PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #66

September 29 and 30, 1992

Comprehensive examination for fall 1992

PART I

General Instructions

This Comprehensive Examination for Fall 1992 (#66) consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Tuesday, September 29 and lasts three hours. The second part (Problems 3-4) will be handed out at 1:30 pm on the same day and will also last three hours. The third and fourth parts will be administered on Wednesday, September 30.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit – especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your assigned student letter (but not your name) is on every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

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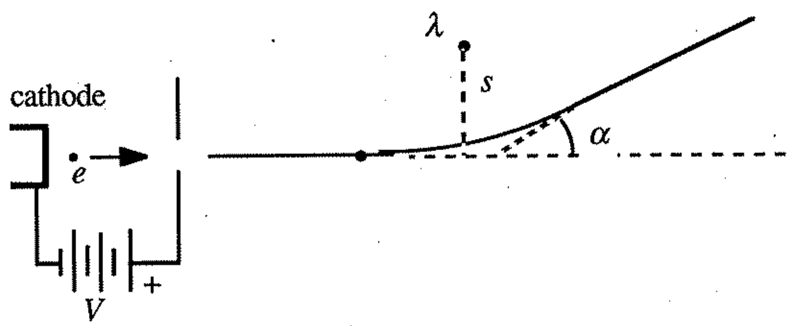
Physics Department Comprehensive Exam # 66, September 29-30, 1992

PART I

Problem #1

Electrons emitted from a hot cathode are accelerated through a potential difference $V = 100$ volts. Travelling in a large evacuated chamber, the electron beam then passes a distance $s = 10$ cm from a long straight wire bearing a uniform charge density $\lambda = 3 \times 10^{-10}$ coulomb/meter, oriented perpendicular to the beam. Calculate the angle α through which the beam is deflected. Assume $\alpha \ll 1$.

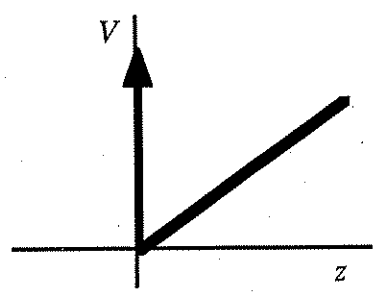


Problem #2

A quantum mechanical particle of mass m moves in the following potential $V(z)$:

$$V = mgz \quad \text{for } z > 0$$

$$V = +\infty \quad \text{for } z < 0$$



(a) Suggest a trial wave function suitable for estimating the ground state energy. Motivate your choice.

(b) Use your function to estimate the ground state energy.

END OF PART I

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PART II

General Instructions

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PART II

Problem #3

Under small applied tension f the elongation of a metal wire L is proportional to the tension according to Hooke's law

$$f = \kappa L \ .$$

As the tension increases, a critical value f_c is reached where an ideal wire becomes plastic and undergoes a sudden jump in its length, the wire behaving more like bubble gum than metal. (At a tension somewhat beyond the critical tension the wire reaches a yield point and breaks.)

This process can be modeled by considering the wire to exist in two phases:

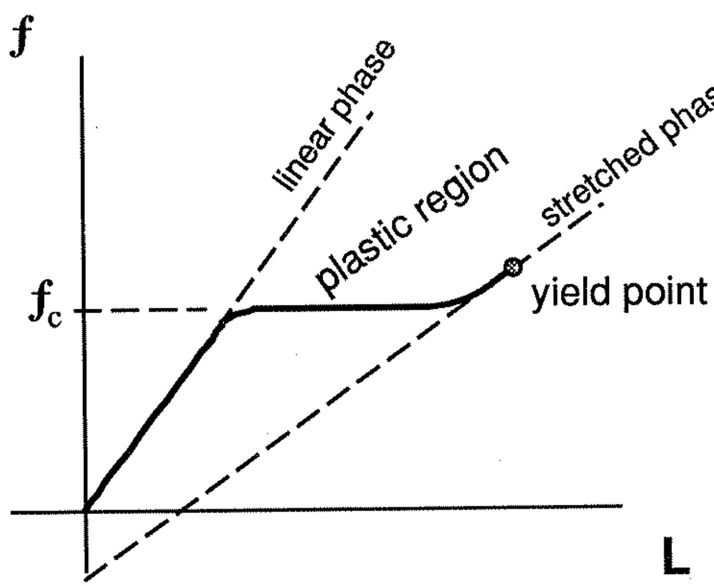
(1) a *linear phase* described by the Helmholtz free energy

