OSU PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #68

March 30 and 31, 1993

Comprehensive examination for Spring 1993

PART I

General Instructions

This Comprehensive Examination for Spring 1993 (#68) consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Tuesday, March 30, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:30 pm on the same day and will also last three hours. The third and fourth parts will be administered on Wednesday, March 31.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit – especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Problem 1

A 1000 kilogram sport car with high-quality tires is able to achieve a lateral (sideways) acceleration of $0.9g$ without sliding on dry, level pavement at a speed of 66 mph = 30 meters per second.

a. What is the coefficient of friction between the tires and the pavement? $\mu = ?$

b. A driver tests the car on an empty parking lot enclosed by a guard rail. She accelerates to 30 meters per second in a straight line perpendicular to the boundary of the lot. To avoid hitting the rail, how far from the rail must she begin to turn? $D = ?$

c. A curve in a highway has a radius $R = 100$ meters, and the highway is level; *i.e.*, the car's center of mass remains in the same horizontal plane. However the pavement is tilted into the curve to help cars stay on the road. The curve is rated at $v = 33$ mph $= 15$ m/s , which means that the force on a car travelling at that speed will be normal to the pavement.What is the pitch of the pavement (the angle between the normal to the pavement and the vertical)? $\tan\theta = ?$

d. What is the maximum speed at which the sport car could negotiate the curve in part (c) without sliding? Assume the coefficient of friction is independent of velocity. $v_{max} = ?$

REMEMBER: You may express your answer to any part of this problems in terms of the answers to the previous parts. You may approximate $g = 10$ m/s^2.

Problem 2

Consider the one-dimensional Schrödinger equation for a slowly varying potential $V(x)$. Assume the solution $\psi(x, t)$ has the form $\psi(x, t) = e^{i(S(x)-Et)/\hbar}$ for $E > V$.

a. Find a differential equation for $S(x)$.

b. Expand S as a series in powers of $\hbar$, *i.e.* $S = S_0 + \hbar S_1 + \ldots$. Find an approximate expression for $S(x)$, to first order in $\hbar$.

c. Find the first-order approximation to $\psi(x, t)$ for the special case of a linear potential $V(x) = \alpha x$ in a region where $E > V(x)$.

d. Where do you expect your approximation to break down?

END OF PART I

OSU Physics Department Comprehensive Exam # 68, March 30-31, 1993 **SOLUTIONS**

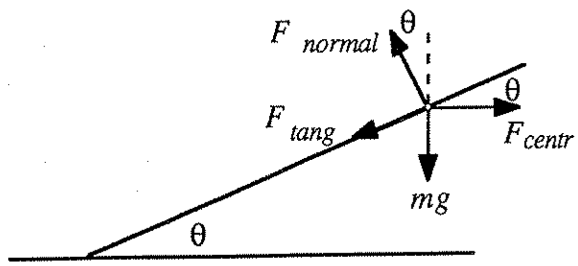
Problem 1

a. $F_{friction} = \mu F_{normal} = F_{tangential}$, $F_{normal} = mg$, $F_{tangential} = m\, a_{tangential} = m \times 0.9g$

$\mu\, m\, g = 0.9\, m\, g \Rightarrow \mu = 0.9$

b. $F_{tang} = m\, v^2/r \le \mu F_{normal} = \mu\, m\, g \Rightarrow r \ge \dfrac{mv^2}{\mu mg} = \dfrac{v^2}{\mu g} = \dfrac{(30\text{ m/s})^2}{0.9\times 10\text{ m/s}^2} = 100\text{ m} = D$

c.



no friction needed $\Rightarrow 0 = F_{tang} = F_{centr} \cos\theta - mg \sin\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{mv^2/R}{mg} = \frac{v^2}{R\ g} = \frac{225\text{ m/s}^2}{100\text{ m}\times 10\text{ m/s}^2} = 0.225 = \tan\theta$$

d. $F_{tang} = F_{frict}^{max} = \mu F_{normal}$

$$F_{tang} = F_{centr} \cos\theta - mg \sin\theta = \frac{mv^2}{R} \cos\theta - mg \sin\theta$$

$$F_{normal} = F_{centr} \sin\theta + mg \cos\theta = \frac{mv^2}{R} \sin\theta + mg \cos\theta$$

