OSU PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #70

January 3 and 4, 1994

Comprehensive examination for Winter 1994

PART I

General Instructions

This Comprehensive Examination for Winter 1994 (#70) consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, January 3, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:30 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, January 4.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit – especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Problem 1 At very high temperatures a gas is composed entirely of non-interacting atoms of mass m. In a container of volume V its grand partition function (Maxwell-Boltzmann statistics) is

$$\mathcal{Z} = \exp[V x^{3/2} \exp(\mu_a/kT)]$$

where $x = (2\pi mkT/h^2)$, and μ_a is the chemical potiential of the atom. At lower temperatures the interactions cause some atoms to combine in pairs to form diatomic molecules, each of mass $2m$, chemical potential μ_m, and binding energy ε. Assume that these molecules have no rotational or vibrational states, so each molecule has energy $-\varepsilon$ plus its translational energy. Neglect all other effects of the interactions except for the binding energy of the molecules.

(a) Write our the expression for the grand partition function (MB statistics) for the mixture of atoms and molecules, as a function of V, x, ε, μ_a, μ_m, and kT. From this, obtain the grand potential $\Omega = -PV$ and, by differentiation, obtain the expressions for N_a and N_m, the mean number of atoms and molecules in the system.

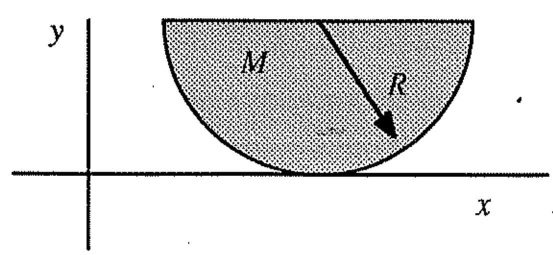
(b) Write out the expression for the Gibbs function $G = \mu_a N_a + \mu_m N_m$ in terms of kT, N_a, N_m, ε, x and V. For thermodynamic equilibrium at constant P and T, what must be the relation between μ_m and μ_a ?

(c) From this ratio compute the value of N_m as a function of N_a, V, x and (ε/kT). Over what range of temperatures is $N_m >> N_a$? What is the equation of state (in terms of $N \equiv N_a + 2N_m$) in this range of temperature? What is the equation of state in terms of N in the range where $N_m << N_a$?

Problem 2 A half cylinder of uniform density, radius R and mass M rests on a horizontal surface.

(a) Assuming the cylinder can rock without slipping, find the frequency of small oscillations about the equilibrium shown in the drawing.

(b) Assuming instead that the surface is perfectly frictionless, find the frequency of small oscillations about the equilibrium shown in the drawing.



END OF PART I

Problem 3 Two identical spinless particles of mass m are contained in a one-dimensional box of length L. The potential is

$$V(x) = \begin{cases} 0 & \text{for} \quad 0 < x < L \\ \infty & \text{for} \quad x \leq 0 \text{ or } x \geq L . \end{cases}$$

(a) Write the wave function $\Psi(x)$ for the lowest energy eigenstate which is antisymmetric under the interchange of the particles.

(b) Find an eigenstate of the same energy which is symmetric.

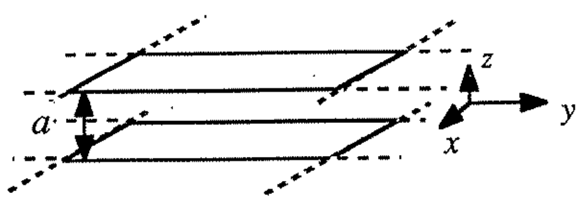
(c) Calculate the expectation value for the square of the interparticle displacement

$$d^2 = (x_1 - x_2)^2$$

for the two cases above. Show that

$$\langle d^2 \rangle_{\text{symmetric}} \leq \langle d^2 \rangle_{\text{antisymmetric}}$$

Problem 4 Consider an electromagnetic wave of frequency ω propagating between two infinite, perfectly conducting parallel planes separated in the z-direction by a distance a in a non-conducting medium of dielectric constant ε and magnetic permeability μ. The wave is propagating in the x-direction and is polarized with the electric field in the y-direction, *i.e.* $E_x = E_z = 0$ everywhere.



