}+\omega_{\perp})t] \cos[(\omega_{||}-\omega_{\perp})t]$$

$$x(t) = a\,[\cos(\omega_{||}t) - \cos(\omega_{\perp}t)] = 2a \cos[(\omega_{||}+\omega_{\perp})t] \sin[(\omega_{||}-\omega_{\perp})t]$$

motion is purely horizontal when $(\omega_{||}+\omega_{\perp})t = \pi/2 = 2\pi t\,(1/T_{||} - 1/T_{\perp})$,

$$t = \frac{T_{||}\,T_{\perp}}{4(T_{\perp} - T_{||})} = 3.6 \text{ s}$$

(d) The maximum acceleration is the same in the horizontal and vertical planes; it is achieved in the vertical plane at $t = 0$, when the accelleration is

$$d^2y/dt^2 = a(\omega_{||}^2 + \omega_{\perp}^2) = 4\pi^2 a\,(1/T_{||}^2 + 1/T_{\perp}^2) = 0.18 \text{ m/s}^2 \approx 0.2\,g\,.$$

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Problem 5

(a) $\{H, L^2, L_z\}$ is the usual c.s.c.o for a central potential, where $\boldsymbol{L} = \boldsymbol{J}$ is the (orbital) angular momentum.

The Hamiltonian is like the hydrogen atom but with $e^2 \to 2e^2$.

So the energies are $E_n = -4\, E_0/n^2$, $n = 1,2,\ldots,\infty$

(b) For each n, L^2 has eigenvalues $l(l+1)\hbar^2$ for $l = 0,1,..n-1$.

For each l, $L_z = m\hbar$ where $m = -l, -l+1, \ldots, +l$

Degeneracy for each combination (n,l) is $2l+1$; sum to $n-1$ gives n^2 degeneracy.

(c) At $t = 0$, it is in the ground state w.f. of the hydrogen atom, which has $l=0$.

This state is a linear superposition of all values of n for the He^+ ion, but it is spherically symmetric so $l = m = 0$.

(d) Only the g.s. of the ion has lower energy than the g.s. of hydrogen

Formula sheet gives g.s. wave function of hydrogen, as well as formula for a_0.

Get g.s. wave function of He^+ by using $a_0(He^+) = a_0(\text{hydrogen})/2$ (because $e^2 \to 2e^2$)

Probability $= |\langle\psi_{gs}(He^+)|\psi_{gs}(\text{hydrogen})\rangle|^2 \Rightarrow$

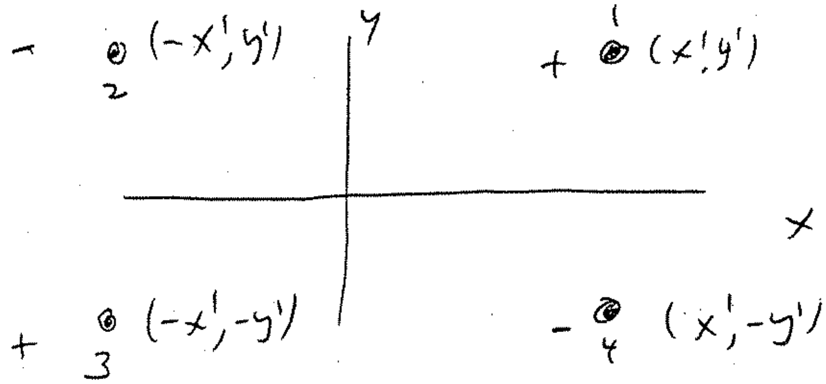
$$\langle\psi_{gs}(He^+)|\psi_{gs}(\text{hydrogen})\rangle = 4\pi \int_0^\infty r^2 dr\ (\pi a_0^3)^{-1} 2^{3/2} \exp[-r/a_0]\, \exp[-2r/a_0]$$

$$= 2^{7/2} a_0^{-3} \int_0^\infty r^2 dr\ \exp(-3r/a_0) = 2^{3/2}(1/3)^3 \text{ (see formula sheet)},$$

probability $= 2^9/3^6$

Problem 6

1. (a) Green's Function is easily found by method of images in 2 dimensions, with four line charges,



$$G(x, y : x', y') = 2 \ln \frac{r_1 r_3}{r_2 r_4}$$

$$r_1^2 = (x-x')^2 + (y-y')^2$$
$$r_2^2 = (x+x')^2 + (y-y')^2$$
$$r_3^2 = (x+x')^2 + (y+y')^2$$
$$r_4^2 = (x-x')^2 + (y+y')^2$$

