OSU PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #69

September 27 and 28, 1993

Comprehensive examination for Fall 1993

PART I

General Instructions

This Comprehensive Examination for Fall 1993 (#69) consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, September 27, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:30 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, September 28.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit – especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Problem 1 An ideal solenoid, infinitely long, carries a current I and is wound with N turns of wire per unit length. Its radius is a.

(a) What is the magnetic field inside and outside the solenoid? The answers are well known, but be careful to justify your conclusions about all three components of both fields.

(b) What is the magnetic vector potential A outside the solenoid? Is your answer unique? If not, how could you change your answer to make another valid potential?

Problem 2 The χ mesons are bound states of a heavy "charmed" D quark with a D antiquark, each with spin $\hbar/2$ and mass $M_D = 1880\ \mathrm{MeV}/c^2$ (twice the proton's mass). The quark interacts with the antiquark by a central potential $V(r) \approx -\frac{4}{3}\,\alpha_s \frac{\hbar c}{r}$ where $\alpha_s \approx 1.0$ measures the strength of the strong interaction. Their relative motion is described by a p state (orbital angular momentum $l = 1$) with no radial nodes.

(a) What values are possible for the total angular momentum of the χ meson?

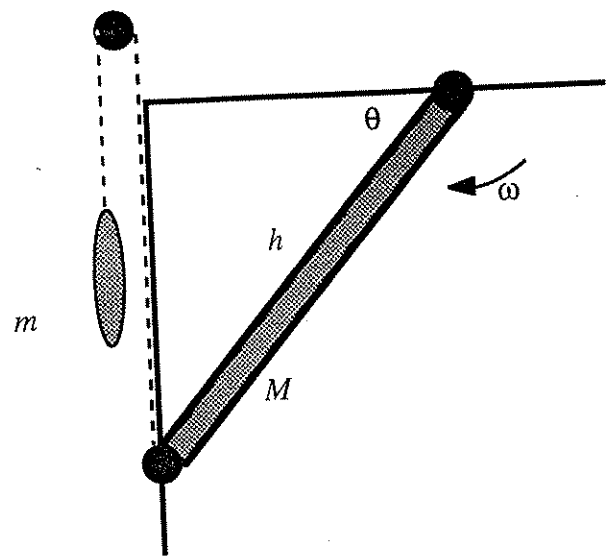
(b) Treating the relative motion of the quarks *non-relativistically*, estimate the binding energy of the χ meson.

(c) In addition to the central potential, the quark and antiquark interact by a spin-spin interaction $H_{ss} = \mathbf{S_1}\bullet\mathbf{S_2}\,\frac{a}{r^3}$ where a is a fairly small positive constant and $\mathbf{S_1}$ and $\mathbf{S_2}$ are the spins of the quark and antiquark respectively. Find the degeneracies and total angular momenta of the ground state and excited states of the χ meson, and give approximate expressions for their energies (you need not evaluate radial integrals).

(d) In addition to the interactions of part (c), the quark and antiquark interact by a spin-orbit interaction $H_{ls} = -\,\mathbf{L}\bullet\mathbf{S}\,\frac{b}{r^3}$ where b is a very small positive constant ($b \ll a$) and $\mathbf{L}$ and $\mathbf{S} \equiv \mathbf{S_1} + \mathbf{S_2}$ are the orbital angular momentum and total spin respectively. Find the degeneracies and total angular momenta of the ground state and excited states of the χ meson, and give approximate expressions for their energies (you need not evaluate radial integrals).

END OF PART I

Problem 3 An ideal garage door is a thin, rigid rectangular plate of mass M distributed uniformly over its height h. It moves without friction on a simple system of tracks which constrain the top edge to move horizontally into the garage while the bottom edge moves vertically from the floor to the ceiling. Its bottom edge is attached to a massless cable over a frictionless pulley which lowers a counterweight m as the door is raised. The mass of the counterweight is $m = M/2$.



(a) Show that the door is in neutral equilibrium, *i.e.* that it can remain motionless at any position – horizontal, vertical, or anywhere in between.

(b) The door is opened by pulling up on its bottom edge until that edge is a distance $b < h$ above the floor and is moving at a velocity v_0 vertically, at which point the door is released and left to coast open. What is the angular velocity ω_h of the door's angle of inclination when it reaches a completely horizontal position? $\omega_h = ?$

(c) How much time elapses after the door is released until it first becomes horizontal? $T_h = ?$

YOU NEED NOT EVALUATE INTEGRALS WHICH ARISE IN THIS PROBLEM

GO ON TO THE NEXT PAGE

Problem 4 The partition function $Z(\beta)$ for a *canonical* ensemble (a system characterized by a constrained mean total energy) is

$$Z(\beta) = Tr\, e^{-\beta H} = \sum_s e^{-\beta E_s}$$

where $\beta = \frac{1}{k_B T}$ with T the temperature and k_B Boltzmann's constant, H is the Hamiltonian, E_s are the energy eigenvalues for the states s of the system, and the sum is over all eigenstates.