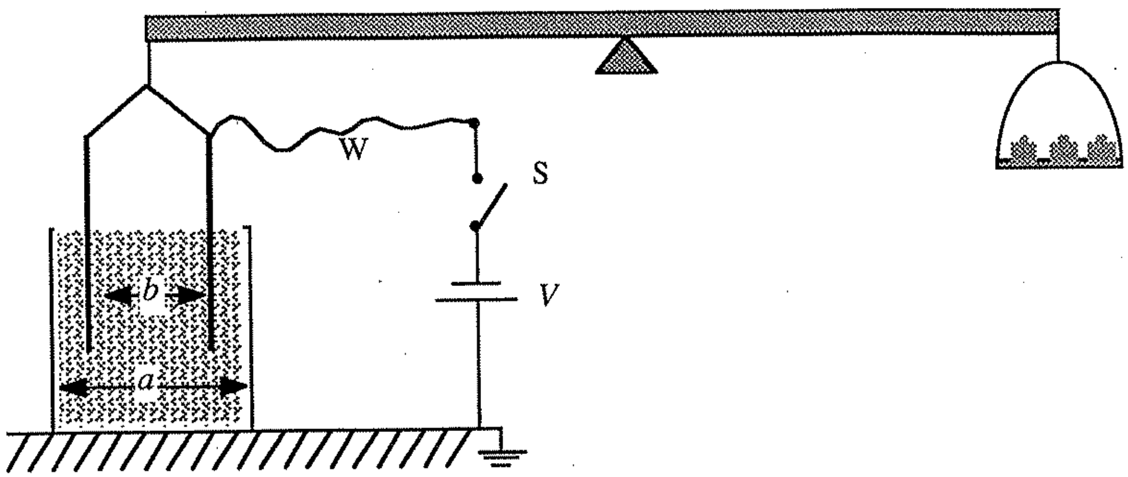
(a) Derive an expression giving the wavelengths in the bounded medium of the possible propagating waves in terms of the frequency ω and separation a.

(b) Show that the lowest frequency mode would have a wavelength $\lambda = 2a$ in the unbounded medium (*i.e.* the same medium without the conducting planes).

(c) What are the implications of your result for propagation of electromagnetic radiation with this polarization (TE) in the space between the (conducting) ocean and the ionosphere (about 100 km high) ?

END OF PART II

Problem 5 A grounded, conducting cylinder of diameter a is filled with an insulating dielectric fluid of dielectric constant ε. A second conducting cylinder of diameter b is suspended from one arm of a laboratory balance and is partially immersed in the fluid. The axes of the two cylinders coincide. A loose wire (W) allows a potential V to be applied to the suspended cylinder when the switch (S) is closed.



(a) Find an expression for the apparent change of weight of the suspended cylinder when the switch is closed.

(b) Evaluate the feasibility of using this apparatus for measurement of the dielectric constants of fluids to an accuracy of ~ 1% using $V \sim 100$ volts and "reasonable" values of a and b.

HINT: $\varepsilon_0 = 8.854...$ Farads per meter

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Problem 6 The eigenenergies of a rotating dumbbell with moment of inertia I,

$$E_l = \frac{\hbar^2 l(l+1)}{2I} \quad ,$$

are $(2l+1)$-fold degenerate.

(a) If we put equal and opposite charges at the ends of the dumbbell it becomes an electric dipole. The interaction energy between such a dipole and a constant, uniform external electric field $\boldsymbol{E}$ is

$$\delta H = -\boldsymbol{d} \cdot \boldsymbol{E}$$

where $\boldsymbol{d}$ is the dipole moment.Calculate the eigenenergies to first order in perturbation theory. You may express your answer in terms of the given quantities together with Clebsch-Gordan coefficients $\langle JM | j_1 j_2 m_1 m_2 \rangle$.

(b) If both ends of the dumbbell have the same charge (*i.e.* same sign) then the rotating dumbbell becomes a magnetic dipole. If the dipole has angular momentum $\boldsymbol{L}$ the corresponding magnetic dipole moment is

$$\mu = \frac{e}{2mc} \boldsymbol{L}$$

where e is the net charge of the dipole. The interaction energy between this magnetic dipole and a constant, uniform external magnetic field $\boldsymbol{B}$ is

$$\delta H = -\mu \cdot \boldsymbol{B} = -\frac{e}{2mc} \boldsymbol{L} \cdot \boldsymbol{B}$$

Find the eigenenergies to first order in perturbation theory.

END OF PART III

Problem 7 A spinless particle of mass m is confined inside a sphere of radius a. There are no forces on the particle when its radial coordinate $r < a$, but $r \geq a$ is impossible.