This Green's Function satisfies the Eq.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) G(x, y : x', y') = \delta(x-x')\,\delta(y-y')$$

with correct B.C. in 1ST Quadrant

(b) From Green's theorem

$$\varphi(x,y) = \int d\vec{x}\,' \, G(\vec{x};\vec{x}\,')\, \rho(\vec{x}\,')$$

$$+ \int dS' \, \hat{n}' \cdot \left[\varphi_0(\vec{r}\,') \nabla' G(\vec{x};\vec{x}\,') \right]_{\Sigma'}$$

Σ' means evaluate $\varphi_0(\vec{r}\,')\nabla' G(x;x')$ on the surface over which the integral is taken

Since there is no charge in the 1st quadrant $\rho(\vec{r}\,') = 0$

so

$$\varphi(x,y) = \int_0^a dy' \, \phi_0 \frac{\partial G}{\partial x'}(x,y;x',y') \Big|_{x'=0}$$

$$+ \int_a^0 dx' \, \phi_0 \frac{\partial G}{\partial y'}(x,y;x',y') \Big|_{y'=0}$$

Problem 6 continued

$$\frac{\partial G(x,y;x',y')}{\partial x'}\bigg|_{x'=0} = -4x\left[\frac{1}{x^2+(y-y')^2} - \frac{1}{x^2+(y+y')^2}\right]$$

$$\frac{\partial G(x,y;x',y')}{\partial y'}\bigg|_{y'=0} = -4y\left[\frac{1}{(x-x')^2+y^2} - \frac{1}{(x+x')^2+y^2}\right]$$

$$\varphi(x,y) = -4x\int_0^a dy'\,\varphi_0\left[\frac{1}{x^2+(y-y')^2} - \frac{1}{x^2+(y+y')^2}\right]$$

$$+4y\int_0^a dx'\,\varphi_0\left[\frac{1}{(x-x')^2+y^2} - \frac{1}{(x+x')^2+y^2}\right]$$

where $\displaystyle\int_0^a dx'\,\frac{1}{(x-x')^2+y^2} = \tan^{-1}\frac{(x-a)}{y} - \tan^{-1}\frac{x}{y}$

etc.

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Problem 7

(a) Since the particle is spinless, total angular momentum $\boldsymbol{J}$ = orbital angular momentum $\boldsymbol{L}$

The wave function is $\psi(r)$ = constant $\times\, y \times \exp(-r^2/b^2)$

$= f(r) \times \sin\theta \sin\phi = g(r) \times [Y_1^{\,1} + Y_1^{\,-1}]$ using math table formulas for Y_l^m.

Both Y's have $l = 1$, so $J^2 = L^2 = 1(1+1)\hbar^2 = 2\hbar^2$ with certainty, probability =1.

(b) We see from part (a) that the state is an equal mixture of $L_z = +\hbar$ and $L_z = -\hbar$. The same would be true for any axis perpendicular to y, such as x.

Thus $L_x = +\hbar$ and $L_x = -\hbar$ with equal probabilities of 1/2 each.

(c) The wave function is symmetric with respect to rotations about the y axis. Thus $J_y = 0$ with certainty, probability =1.

(d) Now the total angular momentum is $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$. In terms of states $|jm\rangle$ quantized along z,

the angular wave function is a product state $|\psi\rangle_{\text{angular}} = |\tfrac{1}{2}\tfrac{1}{2}\rangle \otimes \{\, |1\;1\rangle + |1\;-1\rangle \,\}/\sqrt{2}$.