If the system is characterized not only by the mean total energy, but also by other constants of the motion corresponding to operators Ω_j, then the partition function is

$$Z(\beta,\lambda) = Tr\ \exp \beta\Big[\sum_j \lambda_j \Omega_j - H\Big]$$

where λ_j are a set of constants (Lagrange multipliers) that couple in the additional constants of motion Ω_j.

a. Consider electromagnetic radiation in thermal equilibrium (black-body radiation) at temperature T confined to a very large volume V where the total energy in the black-body rest frame is

$$E = \sum_{k\mu} n_{k\mu}\, \hbar c\, |k| + \text{constant}$$

where μ=1,2 are the two independent polarizations. Write the partition function for the black-body radiation in its rest frame. What is $n_{k\mu}$?

GO ON TO THE NEXT PAGE

Problem 4 (continued)

Consider now, in addition, a small satellite moving with a velocity v through the radiation volume such that in the frame of reference of the satellite both the mean total energy *and the mean z-component of the radiation wave vector*

$$\langle k_z \rangle = \sum_{k\mu} n_{k\mu} |k| \cos\theta$$

are constants of the motion.

b. Taking λ as the Lagrange multiplier which couples the mean z-component of the radiation wave vector k_z, and starting from a partition function $Z(\beta,\lambda)$, show that the relevant free energy of the radiation field measured in the (moving) satellite frame is

$$F = -\frac{\sigma}{3} \frac{VT^4}{[1-(\lambda/\hbar c)^2]^2} \qquad \text{where } \sigma \equiv \frac{\pi^2 k_B{}^2}{15\hbar^3 c^3},$$

and where V and T are the volume and temperature measured in the satellite frame.

c. Since the entropy is invariant and has the same value in all frames of reference, show, by identifying the Lagrange multiplier as $\lambda = \hbar v$, that the temperature transforms according to

$$T = T_0 \sqrt{1-v^2/c^2}$$

where V_0 and T_0 are the volume and temperature measured in the *rest frame* of the radiation field (λ=0).

Mathematical hint: Evaluate the free energy by doing the energy integration first. This entails integrating by parts and then using

$$\int_0^\infty dx \, \frac{x^3}{e^{ax}-1} = \frac{\pi^2}{15a^4}.$$

Then do the angular integration, which is elementary.

END OF PART II

Problem 5 An insulated, evacuated vessel is coupled by a narrow valve to a large feeder pipe filled with an ideal gas maintained at constant pressure P_0 and temperature T_0. The constant pressure and constant volume heat capacities of the gas are C_P and C_V. Initially the valve is closed. The valve is suddenly opened and gas rushes in to fill the evacuated vessel.

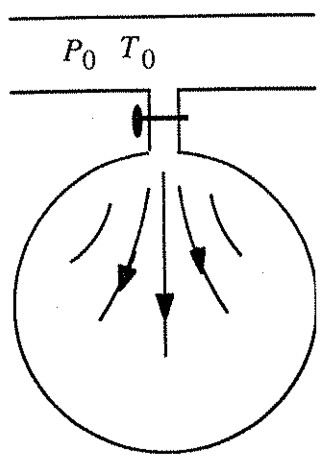
Hint: Replace the open system by a closed system and consider the gas to be leaving the feeder pipe under constant piston pressure (as in throttling) and freely entering the vessel (as in a free expansion).

(a) What is the work per mole done on the gas by the piston?

(b) What is the work per mole done by the gas in filling the evacuated vessel?

(c) Write down the first law of thermodynamics for this process.

(d) What is the temperature of the gas in the vessel immediately after it fills the vessel? (Assume hydrostatic equilibrium is reached, but that the gas in the vessel is not in thermal contact with the gas in the feeder pipe).



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Problem 6 A particle of total angular momentum $s = \hbar$ is subject to an interaction of the form

$$H = A\, s_z^2 .$$

(a) Find the energy levels of the particle, and their degeneracies.

(b) An additional interaction of the form $H_1 = \gamma \boldsymbol{S} \bullet \boldsymbol{B}$ is introduced where $\boldsymbol{B}$ is a vector directed in an arbitrary direction with respect to the z-axis. Derive the cubic equation whose solutions are the exact energy eigenvalues in the presence of this interaction.

(c) Solve for the energy eigenvalues for the case when the magnetic field is along the x-axis, $\boldsymbol{B} = B_0\, \hat{x}$.

(d) Sketch the qualitative dependence of the energy levels on B_0 starting from $B_0 = 0$ for the case $\boldsymbol{B} = B_0\, \hat{x}$.

(e) Find the lowest-energy eigenstate for the case $\boldsymbol{B} = B_0\, \hat{x}$.