(a) Define a potential that describes this situation, and write down the time-independent Schrödinger equation for this case for an eigenstate of energy $E = \frac{\hbar^2 k^2}{2m}$.

(b) In spherical coordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{1}{r^2\hbar^2}\hat{L}^2$$

where $\hat{L}$ is the angular momentum operator. Solve the Schrödinger equation by separation of variables and find the radial wave equation for the reduced radial wave function $u_l(r) = r\,R_l(r)$. Given the fact that the solutions of this equation are called $j_l\,(kr)$, write down the complete solution for the wave function $\Psi(r,\theta,\phi)$.

(c) The functions $j_l(x)$ are equal to zero at special values of x, usually labelled x_{nl} where $n = 1, 2, 3,...$ Derive an expression which determines the energy eigenvalues in terms of the x_{nl}. Carefully define the quantum numbers you use. Is there any degeneracy in your solutions? Explain. (Note that no two x_{nl}'s are equal.)

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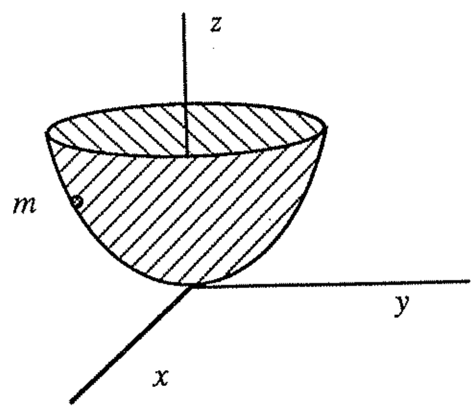


Problem 8 A point particle of mass m moves without friction under the influence of gravity on the surface of a paraboloid of revolution

$$\alpha z = x^2 + y^2$$

(a) Find the Lagrangean of the system.

(b) Find the equations of motion of the particle.

(c) Show that when the particle moves in a horizontal plane with an angular velocity

$$\omega = \sqrt{\frac{2g}{\alpha}}$$

the conditions of vertical and radial equilibrium are fulfilled, *i.e.* the equations of motion are satisfied.

(d) Find the frequency of small oscillations executed about an equilibrium radius.

END OF EXAMINATION

OSU Physics Department Comprehensive Exam # 70, January 3-4, 1994 **SOLUTIONS**

Problem 1

(a) $Z = Z_a Z_m$, $Z_a = \exp[Vx^{3/2}\exp(\mu_a/kT)]$, $x = 2\pi mkT/h^2$

for molecule, $m \to 2m \Rightarrow x \to 2x$, and $\mu_a \to \mu_m + \varepsilon$

$\Rightarrow Z_m = \exp[V(2x)^{3/2}\exp((\mu_m+\varepsilon)/kT)]$

$\Omega = -PV = -kT \ln Z = -kT\ \{\ln Z_A + \ln Z_m\} = -kTVx^{3/2}\{\exp(\mu_a/kT) + 2^{3/2}\exp((\mu_m+\varepsilon)/kT)\}$

$N_a = -\ \partial\Omega/\partial\mu_a = Vx^{3/2}\exp(\mu_a/kT)$, $N_m = -\ \partial\Omega/\partial\mu_m = V(2x)^{3/2}\exp((\mu_m+\varepsilon)/kT)$

(b) eliminate μ's : $\mu_a = kT \ln (N_a/Vx^{3/2}) = -\ kT \ln (Vx^{3/2}/N_a)$

$$\mu_m = -\varepsilon + kT \ln (N_m/V(2x)^{3/2}) = -\{\varepsilon + kT \ln (V(2x)^{3/2}/N_m)\}$$

$G = \mu_a N_a + \mu_m N_m = -N_a\{kT \ln (Vx^{3/2}/N_a)\} - N_m\{\varepsilon + kT \ln (V(2x)^{3/2}/N_m)\}$

for equilibrium of reaction $2a \leftrightarrow m$ need $\mu_m = 2\,\mu_a$

(c) $\mu_m = 2\,\mu_a = -\varepsilon - kT \ln (V(2x)^{3/2}/N_m) = -2\ kT \ln (Vx^{3/2}/N_a) \Rightarrow$

$\varepsilon = kT\ \{2\ \ln (Vx^{3/2}/N_a) -\ \ln (V(2x)^{3/2}/N_m)\ \} = kT \ln \left(\frac{Vx^{3/2}N_m}{2^{3/2}N_a}\right) \Rightarrow$

