PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #67

January 5 and 6, 1993

Comprehensive Examination for Winter 1993

PART I

General Instructions

This Comprehensive Examination for Winter 1993 (#67) consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is administered at 9:00 am on Tuesday, and lasts three hours. The second part (Problems 3-4) will be administered at 1:30 pm on the same day and will also last three hours. The third and fourth parts will be administered on Wednesday.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks. Work each problem in its own numbered bluebook. Write the problem number on the front of the bluebook. Write your student letter (but not your name) inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks, and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

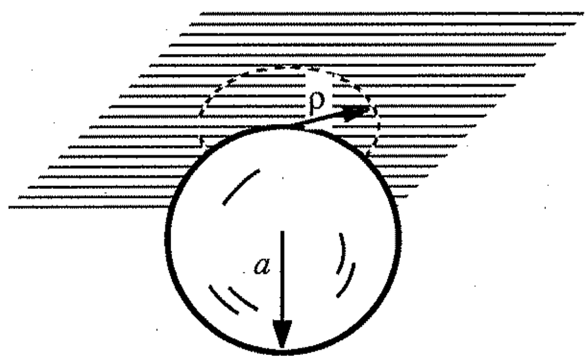
Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Problem #1

An attic ceiling consists of a horizontal foil of aluminum, a conducting metal, with a thin coating of plastic, which is an insulator of dielectric constant 1, i.e. $\epsilon = \epsilon_0$.

A spherical balloon of mass $m = 1$ gm and radius $a = 10$ cm is given an electric charge Q by rubbing it against a wool sweater. When placed in contact with the ceiling, the balloon remains suspended. Assume the charge is, and remains, distributed uniformly on the balloon's surface, which remains spherical:

(a) Find the distribution of surface charge density in the foil, $\sigma(\rho)$ as a function of the distance ρ from the point of contact.



(b) What is the minimum value of Q necessary so the balloon won't fall?

Problem #2

Consider a quantum mechanical system with a two-dimensional state space. An orthonormal basis consists of the states $|1>$ and $|2>$. With respect to this basis the Hamiltonian H and another observable A are represented by the matrices

$$A = a\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \text{and} \qquad H = b\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where a and b are positive constants. At time t = 0 the state of the system is

$$|\psi(0)> = \frac{3}{5}\ |1> + \frac{4}{5}i\ |2> \ .$$

a) Find the energy levels of this system and the corresponding energy eigenstates. Do H and A have common eigenstates?

b) Find the expectation values $<H>$, and the r.m.s. deviation $<\Delta H>$, at time $t = 0$.

c) Observable A is measured at a time t_0, where $t_0 > 0$. What are the possible results of this measurement? What are the corresponding probabilities?

d) It is discovered that the measurement of A (in part c) has yielded the lowest possible value. Find the normalized state $|\psi(t)>$, and the value of $<H>$, at times $t > t_0$.

Problem 3

A particle of mass m, energy $E < 0$, and angular momentum ℓ, is subject to an attractive radial force $f = -\frac{k}{r^2} - \frac{2A}{r^3}$ where k, A are constants. The particle's orbit $r(\theta)$ is given by the equation

$$r^{-1} = \frac{1+\epsilon\cos(\alpha\theta)}{a(1-\epsilon^2)}$$

where ϵ, a, and α are constants.

For this problem you may want to use the integral:

$$\int \frac{dx}{\sqrt{a+bx+cx^2}} = \frac{i}{\sqrt{-c}} \cos^{-1}\left[-\frac{b+2cx}{\sqrt{b^2-4ac}}\right] \quad \text{for } c<0$$

a) Find the value of α in terms of the dimensionless constant $\eta = \frac{A}{ka}$, and the two constants $\epsilon_0 = \left(1 + \frac{2E\ell^2}{mk^2}\right)^{\frac{1}{2}}$, $a_0 = -\frac{k}{2E}$. η is a measure of the relative strengths of the two terms in the force, and ϵ_0 and a_0 are the values of ϵ and a for the case $A = 0$.

b) Estimate to 1 significant figure the value of η sufficient for a force of this form to explain the precession of the perihelion of Mercury. For Mercury $\epsilon = 0.20$, the period is 0.24 years and the rate of precession is 40" per century.