$$A_1(L,T) = \frac{1}{2}\kappa L^2$$

(2) a *stretched phase* described by the Helmholtz free energy

$$A_s(L,T) = \frac{1}{2}\eta\,[L-L_0]^2 + \nu$$

Here L_0 is a characteristic length, T is temperature and κ, η and ν are temperature dependent constants, with $\kappa > \eta$.



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Problem #3 (continued)

(a) Determine the wire's equation of state $f = f(L, T)$ in
   (i) the linear phase
   (ii) the stretched phase.

(b) Find the Gibbs free energy $G(f, T) = A - fL$ in
   (i) the linear phase
   (ii) the stretched phase.

(c) Sketch the Gibbs free energy of both phases on a plot of $G(f, T)$ vs. f.

(d) Find the critical tension f_c at which the wire goes plastic.

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Problem #4

A diatomic molecule is made of two different neutral atoms, each with spin $s_1 = s_2 = \frac{1}{2}$. When present in the molecule, the atoms have magnetic moments $\boldsymbol{\mu}_1 = g_1 \mu_B s_1/\hbar$ and $\boldsymbol{\mu}_2 = g_2 \mu_B s_2/\hbar$ respectively, where $\mu_B \equiv e\hbar/2m_e c$. The magnetic moments interact with each other through an interaction $H_0 = A\, \boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2$.

(a) In the absence of orbital angular momentum, what energies are possible? What is the degeneracy of each energy level?

(b) The molecule is placed in a weak, uniform magnetic field B_0 directed along the z - axis. In the absence of orbital angular momentum, show that the relevant Landé g-factor is

$$g = (g_1 + g_2)/2.$$

(c) Considering only spin-related degrees of freedom, find the possible energies and their degeneracies in the presence of the magnetic field of part (b). What is a CSCO (complete set of commuting observables) for this case?

(d) Now consider, in addition to the above, that the molecule also can rotate rigidly with a Hamiltonian

$$H_{rot} = \boldsymbol{L}^2/2I$$

where $\boldsymbol{L}$ is the rotational angular momentum and I is the moment of inertia. Now find a CSCO, and give the energies and degeneracies.

(e) Define $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{s}_1 + \boldsymbol{s}_2$ where $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are the spins of the atoms. Which of the following quantities are conserved, in the presence of all the above interactions?

$$L^2,\ L_x,\ L_z,\ s_1^2,\ s_{1x},\ s_{2z},\ J^2,\ J_x,\ J_z$$

(f) How would your answers to (a) and (b) change if the atoms were not different, but identical?

END OF PART II

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PART III

General Instructions

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PART III

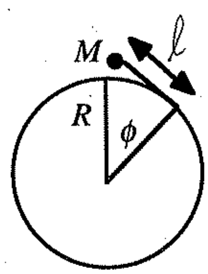
Problem #5

An infinitely thin, massless string, with a point mass M attached to its end, is completely wound aroung a fixed cylinder of radius R so that the mass initially touches the cylinder. At time $t = 0$, a radially directed impulse gives the mass an initial velocity v_0 and the string starts to unwind. No gravity is present.

(a) Find the equation of motion.

(b) Solve the equation of motion subject to the given initial conditions.

(c) Find the angular momentum of the mass about the axis of the cylinder, as a function of time.