$\frac{Vx^{3/2}N_m}{2^{3/2}N_a} = \exp(\varepsilon/kT) \Rightarrow N_m = N_a\ \left(V(x/2)^{3/2}\right) \exp(\varepsilon/kT)$

For $N_m >> N_a$ we must require

$1 << \ln (N_m/N_a) = \varepsilon/kT + \ln V + \frac{3}{2}\ln (x/2)\ = \varepsilon/kT + \frac{3}{2}\ln\, [kTV^{2/3}m\pi/\mathrm{h}^2]$, *i.e.*

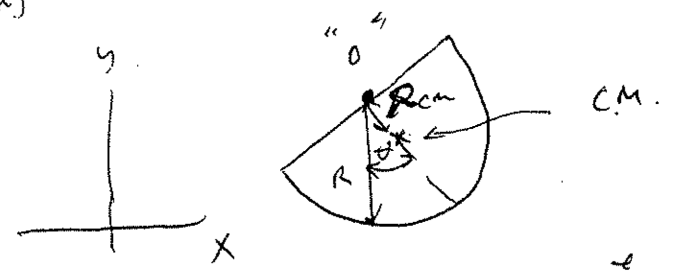
$kT\ (1 - \frac{3}{2}\ln\, [kTV^{2/3}m\pi/\mathrm{h}^2]) << \varepsilon$

In this case we have an ideal gas of $N_m = N/2$ molecules, so $PV = NRT/2$

For $N_m << N_a$, ideal gas of $N_a = N$ atoms, so $PV = NRT$.

Problem 2

(a)



C.M. has no motion in x-direct. due to lack of friction.

$$T = \frac{1}{2} M V_0^2 + M \vec{V}_0 \cdot (\vec{\omega} \times \vec{R}_{cm}) + \frac{1}{2} I_0 \omega^2$$

$\dot{\theta} = \omega$ is angular velocity about body origin "O"

I_0 is moment of inertia about body origin "O".

V_0 is the velocity of the body origin "O"

Potential energy arises due to lifting of C.M

$$V = Mg\,|R|_{cm}\,(1 - \cos\theta)$$

$$y_0 = \text{CONSTANT}$$

$$x_0 = |R_{cm}| \sin\theta$$

$$\dot{x}_0 = |R_{cm}|\,\dot{\theta}\,\cos\theta$$

$$(\dot{x}_0)^2 = |R_{cm}|^2\,\dot{\theta}^2 \cos^2\theta \quad ; \quad |R_{cm}| = \lambda$$

$$L = \frac{1}{2} M V_0^2 + M\vec{V}_0 \cdot (\vec{\omega} \times \vec{R}_{cm}) + \frac{1}{2} I_0 \omega^2 - mg|R_{cm}|(1-\cos\theta)$$

$$= \frac{1}{2} M(\lambda^2 \dot{\theta}^2 \cos^2\theta) - M\lambda\dot{\theta}\cos\theta\,(\dot{\theta}\lambda)\cos\theta + \frac{1}{2} I_0 \dot{\theta}^2$$

$$- mg\lambda(1-\cos\theta)$$

$$= -\frac{1}{2} M \dot{\theta}^2 \lambda^2 \cos^2\theta + \frac{1}{2} I_0 \dot{\theta}^2 - mg\lambda(1-\cos\theta)$$

Problem 2(continued)

Small oscillations

$$\cos\theta \simeq \left(1 - \frac{\theta^2}{2}\right)$$

$$\cos^2\theta \simeq 1 - \theta^2$$

$$\mathcal{L} = \frac{1}{2}(I_0 - M\lambda^2)\dot{\theta}^2 - mg\lambda\frac{\theta^2}{2}$$

Eq. of motion:

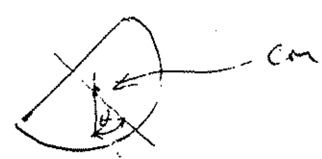
$$2\times\frac{1}{2}(I_0 - M\lambda^2)\ddot{\theta} + mg\lambda\theta = 0$$

$$\omega^2 = \frac{mg\lambda}{I_0 - m\lambda^2} = \frac{mg\lambda}{I_{cm}}$$

$$\lambda \equiv |R_{cm}| = \frac{4}{3\pi}R$$

$$I_0 = \frac{1}{2}MR^2$$

Alternatively:



$$\mathcal{L} = \frac{1}{2}I_{cm}\dot{\theta}^2 - mg\lambda(1 - \cos\theta)$$

etc.