$$\mu\left(\frac{mv^2}{R} \sin\theta + mg \cos\theta\right) = \frac{mv^2}{R} \cos\theta - mg \sin\theta$$

$$\mu g + \mu \frac{v^2}{R} \tan\theta = \frac{v^2}{R} - g \tan\theta\ ,\ \frac{v^2}{R}(1 - \mu \tan\theta) = g\,(\mu + \tan\theta)$$

$$v^2 = Rg\frac{\mu + \tan\theta}{1-\mu \tan\theta} = 100\text{ m} \times 10\text{ m/s}^2 \times \frac{0.9 + 0.225}{1 - 0.9\times 0.225} = 1411\text{ m}^2/\text{s}^2$$

$v = 38$ m/s $= 84$ mph

Problem 3

For an s-state of a one-electron atom, the spherically symmetric function $n(r)$ represents the electron density (probability per unit volume). In this problem $n(r)$ will be treated as a classical distribution for an electron having charge $-e$. Then the charge density is given by $\rho(r) = -en(r)$, and is independent of time.

a. Find an expression for the electric field $\mathbf{E}(\mathbf{r})$ [due to the electron only], in terms of a radial integral over $n(r)$.

b. For the purposes of this classical model, assume that the magnetization vector is proportional to $n(r)$: $\mathbf{M}(r) = \boldsymbol{\mu}_S\, n(r)$, where $\boldsymbol{\mu}_S$ is the electron-spin magnetic moment. Find the associated magnetization current density $\mathbf{J}(\mathbf{r})$, and use it to derive a simple expression for the vector potential $\mathbf{A}(\mathbf{r})$ in terms of $\boldsymbol{\mu}_S$ and $\mathbf{E}(\mathbf{r})$.

c. Again using classical electromagnetic theory, find the magnetic field $\mathbf{B}(0)$ at the nucleus, due to the electron, in terms of $\boldsymbol{\mu}_S$ and $n(0)$, the electron density at the nucleus. From this result find the hyperfine interaction energy, if the nucleus has a magnetic dipole moment $\boldsymbol{\mu}_N$.

GO ON TO THE NEXT PAGE

Problem 4

When ultra-relativistic fermions (f) are in equilibrium with photons (γ) under the reaction

$$f^+ + f^- \leftrightarrow \gamma ,$$

the chemical potentials μ of all constituents are zero:

$$\mu_{f^+} = 0 = \mu_{f^-} ,$$

in which case the partition function for independent spin-$\frac{1}{2}$ fermions is

$$Z_f = \prod_i (1 + e^{-\beta E_i})^{+2}$$

and the partition function for transverse photons is

$$Z_\gamma = \prod_i (1 - e^{-\beta E_i})^{-2} .$$

Assuming that ultrarelativistic free fermions and photons within a volume V have energy levels $E_k = \hbar c k$, where k is the wavenumber, show that

a. the Helmholtz free energy of spin-$\frac{1}{2}$ fermions is

$$F_f = -\frac{V}{3\pi^2}\left(\frac{1}{\beta}\right)^4\left(\frac{1}{\hbar c}\right)^3 \int_0^\infty dz\, z^3\, (e^z+1)^{-1} .$$

b. the Helmholtz free energy of transverse photons is

$$F_\gamma = -\frac{V}{3\pi^2}\left(\frac{1}{\beta}\right)^4\left(\frac{1}{\hbar c}\right)^3 \int_0^\infty dz\, z^3\, (e^z-1)^{-1} .$$

c. the ratio of entropies is

$$\frac{S_f}{S_\gamma} = \frac{7}{8} .$$

END OF PART II

Problem 5

An uncharged conducting spherical shell of mass m floats with one-fourth (1/4) of its volume submerged in a liquid dielectric having dielectric constant ε. To what potential must the sphere be charged if it is to float half-submerged?

Problem 6

A two-dimensional quantum "line" can be modelled by confining electrons in the y-direction with the potential

$$V(x, y) = \frac{1}{2} m \omega_0^2 y^2$$

where m is the electron mass and ω_0 is a frequency that characterizes the confinement dimension, which is in the mesoscopic range (larger than atoms but smaller than macroscopic). The x-coordinate is treated with periodic boundary conditions; *i.e.*, the wave functions satisfy

$$\psi(x+L_x , y) = \psi(x , y) ,$$

where L_x is the period of the x dimension.

a. Find the energy eigenvalues and degeneracies for spin-$\frac{1}{2}$ electrons in this quantum line model.

b. For mesoscopic systems, spin-orbit coupling

$$V_{\text{spin-orbit}} = \frac{\hbar}{4m^2c^2} \, \sigma \cdot (\nabla V \times \boldsymbol{p})$$

may be a factor of some importance. Find the shift of each energy level in this confinement model due to spin-orbit coupling:

i. in first-order perturbation theory

ii. exactly .