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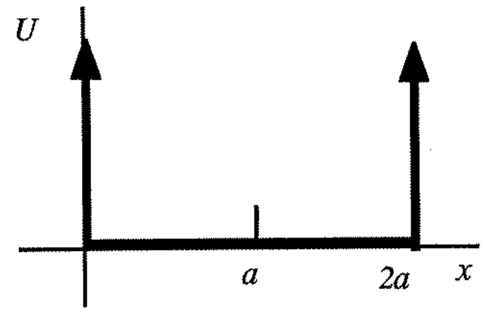
Problem #6

A spinless particle of mass m moves non-relativistically in 1 dimension in a box potential $U(x)$ of width $2a$:

$$U(x) = 0 \quad \text{for } 0 \leq x \leq 2a$$

$$U(x) = +\infty \qquad \text{otherwise}$$



(a) What values can the energy E have?

What is the wave function corresponding to each value of E ?

(b) The particle's energy E is measured, and afterwards its position x is measured. What is the probability of finding $x < a$? Discuss the dependence of your answer on E.

(c) A small perturbing potential δU is added,

$$\delta U(x) = \alpha\,(x - a)$$

Find the change in the energy, to first order in δU ,
(i) in the ground state
(ii) in the first excited state

(d) Find the change in the ground-state wave function, to first order in δU .

(e) How does δU change the probability of finding $x < a$ in the ground state? Explain your result.

END OF PART III

PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #66

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Comprehensive examination for fall 1992

PART IV

General Instructions

This Comprehensive Examination for Fall 1992 (#66) consists of eight problems of equal weight (20 points each). It has four parts. The first three parts (Problems 1-6) were administered in three three-hour periods on Tuesday, September 29 and on Wednesday morning, September 30. The last part (Problems 7-8) is handed out at 1:30 pm on Wednesday, September 30 and will also last three hours.

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PART IV

Problem #7

In the 1950's Fermi suggested that the number of particles produced in a high energy nucleon-nucleon collision could be estimated by a statistical mechanical model in which a key role is played by an effective temperature T_c for the collision.

Fermi's argument, somewhat simplified, can be applied to the case of pure pion (π^+, π^0, π^-) production, by assuming that

(i) the total collision energy E is equilibrated among a large number of pions
(ii) pions are produced in an interaction volume V
(iii) only extremely relativistic pions $\omega_\pi = ck$ are produced
(iv) angular momentum considerations can be ignored.

(a) Find the effective temperature for the collision, as a function of E and V.

(b) Find the number of particles (pions) produced in the collision.

Hint: You may need the integral $\int_0^\infty dz \, \frac{z^n}{e^z - 1} = n! \, \zeta(n+1)$
where $\zeta(n)$ is the Riemann Zeta-function.

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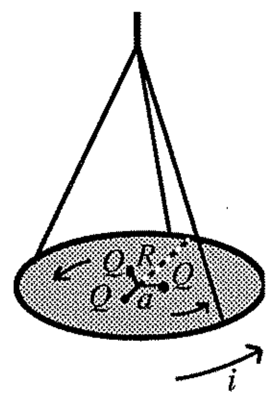
Problem #8

A circular loop of radius R is made from superconducting wire and suspended in a horizontal plane by means of thin non-conducting threads. Three point charges Q are placed symmetrically in the plane of the loop, at a distance a from its center, with $a \ll R$. The charges are attached to the loop and can rotate with it. Initially the system is at rest, with a steady current i flowing in the loop. The loop then warms up until its superconductivity ceases and the current drops to zero.

(a) Calculate the final angular velocity of the system, if its moment of inertia is I about the axis of rotation..

(b) Give a qualitative argument to resolve the apparent nonconservation of angular momentum in this problem.