Problem 4

Relativistic electrons and positrons ($E = pc$) are in thermal equilibrium with black-body radiation at a temperature T.

Assume the equilibrium reaction to be

$$e^+ + e^- \rightleftharpoons h\nu$$

and that e^+ and e^- are present in equal numbers.

a) Show that the chemical potential for the electrons and positrons satisfies

$$\mu^+ = 0 = \mu^-$$

b) Find the electron (positron) density at temperature T.

You will need $\int_0^\infty dx \frac{x^2}{(e^x+1)} = 2\zeta(3)$, where $\zeta(n)$ is the Riemann ζ-function.

Problem #5

A thermodynamic model for gravitational collapse of a huge matter cloud can be constructed by a mean field approximation in which the gravitational self energy is added to the Helmholtz free energy of a non-relativistic, classical ideal gas to give the Helmholtz free energy of the cloud.

Assuming a mass m of dust and gas distributed uniformly in a sphere of radius r, the mean field gravitational cloud energy is

$$U_{mf} = -\frac{3}{5}\frac{m^2 G}{r}$$

a) Find the equation of state of the cloud.

b) Find an expression for the isothermal compressibility.

c) Discuss the conditions at which the cloud is no longer thermodynamically stable.

Problem #6

A single-particle system is in an energy eigenstate described by a wave function of the form

$$\psi(x,y,z) = Kye^{-(x^2+y^2+z^2)/a^2}$$

in cartesian coordinates, where K and a are real constants.

a) If $\vec{L}$ denotes the orbital angular momentum for this system, find the possible values of L^2, L_y, and L_z. In each case, what are the corresponding probabilities?

b) This system has a magnetic moment $\vec{\mu} = \gamma\vec{L}$, where γ is a known constant (the gyromagnetic ratio). If a weak, uniform, static magnetic field $\vec{B} = B\,\hat{y}$ is applied, find the Zeeman level shift ΔE using first-order perturbation theory.

c) What model problem can have the given state ψ as an energy eigenstate? What are the quantum numbers for this state of the system? No calculation is necessary.

Problem #7

a) A particle of rest mass m and 4-momentum p is examined by an observer with 4-velocity u. Show that the energy the observer measures is

$$E = - p \bullet u$$

b) An observer receives light from a source which is moving with relativistic velocity $\vec{v}$ directly away from the observer. What is the ratio of the frequency of light in the observer's frame to the frequency of light in the source's frame?

Problem #8

An oscillating electric dipole near the origin produces, in vacuum, a time-dependent magnetic field $\vec{B}(\vec{r},t)$ which, for large $\vec{r}$, has the form

$$\vec{B} = k^2 \frac{(\vec{r}\times\vec{p})}{r^2} \operatorname{Re} e^{i(kr-\omega t)} \quad \text{where } \vec{p} \text{ is a fixed vector and } \omega = ck$$

a) Find $\vec{E}(\vec{r},t)$ for large $\vec{r}$

b) Find the time-averaged intensity of the radiated power per unit area at $\vec{r}$.

c) What is the total radiated power?

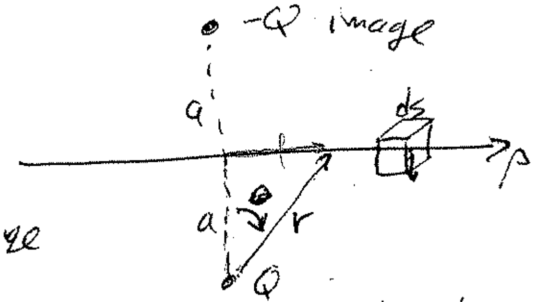
Problem 1 method of images

- spherically-symmetric charge distribution acts (outside its radius) as point charge.
- conducting plane $\Rightarrow$ equipotential $\Rightarrow \vec{E}_{\parallel}(\text{surface}) = 0$

image charge distribution gives $\vec{E}$ below plane, $\vec{E}_{\text{above}} = 0$