END OF PART III

Problem 7

An ideal piano produces a low-frequency note when a hammer simultaneously, at time $t = 0$, strikes two identical strings, each of mass m and length L, at the same distance $l < L$ from the ends of both strings, which are fixed to the piano frame. The tensions in the strings, T_1 and T_2, are not necessarily the same. The displacement of the first string is given by $y_1(x, t)$, and the displacement of the second string is given by $y_2(x, t)$. At time $t = 0$, both strings have an identical displacement and velocity, which is determined by the motion of the hammer:

$$y_1(x, t{=}0) = y_2(x, t{=}0) = 0,$$

$$\frac{\partial}{\partial t} y_1(x, t{=}0) = \frac{\partial}{\partial t} y_1(x, t{=}0) = v_0(x).$$

a. Show that the normal modes of the 2-wire system consist of vibrations in which the strings vibrate with various numbers of nodes n. How many modes are there for each n?

b. Find an expression for the amount of energy in each mode. Show that the amount of energy only depends on n; *i.e.*, it is the same for all modes with n nodes. How does this energy depend on time?

GO ON TO THE NEXT PAGE

Problem 8

A spin-$\frac{1}{2}$ particle, the lambda Λ^0, disintegrates into a (spin-$\frac{1}{2}$) proton , p , and a (spin-0) pion, π^-, via the weak interaction.

a. Suppose the Λ^0 is initially polarized with spin up along the z-axis in the center-of-mass frame. Also suppose the disintegration occurs so that the proton moves straight up the z-axis (within a small solid angle $\Delta\Omega$). What happens to the pion?

b. Under the conditions of part **a**, the amplitude for the disintegration to take place with the proton spin up along the z-axis is A. What is the amplitude for the proton to have spin down?

c. Suppose the Λ^0 is initially polarized with spin down along the z-axis in the center-of-mass frame. The amplitude for the proton to go straight up the z-axis with spin down is B. If $B \neq A$, what can you say about parity? Why?

d. If a spin-$\frac{1}{2}$ particle has spin up along the z-axis, its amplitude to have spin up along an axis z' at an angle θ with respect to the original z-axis is $\cos\frac{\theta}{2}$. What is the magnitude of the amplitude that it will have spin down with respect to z' ?

e. Suppose the Λ^0 is polarized with spin up along the original z-axis. What is the probability $P(\theta)$ of finding the decay proton in a small solid angle $\Delta\Omega$ at angle θ with respect to the original z-axis? Express your answer in terms of A, B, and θ .

END OF EXAMINATION

OSU Physics Department Comprehensive Exam # 68, March 30-31, 1993 **SOLUTIONS**

Problem 2

a) $\psi'' + \frac{2m}{\hbar^2}(E-V)\psi = 0$

$\psi = \exp\left[\frac{i}{\hbar}(S - Et)\right]$

$\psi' = \frac{i}{\hbar} S' \psi \qquad \psi'' = -\frac{1}{\hbar^2}(S')^2\psi + \frac{i}{\hbar} S''\psi$

$-\frac{1}{\hbar^2}(S')^2 + \frac{i}{\hbar}S'' + \frac{2m}{\hbar^2}(E-V) = 0 \quad \Rightarrow \quad (S')^2 - i\hbar S'' - 2m(E-V) = 0$

b) $S = S_0 + \hbar S_1 + \ldots$

$(S_0')^2 - 2m(E-V) = 0$ zeroth order

$\Rightarrow S_0 = \pm\sqrt{2m}\int^x (E-V)^{\frac{1}{2}} dx' \equiv \pm\int^x k(x')dx'$ where $k = \sqrt{2m}(E-V)^{\frac{1}{2}}$