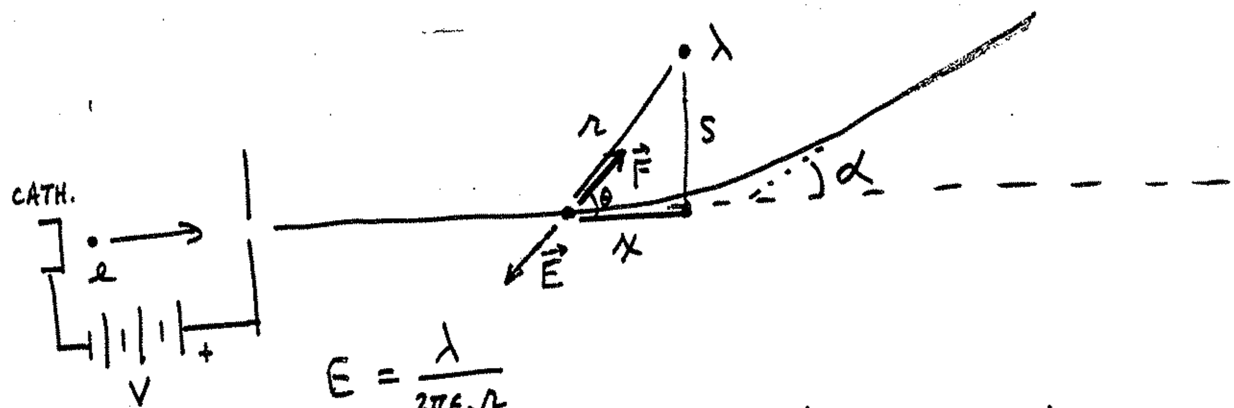


END OF EXAMINATION

Physics Department Comprehensive Exam # 66, September, 1992

SOLUTIONS

Problem #1



$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

$$F_\perp = q E_\perp = q E \sin\theta = e E \frac{s}{r} = \frac{\lambda e s}{2\pi\epsilon_0} \cdot \frac{1}{x^2+s^2}$$

$$F_\perp = m \frac{dv_\perp}{dt}$$

$$v_\perp = \frac{1}{m}\int F_\perp dt = \frac{1}{mv}\int F_\perp dx = \frac{\lambda e s}{2\pi\epsilon_0 m v}\int_{-\infty}^{\infty}\frac{dx}{x^2+s^2}$$

$$= \frac{\lambda e s}{2\pi\epsilon_0 m v}\cdot\frac{\pi}{s} = \frac{\lambda e}{2\epsilon_0 m v}$$

INDEPENDENT OF S.

$$\frac{1}{2}mv^2 = e\cdot V$$

$$\alpha = \frac{v_\perp}{v} = \frac{\lambda e}{2\epsilon_0 \cdot m v^2} = \frac{\lambda e}{4\epsilon_0 e V} = \frac{\lambda}{4\epsilon_0 V}$$

$$\alpha = \frac{3\times10^{-10}\ C/m}{4\times9\times10^{-12}\ C^2/N\cdot m^2 \times 100\ J/C} = \frac{1}{12}\ \text{radian} \cong 5^\circ$$

Physics Department Comprehensive Exam # 66, September, 1992

SOLUTIONS

Problem #2

(a) The trial wave function $\phi(z)$ should be continuous, normalizable and should meet the boundary conditions

$\phi = 0$ for $z \le 0$, $\phi \to 0$ for $z \to \infty$

We also expect that the ground state wave function will have no other zeroes, since extra nodes usually mean increased energy.

It should also have at least one parameter a that can be varied to minimize the energy.

examples: $C\, z \exp(-az)$, $C\, z \exp(-az^2)$, $Cz / (a^2 + z^2)$

(b) variational principle: consider trial wave function as function of a parameter a, minimize expectation value of energy: $\dfrac{\partial}{\partial \alpha} \dfrac{< \phi \mid H \mid \phi >}{< \phi \mid \phi >} = 0.$

example: choose $\phi(z) = z \exp(-az)$,

$$\int_0^\infty dz\ \exp(-2az) = -\frac{1}{2a}\, e^{-2az} \Big|_0^\infty = \frac{1}{2a}$$

$$\int_0^\infty dz\, z \exp(-2az) = -\frac{1}{2}\frac{d}{da}\int_0^\infty dz\ \exp(-2az) = \frac{1}{4a^2}$$

$$< \phi \mid \phi > = \int_0^\infty dz\, z^2 \exp(-2az) = \frac{1}{4}\frac{d^2}{da^2}\int_0^\infty dz\ \exp(-2az) = \frac{1}{4a^3}$$