$\vec{E}_{\parallel}(\text{image}) = -\vec{E}_{\parallel}(\text{original})$

$\vec{E}_{\perp}(\text{image}) = +\vec{E}_{\perp}(\text{original})$



(a) Look at small pillbox enclosing ford area ds, distance ρ

Gauss' law $\frac{1}{4\pi}\oint \vec{E}\cdot d\vec{s}$ = enclosed charge

$\vec{E}$ up (below), $d\vec{s}$ down $\Rightarrow -\frac{1}{4\pi} E\,ds = \sigma\,ds$, $\sigma = -E_{\perp}/4\pi = -E/4\pi$

$$\Rightarrow \vec{E}_{\text{below}} = \frac{Q}{r^2}\hat{r} - \frac{Q}{r^2}(-\hat{r}) = \frac{2Q}{r^2}\cos\theta = \frac{2Q}{r^2}\frac{a}{r} = \frac{2Q\,a}{(a^2+\rho^2)^{3/2}}$$

$$\boxed{\sigma = -\frac{Q}{2\pi}\frac{a}{(\rho^2+a^2)^{3/2}}}$$ Gaussian units

(b) $\vec{F} = Q\vec{E}$(center of sphere, due to image) $= Q\frac{-Q}{(2a)^2}(-\hat{r})$

but balance of forces $\Rightarrow \vec{F} = -mg\hat{z}$ down

$$\Rightarrow mg = \frac{Q^2}{(2a)^2}, \quad \boxed{Q = 2a\sqrt{mg}}$$

Problem 2

$$A = a\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad ; \quad H = b\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Since H is diagonal, the eigenvalues are the diagonal elements

$E_1 = b$ (eigenstate is $|1\rangle$)

$E_2 = -b$ (eigenstate is $|2\rangle$).

A and H do not commute. They do not have common eigenstates.

(b) $|\psi(0)\rangle = \frac{3}{5}|1\rangle + \frac{4}{5}i|2\rangle$, which is of the form $\sum_n c_n |n\rangle$

$\langle H\rangle = \sum_n |c_n|^2 E_n = \left(\frac{3}{5}\right)^2 E_1 + \left(\frac{4}{5}\right)^2 E_2 = \frac{9}{25}b + \frac{16}{25}(-b$

$$\boxed{\langle H\rangle = -\frac{7}{25}b}$$

(continued)

$$\langle H^2\rangle = \sum_n |c_n|^2 E_n^2 = \frac{9}{25}b^2 + \frac{16}{25}(-b)^2 = b^2$$

$$\Delta H = \sqrt{\langle H^2\rangle - \langle H\rangle^2} = \sqrt{b^2 - \left(-\frac{7}{25}b\right)^2} = b\sqrt{1-\frac{49}{625}}$$

$$\boxed{\Delta H = \frac{24}{25}b}$$

(c.) The eigenvalue equation for $\underline{A}$ is $\begin{vmatrix} -A' & ia \\ -ia & -A' \end{vmatrix} = 0$

$A'^2 - a^2 = 0 \implies A' = \pm a$ eigenvalues

The eigenstates are found from $A\phi = A'\phi$

or $\quad a\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \pm a \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad \therefore\ i c_2 = \pm c_1$

Let $c_1 = 1$. Then $c_2 = \mp i c_1 = \mp i$.