END OF PART III

Problem 7 A hydrogen atom in its ground state is subjected to a uniform electric field

$$\boldsymbol{E} = E_0\,\hat{z}\cos\omega t\ .$$

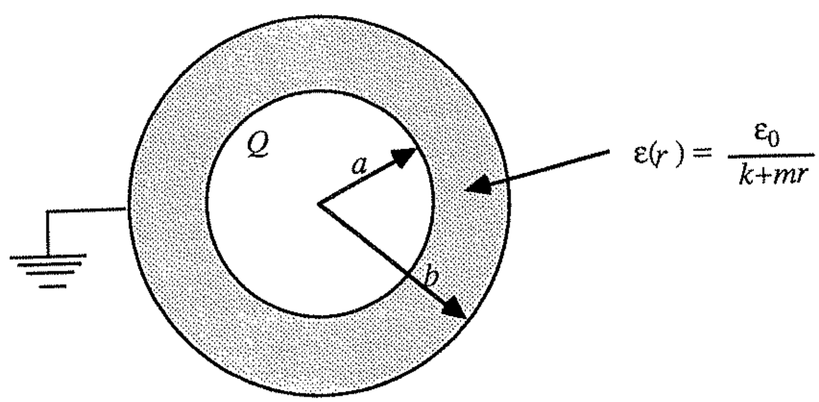
For the case $\hbar\omega > |E_B|$ where E_B is the ground state binding energy, derive an expression giving, as completely as you can, the probability per second that an electron will be photo-ionized into a small solid angle $d\Omega$ at an angle θ with respect to the polarization direction $\hat{z}$. Assume the ionized electron to be free, i.e. a plane wave.

YOU NEED NOT EVALUATE RADIAL INTEGRALS FOR THE ABOVE PROBLEM

Problem 8 The region between two concentric conducting spheres of radii a and b is filled with an inhomogeneous dielectric whose permittivity at a distance r from the center is given by

$$\varepsilon(r) = \frac{\varepsilon_0}{k+mr}$$

where k and m are known constants. A charge Q is placed on the center conductor. The outer conductor is connected to ground.



Derive expressions for:

(a) The displacement $\boldsymbol{D}$ at any point between the spheres.

(b) The capacitance between the two dielectrics.

(c) The polarization volume charge density at any radius r between the spheres.

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Problem 1

Since $\vec{J} = 0$ inside and outside, $\vec{B}$ can be derived from a scalar potential.

(a) $\nabla^2 \Phi_m = 0$ and $-\vec{\nabla} \Phi_m = \vec{B}$

By symmetry, $\vec{B} = \vec{B}(\rho)$ so

$$\frac{\partial B_z}{\partial z} = -\frac{\partial^2 \Phi_m}{\partial z^2} = 0 \qquad \frac{\partial B_\phi}{\partial \phi} = -\frac{1}{\rho} \frac{\partial^2 \Phi_m}{\partial \phi^2} = 0$$

so $\nabla^2 \Phi_m = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi_m}{\partial \rho} \right) = 0$

Separate variables: $\Phi_m = R(\rho) \Phi(\phi) Z(z)$

$$\ddot{\Phi} = 0, \quad \ddot{Z} = 0, \quad \text{and} \quad \frac{d}{d\rho}(\rho \dot{R}) = 0$$

$$\Phi_m = (a \ln \rho + b)(c + dz)$$

$$B_\rho = -\frac{\partial \Phi_m}{\partial \rho} = -\frac{a}{\rho}(c + dz)$$

$$B_\phi = -\frac{1}{\rho} \frac{\partial \Phi_m}{\partial \phi} = 0$$

$$B_z = -(a \ln \rho + b) d$$

Inside solenoid: $\rho \to 0$ so $a = 0$, $B_\rho = 0$, and $B_z = \text{const.}$

Outside solenoid: $\vec{B}$ must not depend on z, so $d = 0$. This leaves $B_\rho \propto \rho^{-1}$, which can be ruled out in several different ways. Finally $\vec{B}_{out} = 0$

Problem 1(continued)

The inside field is found from ampere's law

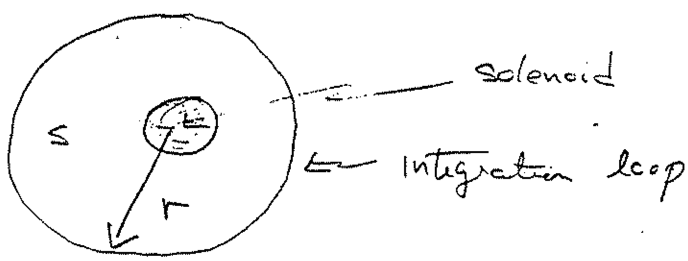
$$\oint \vec{B}\cdot d\vec{l} = B_z l = \frac{4\pi}{c} N l I$$

$$B_z = \frac{4\pi N I}{c}$$

(b) $$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\int_S \vec{B}\cdot\vec{n}\, da = \oint \vec{A}\cdot d\vec{l}$$

Choose S as follows:



$$\text{Flux} = \int_S \vec{B}\cdot\vec{n}\, da = \frac{4\pi N I}{c}(\pi a^2) = \oint \vec{A}\cdot d\vec{l} = 2\pi r A_\phi$$

so $$A_\phi = \frac{2\pi N I a^2}{r c}$$

Since $\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\times\vec{A} = 0$ there is nothing wrong with setting $A_\rho = A_z = 0$. Since $\vec{B} = \vec{\nabla}\times\vec{A}$, we can add the gradient of any scalar to $\vec{A}$ without changing $\vec{B}$.