$$\int_0^\infty dz\, z^3 \exp(-2az) = -\frac{1}{8}\frac{d^3}{da^3}\int_0^\infty dz\ \exp(-2az) = \frac{3}{8a^4}$$

$$\frac{d}{dz}\left(z\, e^{-az}\right) = e^{-az} - az\, e^{-az}, \qquad \frac{d^2}{dz^2}\left(z\, e^{-az}\right) = -2a\, e^{-az} + a^2 z\, e^{-az}$$

$$< \phi \mid H \mid \phi > = -\frac{\hbar^2}{2m}\int_0^\infty dz\, z \exp(-az)\frac{d^2}{dz^2}\left(z\, e^{-az}\right) + mg\int_0^\infty dz\, z^3 \exp(-2az)$$

$$= -\frac{\hbar^2}{2m}\left(-\frac{1}{4a}\right) + mg\left(\frac{3}{8a^4}\right)$$

$$< E > = < \phi \mid H \mid \phi > / < \phi \mid \phi > = \frac{\hbar^2 a^2}{2m} + \frac{3mg}{2a}$$

$$0 = \frac{d<E>}{da} = \frac{\hbar^2 a}{m} - \frac{3mg}{2a^2} \Rightarrow a^3 = \frac{3m^2 g}{2\hbar^2}$$

$$<E>_{\min} = \frac{\hbar^2}{2m}\left(\frac{9g^2m^4}{4\hbar^4}\right)^{1/3} + \frac{3mg}{2}\left(\frac{2\hbar^2}{3m^2 g}\right)^{1/3} = (\hbar^2 g^2 m)^{1/3}\left[\left(\frac{9}{2^5}\right)^{1/3} + \left(\frac{9}{2^2}\right)^{1/3}\right]$$

$$= \left(\frac{9\hbar^2 g^2 m}{4}\right)^{1/3}\left(\frac{9}{8}\right)^{1/3}$$

Physics Department Comprehensive Exam # 66, September, 1992

SOLUTIONS

Problem #3

(a) **The infinitesimal work done *by* a one dimensional elastic system is**

$$đW = -f dL$$

The first law of thermodynamics for a *quasistatic process* is then

$$TdS = dU - f dL$$

where U is the internal energy and S is the entropy. The Helmholtz free energy is

$$A = U - TS$$

so that

$$dA = -SdT + f dL$$

The equation of state for the wire in either phase is found from

$$f = \left(\frac{\partial A}{\partial L}\right)_T.$$

giving in the linear phase

$$f = \kappa L$$

and in the stretched phase

$$f = \eta[L - L_0]$$

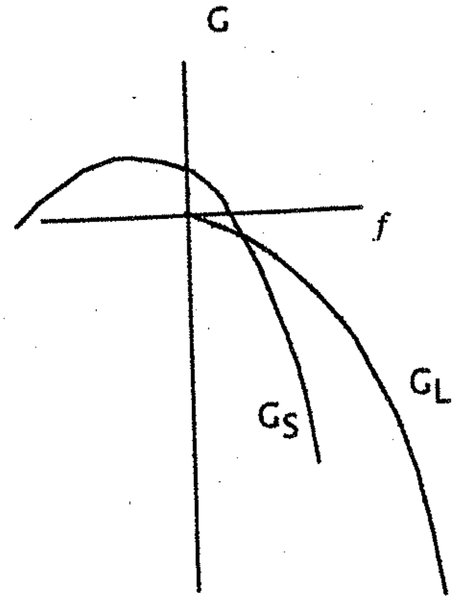
(b) The Gibbs free energy $G(f,T)$ is, in the linear phase

$$G_\ell = \frac{1}{2}\kappa L^2 - fL = -\frac{1}{2}\frac{f^2}{\kappa}$$

and in the stretched phase

$$G_s = \frac{1}{2}\eta[L - L_0]^2 - fL = -\frac{1}{2}\frac{f^2}{\eta} + \nu - fL_0$$

where the equation of state has been used. Plotting roughly:



so that for $f > f_c$ the stretched phase is the lower free energy state while for $f < f_c$ the the linear phase is lower free energy state. At the intersection, $G_s = G_\ell$ and is the point where plastic behavior sets in (a transition takes place between the phases).