Problem 2(continued)

(b) Similarly to (a)

$$x_0 = R\theta$$

$$y_0 = \text{CONSTANT}$$

$$\dot{x} = R\dot{\theta}$$

$$(\dot{x}_0)^2 = R^2\dot{\theta}^2$$

$$\mathcal{L} = \frac{1}{2} m\,(R^2\dot{\theta}^2) - mR\dot{\theta}(\dot{\theta}\lambda)\cos\theta + \frac{1}{2} I_0 \dot{\theta}^2 - mg\lambda\,(1-\cos\theta)$$

use $\cos\theta \simeq 1 - \frac{\theta^2}{2}$

$$I_0 = I_{cm} + m\lambda^2$$

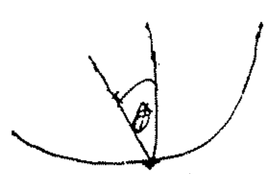
$$\mathcal{L} = \frac{1}{2} m\left[R^2 - 2R\lambda + \lambda^2 + \frac{I_{cm}}{m}\right]\dot{\theta}^2 - mg\frac{\lambda}{2}\theta^2$$

Eq of Motion

$$\left\{ m\left[(R-\lambda)^2\right] + I_{cm} \right\}\ddot{\theta} + mg\lambda\,\theta = 0$$

$$\omega^2 = \frac{mg\lambda}{m(R-\lambda)^2 + I_{cm}}$$

Alternatively



Rotate about base: $I_{BASE} = I_{cm} + (R-\lambda)^2 m$

$I_{BASE}\,\ddot{\theta} + mg\lambda\,\theta = 0$ etc.

OSU Physics Department Comprehensive Exam # 70, January 3-4, 1994 **SOLUTIONS**

Problem 3

(a) single particle states, normalized are

$\langle x|n\rangle = \phi_n(x) = (2/L)^{1/2} \sin(n\pi x/L)$, $n = 1,2,..$ $E_n = \frac{(n\pi\hbar)^2}{2mL^2}$.

2-particle product basis $|nm\rangle$, wave functions $\langle x_1 x_2|nm\rangle = \phi_m(x_1)\phi_n(x_2)$

For antisymmetry need 2 different s.p. states (Pauli principle)

$\Rightarrow$ minimum energy for $n = 1$ antisymmetrized with n=2,$|\psi\rangle = (2)^{-1/2}((|12\rangle \pm |21\rangle)$

$\psi(x_1, x_2) = 2^{-1/2}\{\phi_1(x_1)\,\phi_2(x_2) - \phi_2(x_1)\,\phi_2(x_2)\}$

(b) $\psi(x_1, x_2) = 2^{-1/2}\{\phi_1(x_1)\,\phi_2(x_2) + \phi_2(x_1)\,\phi_2(x_2)\}$

(c) $d^2 = (x_1 - x_2)^2 = x_1{}^2 + x_2{}^2 - 2x_1x_2 \Rightarrow \langle d^2\rangle = \langle x_1{}^2\rangle + \langle x_2{}^2\rangle - 2\langle x_1x_2\rangle$

$\langle x_1{}^2\rangle_\pm = \frac{1}{2}((\langle 12| \pm \langle 21|)x_1{}^2(|12\rangle \pm |21\rangle)$

$\langle 12|x_1{}^2|12\rangle = \langle 1|x^2|1\rangle\langle 2|2\rangle = \langle 1|x^2|1\rangle$, $\langle 12|x_1{}^2|21\rangle = \langle 1|x^2|2\rangle\langle 2|1\rangle = 0 \Rightarrow$

$\langle x_1{}^2\rangle_\pm = \frac{1}{2}(\langle 1|x^2|1\rangle + \langle 2|x^2|2\rangle) = \langle x_1{}^2\rangle_\pm$

$\langle x_1x_2\rangle_\pm = \frac{1}{2}((\langle 12| \pm \langle 21|)x_1x_2(|12\rangle \pm |21\rangle) = \langle 1|x|1\rangle\langle 2|x|2\rangle \pm \langle 1|x|2\rangle\langle 2|x|1\rangle \Rightarrow$

$\langle d^2\rangle_\pm = \langle 1|x^2|1\rangle + \langle 2|x^2|2\rangle - 2\,\langle 1|x|1\rangle\langle 2|x|2\rangle \mp 2\,\langle 1|x|2\rangle^2$

this shows $\langle d^2\rangle$ is less for the symmetric state because the last term is negative in that case.