OSU Physics Department Comprehensive Exam # 69, September 27-28, 1993 **SOLUTIONS**

Problem 2

(a) J=0, 1, $2\hbar$

(b) $E_{\text{binding}} = \frac{1}{2}\alpha^2\mu c^2$ where μ = reduced mass = $M_D/2 = m_p$, $\alpha = \frac{4}{3}\alpha_s$, E_{binding} = 800 MeV

(c) Degenerate perturbation $\Rightarrow$ diagonalize perturbation in subspace, H_{ss} is diagonal in $S^2 = S_1^2 + S_2^2 + 2\, S_1 \cdot S_2$

$\Delta E = \frac{a}{2}(S(S+1) - \frac{3}{2})\hbar^2 \int_0^\infty r^2 dr\, R_{11}(r)^2/r^3$, ground state has S=0, J=1, deg=3; excited states have S=1, deg=9

(d) ground state has S=0 $\Rightarrow$ energy unaffected, deg=3

excited states have S=1 $\Rightarrow$ must diagonalize $\mathbf{L \cdot S}$, diagonalized by $\mathbf{J}^2 = \mathbf{L}^2 + \mathbf{S}^2 + 2\,\mathbf{L \cdot S}$

$\Rightarrow$ additional $\Delta E = \frac{a}{2}(J(J+1) - 6)\hbar^2 \int_0^\infty r^2 dr\, R_{11}(r)^2/r^3$ for J=0, 1, 2 , deg = 2J+1

Problem 3

(a) let y = distance of door bottom from ceiling, then door's center of graivity is y/2 below ceiling, gravitational potential energy of door = $Mgy/2$ + constant, potential energy of weight = $-Mgy/2$ + constant , totatl potential energy is independent of y $\Rightarrow$ there is no force $F_y = -\partial V/\partial y = 0$

(b) Kinetic energy = $\frac{1}{2}m(dy/dt)^2 + \frac{1}{2}M(d\frac{1}{2}y/dt)^2 + \frac{1}{2}M(d\frac{1}{2}x/dt)^2 + \frac{1}{2}I(d\theta/dt)^2$ where x is horizontal displacement of top of door and θ is the angle of the door from horizontal, I = moment of inertia = $h^2M/12$..

$y = h\sin\theta$, $x = h\cos\theta$, $dy/dt = h\cos\theta\, d\theta/dt$, $dx/dt = -h\sin\theta\, d\theta/dt$ $\Rightarrow$

K.E. = $\frac{1}{2}\mu(\theta)(d\theta/dt)^2$ where $\mu(\theta) = Mh^2(\frac{1}{2}\cos^2\theta + \frac{1}{4} + \frac{1}{12})$.

K.E. = constant = initial value, initial angle $\sin\theta_0 = 1 - b/h$, initial angular velocity $\omega_0 = (d\theta/dt)_0 = -v_0/(h\cos\theta_0) = -v_0/(2hb - b^2)^{1/2}$, final angle $\theta_{\text{final}} = 0$ $\Rightarrow$

$\omega_{\text{final}}^2 = [1 - \frac{3}{5}(1 - \frac{b}{h})^2]v_0^2/(2hb - b^2)$

(c) $(d\theta/dt)^2 = 2\,(\text{K.E.}_{\text{initial}})/\mu(\theta) = v_0^2[-5(1 - \frac{b}{h})^2 + 3]/[(3\cos^2\theta + 2)(2hb - b^2)]$ $\Rightarrow$

$dt = d\theta\,[2\,(\text{K.E.}_{\text{initial}})/\mu(\theta)]^{1/2}$ $\Rightarrow$

$$T_h = \int_0^{T_h} dt = v_0[-5(1 - \tfrac{b}{h})^2 + 3]^{1/2}(2hb - b^2)^{-1/2} \int_{\theta_0}^{\pi/2} d\theta\,(3\cos^2\theta + 2)^{-1/2}$$

OSU Physics Department Comprehensive Exam # 69, September 27-28, 1993 **SOLUTIONS**

Problem 4

(b)

For two polarizat

$$Z = \prod_{k} \left(\sum_{n} e^{-\beta n \hbar c|k| + \beta \lambda n |k| \cos\theta} \right)^{2}$$

$$= \prod_{k} \left[\frac{1}{1 - e^{-\beta \hbar c |k| \left(1 - \frac{\lambda}{\hbar c}\cos\theta\right)}} \right]^{2}$$

(Infinite contribution from $\frac{1}{2}\hbar\omega$ is ignored) In satellite frame.

(for (a) set $\lambda = 0$)

$$F = -\frac{1}{\beta} \log Z$$

$$= +\frac{1}{\beta} \, 2 \sum_{k} \log\left(1 - e^{-\beta \hbar c|k| \left(1 - \frac{\lambda}{\hbar c}\cos\theta\right)}\right)$$

Change $\sum_{k} \to \frac{V}{(2\pi)^3} \int d\vec{k} \to \frac{V}{(2\pi)^3} \int k^2 dk \int d\Omega$

$$F = \frac{1}{\beta} \frac{V}{(2\pi)^3} \, 2 \, (2\pi) \int_0^\infty k^2 dk \int_0^\pi \sin\theta \, d\theta$$

$$\times \log\left(1 - e^{-\beta \hbar c |k| \left(1 - \frac{\lambda}{\hbar c}\cos\theta\right)}\right)$$

$$= \frac{V}{2\beta\pi^2} \int_0^\infty k^2 dk \int_0^\pi \sin\theta \, \log\left(1 - e^{-\beta \hbar c|k| \left(1 - \frac{\lambda}{\hbar c}\cos\theta\right)}\right)$$