(d) At phase equilibrium $G_s = G_\ell$, the system goes plastic, so the critical tension f_c satisfies

$$-\frac{1}{2}\frac{f_c^2}{\kappa} = -\frac{1}{2}\frac{f_c^2}{\eta} + \nu - f_c L_0$$

Defining for convenience

$$\frac{1}{\delta} = \left(\frac{1}{\eta} - \frac{1}{\kappa}\right)$$

we have, upon solving the quadratic equation (and taking the positive root)

$$f_c = L_0\delta\left(\sqrt{1 + \frac{2\nu}{L_0^2\delta}} - 1\right)$$

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SOLUTIONS

Problem #4

(a) $H_0 = A\, g_1\, g_2\, \mu_B{}^2\, \mathbf{s}_1{\bullet}\mathbf{s}_2/\hbar^2$ diagonal in basis of S^2, S_z where $S = s_1 + s_2$

$\mathbf{s}_1{\bullet}\mathbf{s}_2 = \frac{1}{2}(S^2 - s_1{}^2 - s_2{}^2)$, $s_1{}^2 = s_2{}^2 = \frac{1}{2}(\frac{1}{2}+1)\hbar^2$, $S^2 = S(S+1)\hbar^2$ where $S = 0$ or 1

degeneracy = $2S+1 \rightarrow$ levels $S = 1$, $E = \frac{1}{4} A\, g_1\, g_2\, \mu_B{}^2$, deg=3;

S=0, $E = -\frac{3}{4} A\, g_1\, g_2\, \mu_B{}^2$, deg= 1

(b) interaction $\delta H = \boldsymbol{\mu}{\bullet}\boldsymbol{B} = \mu_z B_0$. $\boldsymbol{\mu} = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2$ is a vector operator, so use Wigner-Eckhart theorem $\Rightarrow$ matrix elements of $\boldsymbol{\mu}$ are proportional to those of the total angular momentum $\boldsymbol{J}$. Find constant of proportionality to get Landé g-factor, $\boldsymbol{\mu} = g\, \mu_B \boldsymbol{S}$; here $\boldsymbol{J} = \boldsymbol{S}$. need only case $S = 1$ because the matrix element is necessarily 0 for $S = 0$

easy method: compare matrix element $<SM \mid \mu_z \mid SM> = g\, \mu_B <SM \mid S_z \mid SM>$ for S=M=1, note $|1 1> = |\frac{1}{2}\, \frac{1}{2}> \times\, |\frac{1}{2}\, \frac{1}{2}> \rightarrow <SM \mid \mu_z \mid SM> = g_1 \mu_B(\frac{1}{2}\hbar) + g_2 \mu_B(\frac{1}{2}\hbar) = g\mu_B(\hbar)$

harder method: consider arbitrary matrix element of $\boldsymbol{\mu}{\bullet}\boldsymbol{S}$ (see Cohen-Tannoudji *et al*)

(c) possible CSCO's include (s_{1z}, s_{2z}), or (S^2, S_z); $s_1{}^2$ and $s_2{}^2$ are optional.

get energies from perturbation theory, use eigenstates of H_0 to get first-order energies:
energies $E(S, M)$ are $E(1,1) = \frac{1}{4} A\, g_1\, g_2\, \mu_B{}^2 + g\ \mu_B \hbar$;$E(1,-1) = \frac{1}{4} A\, g_1\, g_2\, \mu_B{}^2 - g\ \mu_B \hbar$;

$E(1,0) = \frac{1}{4} A\, g_1\, g_2\, \mu_B{}^2$, $E(0,0) = -\frac{3}{4} A\, g_1\, g_2\, \mu_B{}^2$, all non-degenerate

(d) H_{rot} has eigenvalues $L(L+1)\hbar^2/2I$ where L = 0, 1, 2, 3...