$2 S_0' S_1' - i S_0'' = 0$ first order

$\Rightarrow S_1' = \frac{i}{2}\frac{S_0''}{S_0'}$

$S_1 = \frac{i}{2}\ln S_0' = \frac{i}{2}\ln k$

$\psi = \exp\left[\frac{i}{\hbar}\left(S_0 + \frac{i\hbar}{2}\ln k - Et\right)\right] = \frac{1}{\sqrt{k}}\exp\left[\pm\frac{i}{\hbar}\int^x k\,dx' - \frac{i}{\hbar}Et\right]$

c) $S_0 = \pm\sqrt{2m}\int^x (E-\alpha x')^{\frac{1}{2}} dx' = \pm\sqrt{2m}\left(-\frac{1}{\alpha}\right)\frac{2}{3}(E-\alpha x)^{3/2}$

$\psi = \frac{1}{[2m(E-V)]^{\frac{1}{4}}}\exp\left[\mp\frac{2\sqrt{2m}\,i}{3\alpha\hbar}(E-\alpha x)^{3/2} - \frac{i}{\hbar}Et\right]$

d) breaks down for $E \approx V$ or V rapidly varying

Problem 3

(a) Gauss's law : $\oint\!\!\oint \vec{E}\cdot\hat{n}\, d\sigma = 4\pi q$ with spherical symmetry

$$E \cdot 4\pi r^2 = 4\pi \int_0^r \rho(r') \cdot 4\pi r'^2 dr' \qquad \rho = -en$$

$$\boxed{\vec{E}(r) = -\frac{e\hat{r}}{r^2}\, 4\pi \int_0^r n(r')\, r'^2\, dr'}$$

(b)
$$\vec{J} = c\, \vec{\nabla} \times \vec{M}$$
$$= c\, \vec{\nabla} \times (\vec{\mu}_s n) = -c\, \vec{\mu}_s \times \vec{\nabla} n = \frac{c}{e}\, \vec{\mu}_s \times \vec{\nabla}\rho$$

$$\vec{A}(\vec{r}) = \frac{1}{c} \int \frac{\vec{J}(\vec{r}')\, d^3r'}{|\vec{r}-\vec{r}'|} = \frac{1}{e}\, \vec{\mu}_s \times \underbrace{\vec{\nabla} \int \frac{\rho(r')\, d^3r'}{|\vec{r}-\vec{r}'|}}_{\vec{\nabla}\phi(\vec{r}) = -\vec{E}(\vec{r})}$$

$$\boxed{\vec{A}(\vec{r}) = -\frac{1}{e}\, \vec{\mu}_s \times \vec{E}(r)}$$

(c) Near the origin, $\vec{E}(r) = -\frac{e\hat{r}}{r^2} \cdot \frac{4\pi}{3} r^3 n(0) = -\frac{4\pi e}{3} n(0)\, \vec{r}$

" " " $\vec{A}(\vec{r}) = \frac{4\pi}{3} n(0)\, \vec{\mu}_s \times \vec{r}$

$\therefore$

$$\vec{B}(0) = \lim_{r\to 0} \vec{\nabla} \times \vec{A}(\vec{r}) = \frac{4\pi}{3} n(0)\, \vec{\nabla} \times (\vec{\mu}_s \times \vec{r})$$
$$= \frac{4\pi}{3} n(0) \Big[\vec{\mu}_s \underbrace{\vec{\nabla}\cdot\vec{r}}_{3} - \vec{\mu}_s \cdot \underbrace{\vec{\nabla}\vec{r}}_{1} \Big]$$

$$\boxed{\vec{B}(0) = \frac{8\pi}{3} n(0)\, \vec{\mu}_s}$$

$$\Delta E_{HFS} = -\vec{\mu}_N \cdot \vec{B}(0)$$

$$\boxed{\Delta E_{HFS} = -\frac{8\pi}{3} n(0)\, \vec{\mu}_N \cdot \vec{\mu}_s}$$

Problem 4

4. Partition functions ($\mu = 0$)

$$Z_f = \prod_i \left(1 + e^{-\beta E_i}\right)^2$$

$$Z_\gamma = \prod_i \left(1 - e^{-\beta E_i}\right)^{-2}$$

(a) Helmholtz Free energy for ultra-relativistic fermions

$F_f = -\frac{1}{\beta}\sum_{\mathbf{k}} \log\left(1 + e^{-\beta E_{\mathbf{k}}}\right)$ where $E_{\mathbf{k}} = \hbar c k$.