Problem 4 (continued)

use $\omega = c|k|$

$$F = \frac{V}{2\beta\pi^2}\frac{1}{c^3}\int_0^\infty d\omega\,\omega^2\int_0^\pi \sin\theta \log\left(1 - e^{-\beta\hbar\omega\left(1-\frac{\lambda}{\hbar c}\cos\theta\right)}\right)$$

change variable

$$z = 1 - \frac{\lambda}{\hbar c}\cos\theta$$

$$dz = \frac{\lambda}{\hbar c}\sin\theta\, d\theta$$

$$F = \frac{V}{2\pi^2\beta}\frac{1}{c^3}\int_0^\infty d\omega\,\omega^2\left(\frac{\hbar c}{\lambda}\right)\int_{1-\frac{\lambda}{\hbar c}}^{1+\frac{\lambda}{\hbar c}} dz\,\log\left(1-e^{-\beta\hbar\omega z}\right)$$

Do ω- integration

Integrate by parts

$$u = \log\left(1 - e^{-\beta\hbar\omega z}\right)$$

$$du = \frac{1}{1-e^{-\beta\hbar\omega z}} \times (\beta\hbar z)\, e^{-\beta\hbar\omega z}\, d\omega$$

$$= \frac{\beta\hbar z}{e^{\beta\hbar\omega z}-1}$$

$$dv = \omega^2 d\omega$$

$$v = \frac{1}{3}\omega^3$$

Problem 4 (continued)

$$F = -\left(\frac{\hbar c}{d}\right)\frac{1}{3}\frac{1}{c^3}\frac{V}{2\pi^2\beta}\int_{1-\frac{d}{\hbar c}}^{1+\frac{d}{\hbar c}} dz\,(\beta\hbar z)\underbrace{\int_0^\infty d\omega\,\omega^3\frac{1}{e^{\beta\hbar\omega z}-1}}_{\frac{\pi^4}{15(\beta\hbar z)^4}}$$

$$F = -\left(\frac{\hbar c}{d}\right)\frac{\pi^2}{45}\frac{1}{\beta^4}\frac{1}{\hbar^{\cancel{2}}}\left(\frac{V}{2}\right)\frac{1}{c^3}\int_{1-\frac{d}{\hbar c}}^{1+\frac{d}{\hbar c}}\frac{dz}{z^3}$$

$$= -\left(\frac{c}{d}\right)\frac{\pi^2}{45}\frac{1}{\beta^4}\frac{1}{(\hbar c)^3}\frac{V}{2}\hbar\left(-\frac{1}{2}\right)\left[\frac{1}{\left(1+\frac{d}{\hbar c}\right)^2}-\frac{1}{\left(1-\frac{d}{\hbar c}\right)^2}\right]$$

$$= -\frac{c}{d}\frac{\pi^2}{45}\frac{k_B^4T^4}{(\hbar c)^3}\frac{V}{2}\left(-\frac{1}{2}\right)\hbar\left[\frac{\left(1-\frac{d}{\hbar c}\right)^2-\left(1+\frac{d}{\hbar c}\right)^2}{\left[1-\left(\frac{d}{\hbar c}\right)^2\right]^2}\right]$$

$$= -\frac{c}{d}\frac{\pi^2}{45}\frac{k_B^4T^4}{(\hbar c)^3}\frac{V}{2}\left(-\frac{1}{2}\right)\hbar\frac{\left(-4\frac{d}{\hbar c}\right)}{\left[1-\left(\frac{d}{\hbar c}\right)^2\right]^2}$$

$$= -\frac{\pi^2}{45}\frac{k_B^4}{(\hbar c)^3}T^4V\frac{1}{\left[1-\left(\frac{d}{\hbar c}\right)^2\right]^2}$$

$$(d \to \hbar v)$$

$$= -\frac{1}{3}\sigma\frac{VT^4}{\left[1-\left(\frac{v}{c}\right)^2\right]^2}$$

Problem 4 (continued)

(b) $S = -\left(\frac{\partial F}{\partial T}\right)_V$

$$= \frac{1}{3} 6 \times 4 \quad \frac{V T^3}{\left[1-\left(\frac{v}{c}\right)^2\right]^2}$$

$\boxed{\text{But } S_0 = \left(\frac{1}{3}\right) 6 \times 4 \times V_0 T_0^3}$ and $S = S_0$

V is Lorentz contracted

so $V = V_0 \sqrt{1 - \frac{v^2}{c^2}}$

Therefore

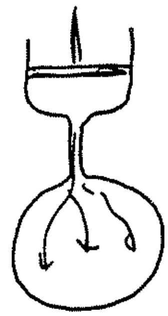
$$\frac{1}{3} 6 \times 4 \times \frac{V_0 \sqrt{1-\frac{v^2}{c^2}} \; T^3}{\left[1-\left(\frac{v}{c}\right)^2\right]^2}$$

$$= \frac{1}{3} 6 \times 4 \times V_0 T_0^3$$

and therefore $\frac{T^3}{\left[1-\left(\frac{v}{c}\right)^2\right]^{3/2}} = T_0^3$

$$T = T_0 \sqrt{1-\left(\frac{v}{c}\right)^2}$$

Problem 5



$P_0 T_0$

(a) As in throttling, the gas is forced in at constant pressure P_0

Work per mole <u>on</u> gas is

$$W = P_0 v_0$$

where v_0 is volume per mole for gas in pipe

(b) Gas freely expands into vessel

<u>No</u> work is done.