$\langle 1|x^2|1\rangle = \frac{2}{L}\int_0^L dx\; x^2 \sin^2(\pi x/L)$, $\langle 2|x^2|2\rangle = \frac{2}{L}\int_0^L dx\; x^2 \sin^2(2\pi x/L)$

$\langle 1|x|2\rangle = \frac{2}{L}\int_0^L dx\; x \sin(\pi x/L)\sin(2\pi x/L)$, need not evaluate integrals to obtain full credit

Problem 4

(a) Start with Maxwell's Eqns:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$E_x = E_z = 0 \qquad E_y = E_0(z)\, e^{i\omega t - kx} \qquad H_z = H_0 e^{i\omega t - kx}$$

$$(\nabla \times E)_x = -\frac{\partial E_y}{\partial z} = -i\omega\mu H_x \qquad (1)$$

$$(\nabla \times E)_y = 0 = -i\omega\mu H_y \qquad (2)$$

$$(\nabla \times E)_z = -k E_y = -i\omega\mu H_z \qquad (3)$$

$$(\nabla \times H)_x = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = 0 \qquad (4)$$

$$(\nabla \times H)_y = \frac{\partial H_x}{\partial z} + k H_z = i\omega\epsilon E_y \qquad (5)$$

$$(\nabla \times H)_z = -k H_y - \frac{\partial H_y}{\partial y} = 0 \qquad (6)$$

Problem 4 (continued)

(2) $\Rightarrow H_y = 0 \Rightarrow$ (6) $\Rightarrow \dfrac{\partial H_x}{\partial y} = 0$

$\Rightarrow$ (4) $\Rightarrow \dfrac{\partial H_z}{\partial y} = 0$

so
$$i\omega\mu H_x = \frac{\partial E_y}{\partial z} \qquad (1)$$
$$i\omega\mu H_z = k E_y \qquad (3)$$
$$i\omega\epsilon E_y = \frac{\partial H_x}{\partial z} + k H_z \qquad (5)$$

Eliminate H_z and H_x:

(5): $$i\omega\epsilon E_y = \frac{1}{i\omega\mu}\frac{\partial^2 E_y}{\partial z^2} + \frac{k^2 E_y}{i\omega\mu}$$

$$-\omega^2\epsilon\mu E_y = \frac{\partial^2 E_y}{\partial z^2} + k^2 E_y$$

$$\frac{\partial^2 E_y}{\partial z^2} = -E_y\left(k^2 + \omega^2\epsilon\mu\right) = -\alpha E_y$$ wave equation for E_y

general solution: $E_y = A\sin\alpha z + B\cos\alpha z$

Boundary conditions on plane: $E_y(0) = E_y(z=a) = 0$

so $E_y = A\sin\alpha a \qquad \alpha = \dfrac{n\pi}{a} = \left(k^2 + \omega^2\epsilon\mu\right)^{1/2}$

so $\boxed{k = \sqrt{\left(\frac{n\pi}{a}\right)^2 - \omega^2\epsilon\mu}} \qquad \boxed{\lambda = \frac{2\pi}{k}}$

Problem 4 (continued)

(b) For propagation, k must be pure imaginary

so $\omega^2 \epsilon\mu > \left(\frac{n\pi}{a}\right)^2$

lowest frequency mode $(n=1)$

$$\omega^2 \epsilon\mu = (\pi/a)^2$$

$\epsilon\mu = \frac{1}{v^2}$ $\quad v$ = velocity in unbounded medium

$$\omega/v = \frac{2\pi\nu}{\lambda_0 \nu} = \frac{2\pi}{\lambda_0} = \pi/a \Rightarrow \boxed{\lambda_0 = 2a}$$

(c) Lowest frequency = $\omega_{min} = v\,\pi/a$

if $a \sim 10^5$ m and $v = 3\times10^8$ m

$$\omega_{min} = 2\pi\nu_{min} = \frac{(3\times10^8)(\pi)}{10^5}$$

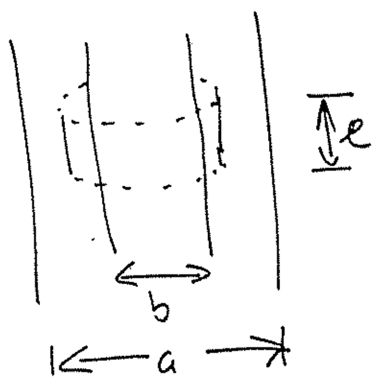
$$\nu_{min} = \frac{(3\times10^8)}{(2\times10^5)} = 1.5\text{ kHz}$$