$$\begin{aligned} F_f &= -\frac{2}{\beta}\sum_{\mathbf{k}} \log\left(1 + e^{-\beta E_{\mathbf{k}}}\right) \\ &= -\frac{2V}{\beta}\left(\frac{1}{2\pi}\right)^3 (4\pi)\int_0^\infty k^2 dk \log\left(1 + e^{-\beta \hbar c k}\right) \\ &= -\frac{V}{\beta \pi^2}\left(\frac{1}{\beta \hbar c}\right)^3 \int_0^\infty dz\, z^2 \log\left(1 + e^{-z}\right) \end{aligned}$$

Integrate by parts:

$$F_f = -\frac{V}{3\beta \pi^2}\left(\frac{1}{\beta \hbar c}\right)^3 \int_0^\infty dz\, z^3 \left(\frac{1}{e^z + 1}\right)$$

(b) For photons $F_\gamma = \frac{V}{\beta \pi^2}\left(\frac{1}{\beta \hbar c}\right)^3 \int_0^\infty dz z^2 \log\left(1 - e^{-z}\right)$. Integrate by parts

$$F_\gamma = -\frac{V}{3\beta \pi^2}\left(\frac{1}{\beta \hbar c}\right)^3 \int_0^\infty dz z^3 \left(\frac{1}{e^z - 1}\right)$$

(c) $-S = \left(\frac{\partial F}{\partial T}\right)_V$

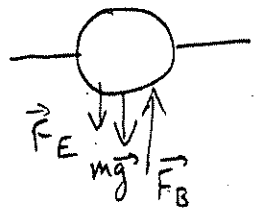
$$\frac{S_f}{S_\gamma} = \frac{\int_0^\infty dz\, z^3 \left(\frac{1}{e^z + 1}\right)}{\int_0^\infty dz\, z^3 \left(\frac{1}{e^z - 1}\right)} = \frac{(1 - 2^{-3})\varsigma(4)\Gamma(4)}{\varsigma(4)\Gamma(4)} = \frac{7}{8}$$

Problem 5

When $V = 0$ (← Potential), the sphere floats with $\frac{1}{4}$ of its volume V submerged.

Buoyant force: $\rho g \cdot \frac{1}{4} V = mg \implies \rho g V = 4mg$.

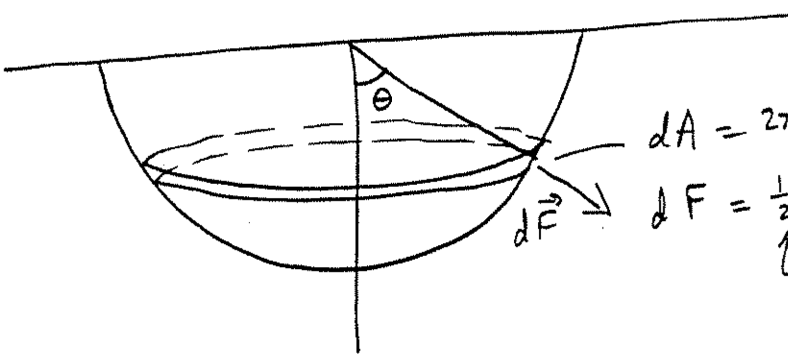
For charged sphere:



$F_B = F_E + mg$

$\therefore F_E = F_B - mg$
$= \rho g \cdot \frac{1}{2} V - mg$
$= \frac{1}{2}(4mg) - mg = mg$

$\therefore F_E = mg$ Downward electric force due to dielectric.



$dA = 2\pi a^2 \sin\theta \, d\theta$

$dF = \frac{1}{2} dq_p \cdot E = \frac{1}{2} E \sigma_p \, dA$

↑ Factor of $\frac{1}{2}$ arises when $q_p \propto E$.
↑ Polarization charge.

Downward force:

$$F_E = \int dF \cos\theta = \frac{1}{2} E \sigma_p \int_0^{\pi/2} \cos\theta \cdot 2\pi a^2 \sin\theta \, d\theta$$

$$= \pi a^2 E \sigma_p \cdot \frac{\sin^2\theta}{2} \Big|_0^{\pi/2} = \frac{1}{2} \pi a^2 E \sigma_p$$

$\vec{D} = \epsilon \vec{E} = \vec{E} + 4\pi \vec{P} \implies \vec{P} = \frac{\epsilon - 1}{4\pi} \vec{E}$ Polarization.