(c) $$đQ = 0 \quad \text{(totally insulated and fast)}$$

$$0 = \epsilon_f - \epsilon_0 - P_0 v_0$$

ϵ is internal energy per mole

$$\text{so } \epsilon_f = \epsilon_0 + P_0 v_0$$

$$\epsilon_f = h_0 \qquad (h_0 \text{ is enthalpy})$$

(d) $$\epsilon_f - \epsilon_0 = C_v (T_f - T_0) \qquad \text{(Ideal Gas)}$$

$$C_v (T_f - T_0) = P_0 v_0$$
$$= R T_0$$

$$T_f = \frac{(C_v + R)}{C_v} T_0 = \frac{C_p}{C_v} T_0$$

$$T_f > T_0$$

OSU Physics Department Comprehensive Exam # 69, September 27-28, 1993 **SOLUTIONS**

Problem 6

$\mathcal{H} = A s_z^2 \qquad s = \hbar = \hbar S \qquad S = 1$

(a) Use basis of s_z eigenstates $|mS\rangle$

$\langle S m' | \mathcal{H}_0 | m S\rangle = A m^2 \delta_{m'm} \qquad m = 0, \pm 1 \quad S = 1$

(let $\hbar = 1$)

$\underline{\qquad\qquad}$ A (degeneracy $= 2$)

$\underline{\qquad\qquad}$ 0 (non-degenerate)

(b) $\mathcal{H}_1 = \gamma \vec{S} \cdot \vec{B} = \gamma B \left[\cos\theta\, S_z + \sin\theta \cos\phi\, S_x + \sin\theta \sin\phi\, S_y \right]$

use $S_x = \frac{1}{2}(S_+ + S_-) \qquad S_y = -\frac{i}{2}(S_+ - S_-)$

$\mathcal{H}_1 = \gamma B \left[\cos\theta\, S_z + \sin\theta\, e^{i\phi} S_+ + \sin\theta\, e^{-i\phi} S_- \right]$

$\equiv c_z(\theta) S_z + c_+(\theta,\phi) S_+ + c_-(\theta,\phi) S_-$

$c_z(\theta) = \gamma B \cos\theta \qquad c_\pm(\theta,\phi) = \gamma B \sin\theta\, e^{\pm i\phi}$

Problem 6 (continued)

matrix of $\mathcal{H} + \mathcal{H}_1$

$$
\begin{array}{c|ccc}
 & |-1\rangle & |0\rangle & |+1\rangle \\
\hline
\langle 1\,-1| & A - C_z & \sqrt{2}\,C_+ & 0 \\
\langle 1\,0| & \sqrt{2}\,C_- & 0 & \sqrt{2}\,C_+ \\
\langle 1\,+1| & 0 & \sqrt{2}\,C_- & A + C_z
\end{array}
$$

$$\langle S, m\pm 1 | S_\pm | m, S\rangle = \left[(S \mp m)(S \pm m + 1)\right]^{1/2} = \sqrt{2}$$

secular equation:

$$(A - C_z - E)\begin{vmatrix} -E & \sqrt{2}C_+ \\ \sqrt{2}C_- & A + C_z - E \end{vmatrix}$$

$$- \sqrt{2}\,C_+ \begin{vmatrix} \sqrt{2}C_- & \sqrt{2}C_+ \\ 0 & A + C_z - E \end{vmatrix} + 0 = 0$$

$$(-E)(A - C_z - E)(A + C_z - E) - 2|C_+|^2 (A - C_z - E)$$

$$- 2|C_+|^2 (A + C_z - E) = 0$$

$$\boxed{E\left[(A-E)^2 - C_z^2\right] + 4|C_+|^2 (A - E) = 0}$$

or $\quad E^3 - 2AE^2 + (A - 4|C_+|^2)E + 4|C_+|^2 A = 0$

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Problem 6 (continued)

(c) $B = B_0 \hat{x} \Rightarrow \theta = \pi/2 \quad C_2(\theta) = 0 \quad |C_1|^2 = (\gamma B_0)^2$