Introducing Clebsch-Gordan coefficients $\langle JM \,|\, j_1 j_2 m_1 m_2 \rangle$ we have

$|\tfrac{1}{2}\tfrac{1}{2}\rangle \otimes |1\;1\rangle = |\tfrac{3}{2}\tfrac{3}{2}\rangle$ (the stretched case) and

$$|\tfrac{1}{2}\tfrac{1}{2}\rangle \otimes |1\;-1\rangle = |\tfrac{3}{2}\;-\tfrac{1}{2}\rangle\langle \tfrac{3}{2}-\tfrac{1}{2}|\,1\,\tfrac{1}{2}-1\,\tfrac{1}{2}\rangle + |\tfrac{1}{2}-\tfrac{1}{2}\rangle\langle \tfrac{1}{2}-\tfrac{1}{2}|\,1\,\tfrac{1}{2}-1\,\tfrac{1}{2}\rangle . \qquad (*)$$

The probability is obtained by squaring the coefficients of the states $|jm\rangle$:

$$\text{Probability}\left(J^2 = \tfrac{3}{2}(\tfrac{3}{2}+1)\hbar^2\right) = \tfrac{1}{2}[1 + \langle \tfrac{3}{2}-\tfrac{1}{2}|\,1\,\tfrac{1}{2}-1\,\tfrac{1}{2}\rangle^2]$$

$$\text{Probability}\left(J^2 = \tfrac{1}{2}(\tfrac{1}{2}+1)\hbar^2\right) = \tfrac{1}{2}\langle \tfrac{1}{2}-\tfrac{1}{2}|\,1\,\tfrac{1}{2}-1\,\tfrac{1}{2}\rangle^2 = \tfrac{1}{2}[1 - \langle \tfrac{3}{2}-\tfrac{1}{2}|\,1\,\tfrac{1}{2}-1\,\tfrac{1}{2}\rangle^2]$$

Problem 7 (continued)

To evaluate Clebsch's note that $\langle \tfrac{1}{2} -\tfrac{1}{2} | 1 \tfrac{1}{2} -1 \tfrac{1}{2} \rangle^2 = 1 - \langle \tfrac{3}{2} -\tfrac{1}{2} | 1 \tfrac{1}{2} -1 \tfrac{1}{2} \rangle^2$ from (*)

get $\langle \tfrac{3}{2} -\tfrac{1}{2} | 1 \tfrac{1}{2} -1 \tfrac{1}{2} \rangle^2 = \langle \tfrac{3}{2} \tfrac{1}{2} | 1 \tfrac{1}{2} 1 -\tfrac{1}{2} \rangle^2$ from

$$J_- |\tfrac{3}{2}\tfrac{3}{2}\rangle = J_-\left(|\tfrac{1}{2}\tfrac{1}{2}\rangle \otimes |1\,1\rangle\right) = \left(J_-|\tfrac{1}{2}\tfrac{1}{2}\rangle\right) \otimes |1\,1\rangle + |\tfrac{1}{2}\tfrac{1}{2}\rangle \otimes \left(J_-|1\,1\rangle\right) \rightarrow$$

$$3^{1/2}\,|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = 1^{1/2}\,|\tfrac{1}{2} -\tfrac{1}{2}\rangle \otimes |1\,1\rangle + 2^{1/2}\,|\tfrac{1}{2}\tfrac{1}{2}\rangle \otimes |1 -1\rangle$$

$\rightarrow \langle \tfrac{3}{2}\,\tfrac{1}{2} | 1 \tfrac{1}{2} 1 -\tfrac{1}{2} \rangle = 3^{-1/2}$, so

Probability $\left(J^2 = \tfrac{3}{2}(\tfrac{3}{2} + 1)\hbar^2\right) = \tfrac{1}{2}[1 + \langle \tfrac{3}{2} -\tfrac{1}{2} | 1 \tfrac{1}{2} -1 \tfrac{1}{2} \rangle^2] = 2/3$

Probability $\left(J^2 = \tfrac{1}{2}(\tfrac{1}{2} + 1)\hbar^2\right) = \tfrac{1}{2}[1 - \langle \tfrac{3}{2} -\tfrac{1}{2} | 1 \tfrac{1}{2} -1 \tfrac{1}{2} \rangle^2] = 1/3$

Problem 8

$$E = E_0 + \sum_{r=1}^{3N} n_r \hbar \omega_r \qquad n_r = 0, 1, 2$$

(a)
$$Z = \sum_{n_1 \cdots n_r} e^{-\beta\left[E_0 + n_1\hbar\omega_1 + n_2\hbar\omega_2 + \cdots\cdots n_{3N}\hbar\omega_{3N}\right]}$$

$$= e^{-\beta E_0}\left(\sum_{n_1} e^{-\beta\hbar\omega_1 n_1}\right)\left(\sum_{n_2} e^{-\beta\hbar\omega_2 n_2}\right)\cdots\left(\sum_{n_{3N}} e^{-\beta\hbar\omega_{3N} n_{3N}}\right)$$

$$= e^{-\beta E_0}\left(\frac{1}{1-e^{-\beta\hbar\omega_1}}\right)\left(\frac{1}{1-e^{-\beta\hbar\omega_2}}\right)\cdots\left(\frac{1}{1-e^{-\beta\hbar\omega_{3N}}}\right)$$