= lowest possible TE mode

$$\lambda_{max} = 2\times10^5\text{ m}$$

Problem 5

(a) First, find the capacitance of the two conducting, concentric cylinders



Gaussian "pillbox" of length ℓ and radius r

Gauss's law: $\int \bar{D}_\perp \cdot d\bar{s} = Q$

$$\int D_\perp \cdot ds = \epsilon \int \bar{E}_\perp \cdot d\bar{s} = 2\pi r \ell \epsilon E(r) = Q$$

$$E(r) = \frac{Q}{2\pi r \ell \epsilon}$$

$$V = -\int_{b/2}^{a/2} E(r)\,dr = -\frac{Q}{2\pi \ell \epsilon} \int_{b/2}^{a/2} \frac{dr}{r} = \frac{Q}{2\pi \ell \varepsilon} \ln(b/a) .$$

so $C = \dfrac{Q}{V} = \dfrac{2\pi \ell \epsilon}{\ln(b/a)}$

Problem 5 (continued)

Now, if immersed portion increases from l to $l+dl$:

energy change dw =

increase in stored energy − work done by battery

$$dw = d\left(\tfrac{1}{2}CV^2\right) - V\,dQ = \tfrac{1}{2}V^2 dC - V^2 dC = -\tfrac{1}{2}V^2 dC$$

$(dQ = V\,dC)$

$$dw = -\frac{\pi \epsilon V^2}{\ln(b/a)}\, l \Rightarrow \text{force } F = -\frac{dw}{dl} = \frac{\pi \epsilon V^2}{\ln(b/a)}$$

Force is in direction of increasing L (downward)

$$\therefore\ F = \Delta m\, g = \frac{\pi \epsilon V^2}{\ln(b/a)}$$

$$\boxed{\Delta m = \frac{\pi \epsilon V^2}{g \ln(b/a)}}$$

(b) $$\frac{\delta(\Delta m)}{\delta \epsilon} = \frac{\pi V^2}{g \ln(b/a)} \simeq \frac{\pi \cdot 10^4}{(9.8)(0.095)} = 3.4 \times 10^4$$

Suppose $b/a \sim 1.1$

$\delta\epsilon \sim 10^{-3}\epsilon_0 \sim 9 \times 10^{-14}$

$\delta(\Delta m) = 3 \times 10^{-9}$ kg $\sim 3\,\mu g$ possible with best balances

Problem 6

Problem # 6

(a) Classically $\vec{d} = g\vec{r}$ $\quad \ominus \xrightarrow{\vec{r}} \oplus$

To get QM just replace $-\vec{d}\cdot\vec{E} = -g r E \cos\theta$

(assuming that $\vec{E}$ is in $\hat{z}$ direction and θ is polar angle of $\vec{r}$.)

The wave function for the rigid rotator is Y_ℓ^m. Since the levels are $(2\ell+1)$-fold degenerate in lowest order we need to look at the matrix

$$V_{mm'} = \langle \ell m | (-\hat{d}\cdot\vec{E}) | \ell m' \rangle$$

but $-\vec{d}\cdot\vec{E} \propto Y_1^0$ and

$$\langle \ell m | Y_1^0 | \ell m' \rangle = 0$$

(Because of parity, for example). So the 1st order perturbation does not change the energy.

(b) Now $V_{mm'} = \langle \ell, m | \left(-\frac{e}{2mc} B \hat{L}_z \right) | \ell, m' \rangle$

assuming $\vec{B}$ is in $\hat{z}$ direction

$$V_{mm'} = -\frac{eB}{2mc} \langle \ell, m | \hat{L}_z | \ell, m' \rangle$$

$$= -\frac{e\hbar m B}{2mc} \delta_{mm'}$$

The matrix is already diagonal so the problem is solved.