Normalization

Eigenvalue $\underline{\underline{A' = +a}}$ has eigenstate $|a\rangle = \frac{1}{\sqrt{2}}(|1\rangle - i|2\rangle)$

and probability $P_a = |\langle a|\Psi(t_0)\rangle|^2$

$$|\Psi(t)\rangle = \frac{3}{5}e^{-i\frac{bt}{\hbar}}|1\rangle + \frac{4}{5}i\,e^{i\frac{bt}{\hbar}}|2\rangle$$

$$= \frac{1}{2}\left|\langle 1|\Psi(t_0)\rangle + i\langle 2|\Psi(t_0)\rangle\right|^2$$

$$= \frac{1}{2}\left|\frac{3}{5}e^{-i\frac{bt_0}{\hbar}} + i\cdot\frac{4}{5}i\,e^{i\frac{bt_0}{\hbar}}\right|^2$$

$$= \frac{1}{2}\left[\frac{9}{25} + \frac{16}{25} - \frac{12}{25}\underbrace{\left(e^{2i\frac{bt_0}{\hbar}} + e^{-2i\frac{bt_0}{\hbar}}\right)}_{2\cos\frac{2bt_0}{\hbar}}\right]$$

$$\underline{P_a = \frac{1}{2} - \frac{12}{25}\cos\frac{2bt_0}{\hbar}}$$

(continued)

The other eigenvalue is $\underline{A' = -a}$.

Since $P_a + P_{-a} = 1$, it follows that $\underline{P_{-a} = \frac{1}{2} + \frac{12}{25}\cos\frac{2b t_0}{\hbar}}$.

(d) The lowest possible measured value of A is the eigenvalue $-a$. The state after this measurement is therefore the eigenstate of A corresponding to $A' = -a$. From part (c), $i c_2 = -c_1$

$c_2 = +i$ if $c_1 = 1$. $\therefore$

$$\therefore |\psi(t_0^+)\rangle = |-a\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + i|2\rangle\right)$$

At later times, $|\psi(t)\rangle$ is again governed by the Schrödinger equation:

$$\underline{|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\frac{b}{\hbar}(t-t_0)}|1\rangle + i\, e^{i\frac{b}{\hbar}(t-t_0)}|2\rangle\right]}$$

Then $\langle H\rangle = \langle\psi(t)|H|\psi(t)\rangle = \sum_n |c_n'|^2 E_n$

$$= \frac{1}{2}b + \frac{1}{2}(-b).$$

$$\underline{\underline{\langle H\rangle = 0}} \quad \text{if } t > t_0$$

Problem 3

$$l = mr^2\dot{\phi}$$

3a) $V = -\frac{k}{r} - \frac{A}{r^2}$ $\qquad \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{l}{mr^2}\frac{dr}{d\phi}$

$$E = T + V = \tfrac{1}{2}(m\dot{r}^2 + r^2\dot{\phi}^2) + V = \tfrac{1}{2}m\left[\dot{r}^2 + r^2\frac{l^2}{m^2r^4}\right] - \frac{k}{r} - \frac{A}{r^2}$$

$$= \frac{l^2}{2mr^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{B}{r^2} - \frac{k}{r} \qquad \text{where } B = \frac{l^2 - 2mA}{2m}$$

$$d\phi = \pm\frac{l}{\sqrt{2m}}\int^r \frac{1}{r^2}\left(E + \frac{k}{r} - \frac{B}{r^2}\right)^{-\frac{1}{2}} dr \qquad \text{let } x = r^{-1},\ dx = -r^{-2}dr$$

$$= \frac{l}{\sqrt{2m}}\int^x (E + kx - Bx^2)^{-\frac{1}{2}} dx$$

$$\phi = \mp\frac{l}{\sqrt{2m}}\frac{1}{\sqrt{B}}\cos^{-1}\left[-\frac{k - 2Bx}{\sqrt{k^2 + 4EB}}\right]$$

$$x = r^{-1} = \frac{1}{2B}\left[k + \sqrt{k^2 + 4EB}\cos\frac{\sqrt{2mB}}{l}\phi\right]$$

cf $\frac{1}{a(1-\varepsilon^2)} + \frac{\varepsilon}{a(1-\varepsilon^2)}\cos(\alpha\phi)$

$$\alpha = \frac{\sqrt{2mB}}{l} = \sqrt{1 - \frac{2mA}{l^2}} \qquad (1-\varepsilon_0^2)a_0 = \frac{l^2}{mk}$$

$$= \sqrt{1 - \frac{2A}{ka_0(1-\varepsilon_0)^2}} = \sqrt{1 - \frac{2\eta}{1-\varepsilon_0^2}} \qquad \text{where } \eta = \frac{A}{ka}$$