The polarization charge is $\sigma_p = \vec{P} \cdot \hat{n} = \frac{\epsilon - 1}{4\pi} E$.

$E = \frac{V}{a}$ for sphere. (← Surface field)

$$\therefore F_E = \frac{\pi a^2}{2} \cdot \frac{\epsilon - 1}{4\pi} \cdot E^2 = \frac{a^2(\epsilon - 1)}{8} \cdot \frac{V^2}{a^2}$$

$mg =$

$$\therefore V^2 = \frac{8mg}{\epsilon - 1}$$

$$\boxed{V = 2\sqrt{\frac{2mg}{\epsilon - 1}}}$$

MKS: $\times \frac{1}{4\pi\epsilon_0}$ inside $\sqrt{\ }$, $\epsilon \to K$

Problem 6

6. (a) $H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega_0{}^2 y^2$

wave function

$$\psi(x,y) = \frac{1}{\sqrt{L_x}} e^{ik_x x}\Phi_n(y)\chi_{spin} \quad \text{with} \quad k_x = \frac{2\pi\nu}{L_x} \quad \nu = 0, \pm 1, \pm 2, \ldots$$

Energies: $E_{k_x,n} = \frac{\hbar^2}{2m}\left(\frac{2\pi\nu}{L_x}\right)^2 + (n + \frac{1}{2})\hbar\omega_0$

Degeneracy: $spin \times (\pm\nu) = 4\,fold$

(b) $V_{s.o.} = \frac{\hbar}{4m^2c^2}\boldsymbol{\sigma}\cdot(\nabla V \times \mathbf{p}) \qquad \nabla V = m\omega_0{}^2 y\,\hat{y}$

$$V_{s.o.} = \frac{\hbar\omega_0{}^2}{4mc^2} y p_x \sigma_z$$

First order perturbation theory: $\langle \psi_{k_x,n} | y | \psi_{k_x,n} \rangle = 0$, since $\psi_{k_x,n}$ is an eigenfunction of parity.

(c) Exact solution: $H = \frac{p_x^2}{2m} \pm \frac{\hbar\omega_0^2}{4mc^2} y p_x + \frac{1}{2}m\omega_0{}^2 y^2 + \frac{p_y^2}{2m}$

Substitute assumed solution (still translationally invariant in x-direction)

$H = \frac{\hbar^2 k_x^2}{2m} \pm \frac{\hbar^2\omega_0^2}{4mc^2} y k_x + \frac{1}{2}m\omega_0{}^2 y^2 + \frac{p_y^2}{2m}$ which is a displaced harmonic oscillator.

Complete the square to get $\Delta E(\nu) = \frac{1}{2m}\left(\frac{2\pi\nu\hbar^2\omega_0}{4mc^2 L_x}\right)^2$.

Problem 7

a. Let the mass density per unit length be $\rho = m/L$.

$$\text{K.E.} = \int_0^L dx\ \tfrac{1}{2}\rho\left[\left(\frac{\partial y_1}{\partial t}(x,t)\right)^2+\left(\frac{\partial y_2}{\partial t}(x,t)\right)^2\right],\ \text{P.E.} = \int_0^L dx\ \left[\frac{T_1}{2}\left(\frac{\partial y_1}{\partial x}(x,t)\right)^2+\frac{T_2}{2}\left(\frac{\partial y_2}{\partial x}(x,t)\right)^2\right]$$

boundary conditions $y_i(x=0,t) = 0 = y_i(x=L,t)$ for $i = 1,2 \Rightarrow$

normal mode expansion $y_i(x,t) = \sum_n a_{in}(t)\sin(n\pi x/L) \Rightarrow$

$$\text{K.E.} = \tfrac{1}{2}\rho \sum_{n\ n'} \int_0^L dx\ \left[\frac{da_{1n}(t)}{dt}\sin(n\pi x/L)\frac{da_{1n'}(t)}{dt}\sin(n'\pi x/L)+(1\to 2)\right],$$