(geometric series)

$$\ln Z = -\beta E_0 - \sum_{r=1}^{3N} \ln\left(1-e^{-\beta\hbar\omega_r}\right)$$

introduce $\sigma(\omega)\,d\omega$ = # modes between ω & $\omega + d\omega$
& convert to integral:

$$\ln Z = -\beta E_0 - \int_0^\infty \ln\left(1-e^{-\beta\hbar\omega}\right)\sigma(\omega)\,d\omega$$

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Problem 8 (continued)

(b) Use result of part (a) with $\sigma(\omega) = A\omega^2 \quad \omega \le \omega_D$

$$\ln Z = -\beta E_0 - A\int_0^{\omega_D} \ln\left(1 - e^{-\beta\hbar\omega}\right)\omega^2\, d\omega$$

To evaluate integral, integrate by parts:

$$\int_0^{\omega_D} \ln\left(1 - e^{-\beta\hbar\omega}\right)\omega^2\, d\omega$$

$$= \ln\left(1 - e^{-\beta\hbar\omega_D}\right)\cdot\frac{\omega_D^3}{3} - \frac{1}{3}\int_0^{\omega_D} \omega^3 \frac{d\omega}{1 - e^{-\beta\hbar\omega}}(-\beta\hbar)$$

$$= \frac{\omega_D^3}{3}\ln\left(1 - e^{-\beta\hbar\omega_D}\right) - \frac{1}{3}\left(\frac{1}{\beta\hbar}\right)^3 \int_0^{\Theta_D/T} \frac{x^3\, dx}{e^{-x} - 1}$$

$$\ln Z = -\beta E_0 - A\frac{\omega_D^3}{3}\ln\left(1 - e^{-\beta\hbar\omega_D}\right) + \frac{A}{3}\frac{1}{(\beta\hbar)^3}\int_0^{\Theta_D/T} \frac{x^3\, dx}{e^{-x} - 1}$$

$$\left\{ \int_0^{\omega_D} A\omega^2\, d\omega = \frac{A\omega_D^3}{3} = 3N \qquad \frac{A}{3}\frac{1}{(\beta\hbar)^3} = \frac{3N}{(\Theta_D/T)^3} \right.$$

$$\ln Z = -\beta E_0 - 3N\ln\left(1 - e^{-\beta\hbar\omega_D}\right) + N\, D(\Theta_D/T)$$

where $D(y) = \dfrac{3}{y^3}\displaystyle\int_0^y \frac{x^3\, dx}{e^{-x} - 1}$

Problem 8 (continued)

(c) $\overline{E} = -\dfrac{\partial \ln Z}{\partial \beta}$

$= -\dfrac{\partial}{\partial \beta}\left[-\beta E_0 - 3N \ln(1-e^{-\beta\hbar\omega_D}) + N D(\Theta_D/T)\right]$

$= E_0 + \dfrac{\partial}{\partial \beta}\left\{3N \ln(1-e^{-\beta\hbar\omega_D}) - \dfrac{\partial}{\partial \beta} N D(\Theta_D/T)\right]$

$\Big\{$ let $y = \Theta_D/T \quad \dfrac{\partial}{\partial \beta} = k\Theta_D \dfrac{\partial}{\partial y}$

$= E_0 + 3Nk\Theta_D \dfrac{\partial}{\partial y}\left[\ln(1-e^{-y})\right] - Nk\Theta_D \dfrac{\partial}{\partial y} D(y)$

$= E_0 - \dfrac{3Nk\Theta_D}{1-e^{-y}} - Nk\Theta_D\left[\dfrac{-9}{y^4}\int_0^y \dfrac{x^3\,dx}{e^{x}-1} + \dfrac{3}{y^3}\dfrac{y^3}{e^{-y}-1}\right]$

$= E_0 + \dfrac{9Nk\Theta}{y^4}\int_0^y \dfrac{x^3\,dx}{e^{-x}-1} = \underline{\underline{E_0 + 3NkT D(y)}}$