Check $a_0 = a$:

$$\varepsilon = \frac{\sqrt{k^2 + 4EB}}{k} = \sqrt{1 + \frac{4EB}{k^2}}$$

$$a(1-\varepsilon^2) = \frac{2B}{k} = a\left(-\frac{4EB}{k^3}\right) \Rightarrow a = -\frac{k}{2E} = a_0$$

b) $2\pi = \alpha\phi \Rightarrow \Delta\phi \sim \frac{2\pi\eta}{1-\varepsilon_0^2}$

$$\eta = \frac{1-\varepsilon_0^2}{2\pi}\Delta\phi = \frac{.96}{2\pi}\ \frac{40\text{ sec}}{100\text{ yr}}\ \frac{1}{3600}\ \frac{\text{deg}}{\text{sec}}\ \frac{1\text{ rev}}{360\text{ deg}}\ \frac{.24\text{ yr}}{\text{rev}} \sim 10^{-8}$$

Problem 4

"Chemical" Reaction

(a) $\mu^+ + \mu^- = \mu^\gamma$

$\mu^\gamma = 0$

(particle number not conserved)

Therefore $\mu^+ = -\mu^-$

However

$$N^+ = 2\sum_k \frac{1}{e^{\beta(\epsilon_k - \mu^+)}+1}$$

$$N^- = 2\sum_k \frac{1}{e^{\beta(\epsilon_k - \mu^-)}+1}$$

where $\epsilon_k = \hbar c|k|$

and $N^+ = N^-$.

So $\mu^+ = \mu^- = 0$

(b)

$$N = 2\times\frac{V}{(2\pi)^3}\,4\pi\int_0^\infty dk\, k^2 \frac{1}{e^{\beta\hbar c k}+1}$$

$$= 2\times\frac{V}{(2\pi)^3}\,4\pi\int_0^\infty dx\,\frac{x^2}{e^x+1}\left(\frac{1}{\beta\hbar c}\right)^3$$

$$\frac{N}{V} = \frac{1}{\pi^2}\left(\frac{1}{\beta\hbar c}\right)^3 \tfrac{3}{2}\,\zeta(3)$$

Problem 5

$$U_{mf} = -\frac{3}{5}\frac{m^2 G}{r}$$

$$r = \left(\frac{3V}{4\pi}\right)^{1/3} \qquad m = \mu N \qquad \begin{array}{l}\mu = \text{mass/molecule} \\ N - \text{number of molecules}\end{array}$$

$$U_{mf} = -\frac{3}{5}\left(\frac{4\pi}{3}\right)^{1/3}\frac{\mu^2 N^2 G}{V^{1/3}}$$

Free energy of ideal gas $= -Nk_BT\log V$ + other stuff indep. of V

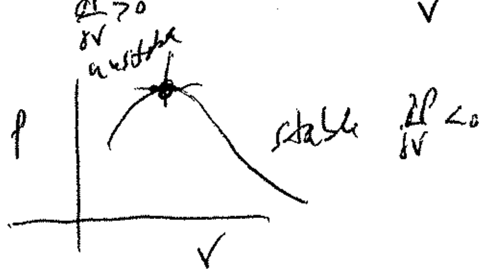
Free energy "total"
$$= -Nk_BT\log V - \frac{3}{5}\left(\frac{4\pi}{3}\right)^{1/3}\frac{\mu^2N^2G}{V^{1/3}}$$
+ other stuff independent of V

$$P = -\left(\frac{\partial F}{\partial V}\right)_T \quad \text{(fixed } N\text{)}.$$

(a) $$P = \frac{Nk_BT}{V} - \frac{3}{5}\left(\frac{1}{3}\right)\left(\frac{4\pi}{3}\right)^{1/3}\frac{\mu^2N^2G}{V^{4/3}}$$