Problem 7

a) we use
$$V(r) = 0 \quad r < a$$
$$= \infty \quad r \geq a$$

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(r) \right] \psi = E \psi$$

b) assume $\psi(r, \theta, \phi) = C\, R(r)\, Y(\theta, \phi)$

For $r < a$

$$-\frac{Y}{r^2} \frac{d}{dr}\left(r^2 \frac{d}{dr} R \right) + \frac{R}{r^2 \hbar^2} \hat{L}^2 Y = \frac{2mE}{\hbar^2} R Y$$

we know $\hat{L}^2 Y_{\ell m} = \hbar^2 \ell(\ell+1) Y_{\ell m}$

$$\frac{2mE}{\hbar^2} = \kappa^2$$

$$\frac{d}{dr}\left(r^2 \dot{R} \right) - R\, \ell(\ell+1) + \kappa^2 r^2 R = 0$$

$$\ddot{R} + \frac{2}{r} \dot{R} + \left[\kappa^2 - \frac{\ell(\ell+1)}{r^2} \right] R = 0$$

Now let $x = \kappa r \qquad \frac{d}{dr} = \kappa \frac{d}{dx}$

$$\frac{d^2 R}{dx^2}(x) + \frac{2}{x} \frac{dR}{dx} + \left[1 - \frac{\ell(\ell+1)}{x^2} \right] R = 0$$

The solutions are $j_\ell(x)$. The complete soln is

$$\psi(r, \theta, \phi) = j_\ell(\kappa r)\, Y_\ell^m(\theta, \phi)$$

Problem 7 (continued)

We must force this to vanish at $r=a$

consequently $j_\ell(\kappa a) = j_\ell(x_{n\ell}) = 0$

$$\kappa \Rightarrow \kappa_{n\ell} = x_{n\ell}/a$$

The most general soln is

$$\psi = \sum_{n=1}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} A_{n\ell m}\, j_\ell(x_{n\ell} r/a)\, Y_\ell^m$$

c) $$E = \frac{\hbar^2 \kappa^2}{2m} = \frac{\hbar^2 x_{n\ell}^2}{2ma^2} = E_{n\ell}$$

Since this doesn't depend on m the levels are $(2\ell+1)$ - fold degenerate.

Problem 8

In cylindrical coordinates, the Lagrangian

(a) $\mathcal{L} = \frac{1}{2} m (\dot\rho^2 + \rho^2 \dot\varphi^2 + \dot z^2) - mgz$

$+ \lambda (\alpha z - x^2 - y^2)$ ← $x^2 + y^2 = \rho^2$

LAGRANGE MULTIPLIER → λ; constraint (on wire) → $(\alpha z - x^2 - y^2)$

Eq of Motion:

(b) $m\ddot\rho - m\rho\dot\varphi^2 - 2\lambda\rho = 0$ ①

$m \frac{d}{dt} (\rho^2 \dot\varphi) = 0$ ②

$m\ddot z + mg + \lambda\alpha = 0$ ③

Constraint condition

$\frac{d}{dt} (\alpha z - \rho^2) = 0$

$\alpha \frac{dz}{dt} - 2\rho \frac{d\rho}{dt} = 0$

$\alpha \dot z - 2\rho\dot\rho = 0$ ④

(c) vertical equilibrium $\ddot{z} = 0$

$$mg + \lambda\alpha = 0$$

$$\lambda = -\frac{mg}{\alpha}$$

(c) Radial equil $\ddot{\rho} = 0$

$$-m\rho\,\dot{\varphi}^2 - 2\lambda\rho = 0$$

$$+m\rho\left(-\dot{\varphi}^2 + 2\frac{g}{\alpha}\right) = 0$$

$$\dot{\varphi} = \sqrt{\frac{2g}{\alpha}}$$

(d)

At equilibrium in ρ, z

$$\rho_0^2\dot{\phi} = \text{constant}$$

$$= \rho_0^2\sqrt{\frac{2g}{\alpha}}$$

Thus $\rho^2\dot{\varphi} = \rho_0^2\sqrt{\frac{2g}{\alpha}}$

From ρ eq.

$$m\ddot{\rho} - m\rho\left(\frac{\rho_0^4\left(\frac{2g}{\alpha}\right)}{\rho^4}\right) + 2\left(\frac{mg}{\alpha}\right)\rho = 0$$

Expand about $\rho = \rho_0$ $\qquad \rho - \rho_0 = \Delta\rho$

$$\rho^{-3} \cong \rho_0^{-3} - 3\rho_0^{-4}(\rho - \rho_0)$$

$$\Delta\ddot{\rho} - \frac{\rho_0^4}{\rho_0^3}\left(\frac{2g}{\alpha}\right)\left[1 - \frac{3}{\rho_0}\Delta\rho\right] + \frac{2g}{\alpha}\Delta\rho = 0$$

$$\Rightarrow \Delta\ddot{\rho} + \frac{8g}{\alpha}\Delta\rho = 0 \qquad \omega^2 = \frac{8g}{\alpha}$$