Secular equation: $E(A-E)^2 + 4(B_0\gamma)^2(A-E) = 0$

$$(A-E) = 0 \Rightarrow \underline{\underline{E_1 = A}}$$

remaining eigenvalues are roots of

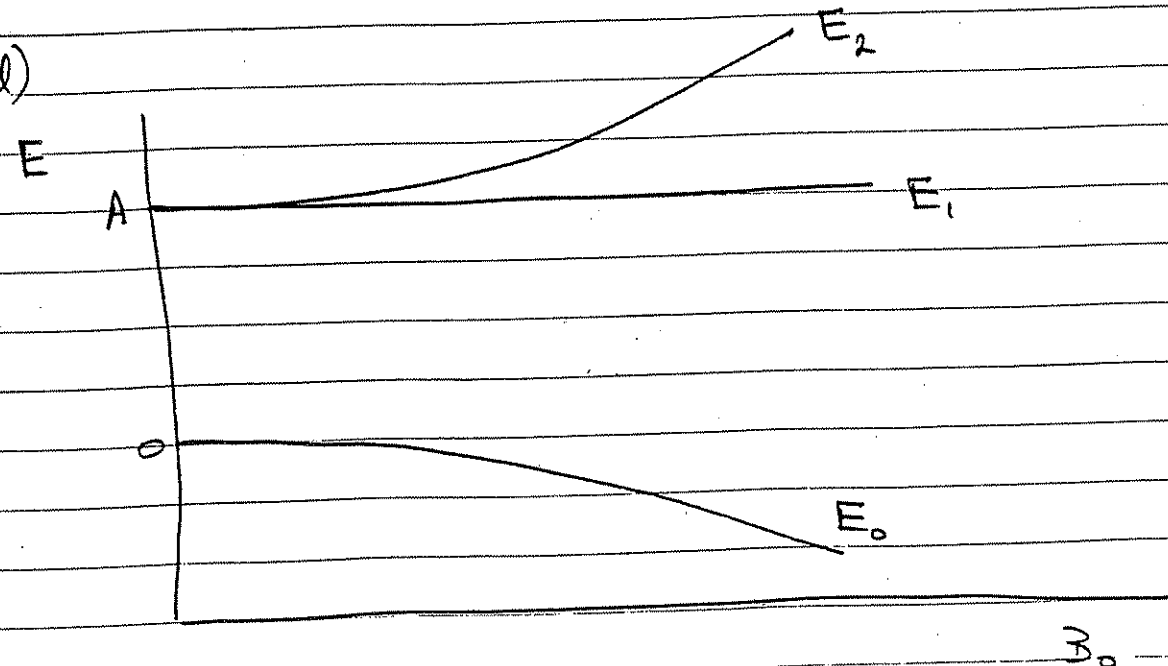
$$E(A-E) + 4(B_0\gamma)^2 = 0$$

$$E^2 - AE - 4(B_0\gamma)^2 = 0$$

or

$$\boxed{E_{2,0} = \frac{A \pm \sqrt{A^2 + 16(B_0\gamma)}}{2}}$$

(d)



Problem 6 (continued)

(e) Eigenstates are of the form:

$$\psi = a_- |-1\rangle + a_0 |0\rangle + a_+ |+1\rangle$$

$$|a_-|^2 + |a_0|^2 + |a_+|^2 = 1 \quad \text{(normalization)}$$

Coefficients $a_0, a_{\pm 1}$ are given by

(i) $(A-E)a_- + \sqrt{2}\,c_+ a_0 + 0 = 0$

(ii) $\sqrt{2}\,c_- a_- - E a_0 + \sqrt{2}\,c_+ a_+ = 0$

(iii) $0 + \sqrt{2}\,c_- a_0 + (A-E)a_+ = 0$

Eqn (i) $\Rightarrow$ $a_- = \dfrac{-\sqrt{2}\,c_+}{(A-E)}\,a_0$

Eqn (iii) $\Rightarrow$ $a_+ = -\dfrac{\sqrt{2}\,c_-}{(A-E)}\,a_0$

normalization condition $\Rightarrow$ $a_0^2\left[1 + \dfrac{2(|c_+|^2+|c_-|^2)}{(A-E)^2}\right] = 1$

$$a_0^2 = \frac{1}{\left[1 + \dfrac{2(|c_+|^2+|c_-|^2)}{(A-E)^2}\right]} = \frac{1}{\left[1 + \dfrac{4(\gamma B_0)^2}{(A-E)^2}\right]}$$

Problem 6 (continued)

lowest energy eigenstate:

$$E_0 = \frac{A - \sqrt{A^2 + 16(\gamma B_0)^2}}{2}$$

$$A - E_0 = \frac{A + \sqrt{A^2 + 16(\gamma B_0)^2}}{2}$$

$$a_0 = \frac{1}{\left[1 + \frac{16(\gamma B_0)^2}{\left(A + \sqrt{A^2 + 16(\gamma B_0)^2}\right)^2}\right]^{1/2}}$$

$$a_0 = \frac{A + \sqrt{A^2 + 16(\gamma B_0)^2}}{\left[2A^2 + 32(\gamma B_0)^2 + 2A\sqrt{A^2 + 16(\gamma B_0)^2}\right]^{1/2}}$$

$$a_\pm = -\frac{\sqrt{2}\,C_\mp}{(A - E_0)}\,a_0 = \frac{-\sqrt{2}(\gamma B_0)e^{\mp i\phi} \cdot 2}{\left[2A^2 + 32(\gamma B_0)^2 + 2A\sqrt{A^2 + 16(\gamma B_0)^2}\right]^{1/2}}$$

note: $B_0 \to 0 \Rightarrow \quad a_0 = 1 \qquad a_\pm = 0$

Problem 7

Use Fermi Golden Rule:

$$\text{probability} = \frac{2\pi}{\hbar} \left| \langle f | \mathcal{H}_{int} | i \rangle \right|^2 \rho(E)\, \delta(\omega - E + E_0)$$