(b) $$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$$

$$= \frac{1}{\dfrac{Nk_BT}{V} - \left(\frac{1}{5}\right)\left(\frac{4\pi}{3}\right)^{1/3}\left(\frac{4}{3}\right)\dfrac{m^2G}{V^{4/3}}}$$

(c) 

Instability at
$$Nk_BT = \left(\frac{1}{5}\right)\left(\frac{4\pi}{3}\right)^{1/3}\left(\frac{4}{3}\right)\frac{m^2G}{V^{1/3}}$$

Problem 6

ψ is of the form $\psi = R(r) \cdot Y(\theta, \phi)$

where $Y(\theta,\phi) = \frac{y}{r} = \sin\theta \sin\phi \; . = \frac{K}{2i} \sin\theta \left[e^{i\phi} - e^{-i\phi} \right]$

$\int Y^* Y \, d\Omega = 1$.

Thus $Y(\theta,\phi) = \frac{1}{\sqrt{2}} \left[Y_{11}(\theta,\phi) + Y_{1,-1}(\theta,\phi) \right]$ (normalized).

(a) L^2: only $l(l+1)\hbar^2 = \underline{\underline{2\hbar^2}}$ is possible. $(P = 1)$.

L_y: $L_y Y = \frac{1}{2i} (L_+ - L_-) Y$

$$= \frac{\hbar}{2\sqrt{2}i} \left[\frac{\sqrt{\overset{l(l+1)-m(m+1)}{1(1+1) - (-1)(-1+1)}}}{\sqrt{2}} Y_{10} - \frac{\sqrt{\overset{l(l+1)-m(m-1)}{1(1+1) - 1(1-1)}}}{\sqrt{2}} Y_{10} \right]$$

$= 0$. $\therefore$ only $\underline{\underline{0}}$ is possible $(P=1)$

L_z: $a_{+1} = \frac{1}{\sqrt{2}}$; $a_{-1} = \frac{1}{\sqrt{2}}$; $a_0 = 0$.

The values $\underline{\underline{\pm\hbar}}$ are possible. $\left(P_{+\hbar} = P_{-\hbar} = \frac{1}{2} \right)$.

(b) $\Delta E = -\gamma B \langle \psi | L_y | \psi \rangle = \underline{\underline{0}}$ since $L_y \psi = 0$.

This result also follows from $L_y = \frac{\hbar}{i} \frac{\partial}{\partial \beta}$, where β is the angle which wraps around the y-axis. $L_y \psi \propto \frac{\partial y}{\partial \beta} = \underline{\underline{0}}$.

(c) Three-dimensional isotropic harmonic oscillator with $n_x = 0$, $n_y = 1$, $n_z = 0$. (or $l = 1$, $m_l = \pm 1$).

#7

a) In frame of observer: $u = (1, \underline{0})$

$p = (E, \underline{p})$

$$-p \cdot u = -p_0 u^0 = E \qquad \text{but } -p \cdot u \text{ is invariant.}$$

b) observer's 4-velocity $\quad u = (1, \underline{0})$ ← choose x-axis appropriately

source's 4-velocity $\quad w = (\gamma, \gamma \underline{v}, 0, 0)$ $\qquad \gamma = \frac{1}{\sqrt{1-v^2}}$

photon's 4-momentum $\quad p = (\underbrace{E, -E}_{\text{lightlike}}, 0, 0)$

$$\frac{\nu_{lab}}{\nu_{source}} = \frac{E_{lab}}{E_{source}} = \frac{-p \cdot u}{-p \cdot w} = \frac{E}{\gamma E + \gamma v E} = \frac{1}{\gamma(1+v)}$$

$$= \sqrt{\frac{1-v}{1+v}}$$

alternatively:

$$k \cdot x - \omega t = k' \cdot x' - \omega' t' \qquad \frac{\omega}{|k|} = \frac{\omega'}{|k'|} = 1$$

$$x' = \gamma (x - vt)$$
$$t' = \gamma (-vx + t)$$

along line of sight

$$\omega(x-t) = \omega' \gamma (x - vt + vx - t) = \omega' \gamma (x-t)(1+v)$$

$$\frac{\omega'}{\omega} = \frac{1}{\gamma(1+v)} = \sqrt{\frac{1-v}{1+v}}$$

Problem 8

(a) Maxwell eqn. $\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}\vec{D}$

free space: $\vec{H} = \vec{B}/\mu_0$, $\vec{D} = \epsilon_0\vec{E}$, $\epsilon_0\mu_0 = 1$, $\vec{J} = 0$ (Gaussian Units)