$$\text{P.E.} = \tfrac{1}{2}\sum_{n\ n'} \int_0^L dx\ \left[T_1 a_{1n}(t)\,\tfrac{n\pi}{L}\cos(n\pi x/L)\ a_{1n'}(t)\,\tfrac{n'\pi}{L}\cos(n'\pi x/L) + (1\to 2)\right]$$

but $\int_0^L dx\cos(n\pi x/L)\cos(n'\pi x/L) = \frac{L}{2}\delta_{nn'} = \int_0^L dx\sin(n\pi x/L)\sin(n'\pi x/L) \Rightarrow$

$$\text{K.E.} = \frac{\rho L}{4}\sum_n\left[(da_{1n}/dt)^2 + (da_{2n}/dt)^2\right],\ \text{P.E.} = \frac{\pi^2}{4L}\sum_n n^2\left[T_1 a_{1n}(t)^2 + T_2 a_{2n}(t)^2\right]$$

We see that the kinetic and potential energies separate into a sum of terms with n nodes.

There are two modes for each n, since a_{1n} and a_{2n} are independent.

Problem 7(continued)

b. Energy (n) = K.E.(n) + P.E.(n) $= \frac{m}{4}[(da_{1n}/dt)^2 + (da_{2n}/dt)^2)] + \frac{\pi^2}{4L}n^2[T_1 a_{1n}(t)^2 + T_2 a_{2n}(t)^2]$

$= \sum_i [\frac{\mu}{2}(da_{in}/dt)^2 + \frac{K_{in}}{2}a_{in}{}^2]$ with $\mu = \frac{m}{2}$, $K_{in} = \frac{\pi^2 n^2 T_i}{2L}$, harmonic oscillator with $\omega_{in}{}^2 = \frac{K_{in}}{\mu}$

initial condition $y_i(t=0) = 0 \Rightarrow$ $a_{in}(t) = a_{in} \sin \omega_{in}t \Rightarrow$

$$\text{Energy}(n) = a_{1n}{}^2[\frac{m}{4}\omega_{1n}{}^2 \sin^2\omega_{in}t + \frac{\pi^2}{4L}n^2 T_1 \cos^2\omega_{in}t\,] + (1 \rightarrow 2) = a_{1n}{}^2 \frac{m}{4}\omega_{1n}{}^2 + (1 \rightarrow 2)$$

independent of time

Insert normal mode expansion, differentiate to find initial condition for velocity:

$$v_0(x) = \partial y_i/\partial t\,|_{t=0} = \sum_n \frac{\partial}{\partial t} a_{in}(t)|_{t=0} \sin(n\pi x/L) = \sum_n a_{in}\omega_{in} \sin(n\pi x/L) \text{ for } i=1,2$$

orthogonality $\Rightarrow a_{1n}\omega_{1n} = a_{2n}\omega_{2n} \Rightarrow$ energy same for $i = 1,2$

to find a_{in}, use orthogonality:

$$\int_0^L dx\, v_0(x) \sin(n\pi x/L) = \int_0^L dx \sum_n a_{in}\omega_{in} \sin(n\pi x/L) \sin(n\pi x/L) = \frac{L}{2} a_{in}\,\omega_{in}$$

$$\Rightarrow a_{1n}\omega_{1n} = a_{2n}\omega_{2n} = \frac{2}{L}\int_0^L dx\, v_0(x) \sin(n\pi x/L)$$

$$\Rightarrow \text{Energy(each string, mode } n) = \frac{2m}{L^2}\left[\int_0^L dx\, v_0(x) \sin(n\pi x/L)\right]^2 \text{ independent of time}$$

Problem 8

a. pion goes straight down z-axis to con3erve momentum

b. Amplitudes must be zero or angular momentum would not be conserved

c. spin is axial vector, flips under reflection. Parity not conserved if $A \neq B$

d. total probability = 1 $\Rightarrow$

$$? = \left(1 - \cos^2\frac{\theta}{2}\right)^{1/2} = \sin\frac{\theta}{2}$$

e. Amplitude of up-spin emitted along z' axis $= B \sin\frac{\theta}{2}$

These states are distinguishable $\Rightarrow$ total probability need to square + add

$$f(\theta) = |A|^2 \cos^2\frac{\theta}{2} + |B|^2 \sin^2\frac{\theta}{2}$$

$$= \frac{1}{2}\left(|A|^2 + |B|^2\right) + \frac{1}{2}\left(|A|^2 - |B|^2\right)\cos\theta$$