$$\rho(E) = p^2 \frac{dp}{dE} = m\sqrt{2mE} \quad (\text{free electrons})$$

$$\mathcal{H}_{int} = -ezE_0 \quad (\text{electric dipole transition})$$

initial state: $\psi_i(\vec{r}) = \frac{1}{\sqrt{2}} \left(\frac{Z}{a}\right)^{3/2} e^{-r/a}$

final (free electron) state: $\psi_f(\vec{r}) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}}$

$$k = \sqrt{2mE/\hbar^2}$$

matrix element: $\langle f | \mathcal{H}_{int} | i \rangle = \int d^3r \frac{1}{\sqrt{2}} \left(\frac{Z}{a}\right)^{3/2} e^{-r/a} (-ezE_0) \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}}$

$$= \frac{1}{\sqrt{2}} \left(\frac{Z}{a}\right)^{3/2} \left(\frac{1}{2\pi}\right)^{3/2} (-eE_0) \int d^3r\, e^{-r/a} \cdot r\cos\theta_r \cdot e^{i\vec{k}\cdot\vec{r}}$$

introduce spherical harmonics:

$$\cos\theta_r = \sqrt{\frac{4\pi}{3}}\, Y_1^0(\theta_r, \phi_r)$$

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l j_l(kr)\, Y_l^{m*}(\theta_k, \phi_k)\, Y_l^m(\theta_r, \phi_r)$$

Problem 7 (continued)

$$\langle 1|1\rangle = \frac{1}{\sqrt{2}}\left(\frac{2}{a}\right)^{3/2}\left(\frac{1}{2\pi}\right)^{3/2}(-eE_0)\int dr\, r^3 e^{-r/a}\cdot 4\pi \sum_{\ell}\sum_{m} i^{\ell} j_{\ell}(kr)$$

$$\cdot\, Y_{\ell}^{m*}(\theta_k,\phi_k)\cdot \underbrace{\int d\Omega_r \sqrt{\frac{4\pi}{3}}\, Y_1^0(\theta_r,\phi_r)\, Y_{\ell}^{m}(\theta_r,\phi_r)}_{=\sqrt{\frac{4\pi}{3}}\,\delta_{\ell,1}\,\delta_{m,0}}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{2}{a}\right)^{3/2}\left(\frac{1}{2\pi}\right)^{3/2}(-eE_0)\cdot 4\pi\,(i)\sqrt{\frac{4\pi}{3}}\cdot\sqrt{\frac{3}{4\pi}}\cos\theta_k$$

$$\times \underbrace{\int_0^\infty dr\, r^3 e^{-r/a} j_1(kr)}_{\equiv I_{radial}}$$

$$\langle 1|1\rangle = i\pi\left(\frac{2}{a\pi}\right)^{3/2}(-eE_0)\cos\theta_k\cdot I_{radial}$$

$$\text{probability} = \frac{2\pi}{\hbar}\cdot\pi^2\left(\frac{2}{a\pi}\right)^3(eE_0)^2\cos^2\theta\, I_{radial}^2\, m\sqrt{2m(E_B+\omega)}$$

$$= \frac{16}{a^3\hbar}(eE_0)^2\cos^2\theta\, I_{radial}^2\, m\sqrt{2m(E_B+\omega)}$$

$$I_{radial} = \frac{2a}{k^3}\,\frac{1}{1+(ka)^2}$$ (not necessary to evaluate)

$$k = \frac{\sqrt{2m(E_B+\omega)}}{\hbar}$$

Problem 8

(a) $\vec{\nabla} \cdot \vec{D} = 4\pi \rho \text{ (free)}$

$\int \vec{D} \cdot \hat{n} \, da = 4\pi Q \text{ (free)}$

Draw a spherical Gaussian surface of radius r:

$D \times 4\pi r^2 = 4\pi Q$

so $D = Q/r^2$

(b) $C = Q/V$ where $V \equiv \int_a^b \vec{E} \cdot d\vec{\ell}$

But $E = \dfrac{D}{\epsilon} = \dfrac{Q\,(K + mr)}{\epsilon_0\, r^2}$

$$V = \int_a^b \vec{E} \cdot d\vec{r} = \int_a^b E \, dr = \frac{Q}{\epsilon_0} \int_a^b \frac{(K + mr)\, dr}{r^2}$$

$$= \frac{Q}{\epsilon_0} \left[-\frac{K}{r} + m \ln r \right]_a^b$$

$$= \frac{Q}{\epsilon_0} \left[\frac{K}{a} - \frac{K}{b} + m \ln(b/a) \right]$$

So $$C = \epsilon_0 \left[\frac{K}{a} - \frac{K}{b} + m \ln(b/a) \right]^{-1}$$

(c) Since $D = E + 4\pi \vec{P}$ and $D = \epsilon E$

$$\vec{P} = \frac{1}{4\pi} \vec{D} \left(1 - 1/\epsilon \right)$$

Inside the dielectric $\vec{\nabla} \cdot \vec{D} = 4\pi \rho \text{ (free)} = 0$ so

$$\rho_P = -\vec{\nabla} \cdot \vec{P} = \frac{1}{4\pi} \vec{D} \cdot \vec{\nabla}\left(\frac{1}{\epsilon}\right) = \frac{mQ}{4\pi \epsilon_0 r^2}$$