$\Rightarrow \vec{\nabla}\times\vec{B} = \frac{1}{c}\partial\vec{E}/\partial t$

$$\partial\vec{E}/\partial t = c\vec{\nabla}\times\vec{B} = k^2 c\,\vec{\nabla}\times\left[\frac{\vec{r}\times\vec{p}}{r^2}\,\mathrm{Re}\, e^{i(kr-\omega t)}\right]$$

$$= k^2 c\left\{\frac{\mathrm{Re}\, e^{i(kr-\omega t)}}{r^2}\vec{\nabla}\times(\vec{r}\times\vec{p}) + \left(\vec{\nabla}\frac{\mathrm{Re}\, e^{i(kr-\omega t)}}{r^2}\right)\times(\vec{r}\times\vec{p})\right.$$

$$= \frac{k^2 c}{r^2}\left(\vec{\nabla}\,\mathrm{Re}\, e^{i(kr-\omega t)}\right)\times(\vec{r}\times\vec{p}) + \text{terms that fall off faster for large } r$$

$$= \frac{k^2 c}{r^2}\left(\frac{k\hat{r}}{\omega}\left(-\frac{\partial}{\partial t}\right)\mathrm{Re}\, e^{i(kr-\omega t)}\right)\times(\vec{r}\times\vec{p})$$

$$= -\frac{\partial}{\partial t}\left\{\frac{k^3 c}{r^2\omega}\,\mathrm{Re}\, e^{i(kr-\omega t)}\right\}\hat{r}\times(\vec{r}\times\vec{p})$$

$$= -\frac{\partial}{\partial t}\frac{k^2 c}{r}\,\mathrm{Re}\, e^{i(kr-\omega t)}\underbrace{\hat{r}\times(\hat{r}\times\vec{p})}_{=\hat{r}(\hat{r}\cdot\vec{p})-\vec{p}(\hat{r}\cdot\hat{r})}$$

$$\boxed{\vec{E} = \frac{k^2 c}{r}\left(\vec{p}-\hat{r}(\vec{p}\cdot\hat{r})\right)\mathrm{Re}\, e^{i(kr-\omega t)} + \text{const.}}$$ Gaussian Units

(b) Poynting vector $\vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{H}) = \frac{k^4c^2}{4\pi r^2}\left(\mathrm{Re}\, e^{i(kr-\omega t)}\right)^2(\vec{p}-\hat{r}(\vec{p}\cdot\hat{r}))\times(\hat{r}\times\vec{p})$

$$= \frac{k^4c^2\hat{r}}{4\pi r^2}\left(\vec{p}^{\,2}-(\hat{r}\cdot\vec{p})^2\right)\underbrace{\cos^2(kr-\omega t)}_{\text{time average } = 1/2}$$

$$\boxed{\vec{S}_{av} = \hat{r}\frac{k^4c^2}{8\pi r^2}\vec{p}^{\,2}\sin^2\vartheta_{\hat{r}\hat{p}}}$$

(c) total power $= \oint_{\text{sphere}} d\vec{s}\cdot\vec{S}_{av} = \frac{c^2k^4}{8\pi}\underbrace{\frac{1}{r^2}\int_0^{2\pi} r^2 d\varphi}_{=2\pi}\underbrace{\int_{-1}^{1} d\cos\vartheta_{rp}\,\sin^2\vartheta_{rp}}_{=\int_{-1}^{1}du\,(1-u^2) = 2-\frac{2}{3}}$

$$= \boxed{\frac{c^2k^4\vec{p}^{\,2}}{3}}$$