Energies are $E(L, S, M) = L(L+1)\hbar^2/2I + E(S, M)$, degeneracy $= 2L+1$

CSCO is, e.g., (L^2, L_z, S^2, S_z); note energy independent of $L_z \rightarrow 2L+1$ degeneracy

(e) conserved: $L^2, L_x, L_z, , s_1{}^2, J_z$; note L^2, J^2, L_x, J_x are not conserved because of external $\boldsymbol{B}$ -field; note components of s_1 and s_2 are not conserved because of H_0 .

(f) Since the atoms have half-integer spins they are fermions; if they are identical, the wave functions must be antisymmetric. Only the $S = 0$ state is allowed.

note: if L were included as in part (d), then even L have symmetric spatial wave functions, odd L antisymmetric spatial w.f.'s, so could have even L with S=0, odd L with S=1.

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SOLUTIONS

Problem #5

(a) Use Lagrangian $\mathcal{L} = \frac{1}{2} M\, v^2 = \frac{1}{2} M\, l^2\, \dot{\phi}^2 = \frac{m^2 l^2 \dot{l}^2}{2R^2}$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial l/\partial t}\right) - \frac{\partial \mathcal{L}}{\partial l} = 0 \Rightarrow \frac{m}{R^2}\left(\frac{d}{dt}(l^2\, \dot{l}) - l\, \dot{l}^2\right) = 0 \Rightarrow 2l\, \dot{l}^2 + l^2 \ddot{l} - \dot{l}^2\,)$$

$$\Rightarrow 0 = \dot{l}^2 + l\, \ddot{l} = \frac{d}{dt}(l\, \dot{l}) = 0$$

(b) $\frac{d}{dt}(l\, \dot{l}) = 0 \Rightarrow l\, \dot{l} = A = \text{constant} \Rightarrow l\, dl = A\, dt \Rightarrow \frac{1}{2} l^2 = At + B$

$l = 0$ at $t = 0 \Rightarrow B = 0$

$v = v_0$ at $t = 0 \Rightarrow A = R v_0$

(c) $J = m v\, l = \frac{m\, \dot{l}\, l^2}{R} = m\,(2R\, v_0{}^3 t\,)^{1/2}$

$v = \frac{l \dot{l}}{R}$

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SOLUTIONS

Problem #6

(a) wave functions are free-particle→linear combinations of sines and cosines
$\psi(x{=}0) = \psi(x{=}2a) = 0 \rightarrow \psi_n(x) = C \sin(n\pi x/2a)$ where $n = 1,2,...$
normalization $\int |\psi|^2\, dx = 1 \rightarrow C = a^{-1/2}$. $E = \hbar^2k^2/2m = n^2\,(\hbar^2\pi^2/8ma^2)$

(b) Potential is symmetric about $x = a$ → wave functions have even or odd parity with respect to reflection about $x = a$, actual parity is $(-)^n$. In either case $|\psi|^2$ is even → equal probabilities for $x<a$ and $x>a$ → probability = 1/2 independent of E.

(c) To first order in δU, the change in the energy is just the mean value of δU in each state. In each state, the probability distribution is symmetric about $x{=}a$, but δU is antisymmetric, so there is no change in the energy of any state, to first order in perturbation theory.

(d) $\delta\psi_1(x) = \sum c_{n1}\psi_n(x)$ where, to first order in δU, for $n>1$,

$$c_{n1} = \int dx\ \psi_n(x)^*\ \delta U(x)\ \psi_1(x)/(E_1-E_n) = \alpha a^{-1}\int_0^{2a} dx\ \sin(\pi x/2a)\ \sin(n\pi x/2a)\ (x-a)/(E_1-E_n)$$

for n odd, the integral vanishes; for n even, it is (with $y \equiv \pi x/2a$)

$$4\pi^{-2}a^2\int_0^{\pi} dy\ (y-\tfrac{\pi}{2})\ \sin y\ \sin ny\ = 4\pi^{-2}a^2\int_0^{\pi} dy\ y\ \sin y\ \sin ny\ y$$

evaluate integral using partial integration or trick: $-\partial/\partial n \int dy\, \sin y\ \cos ny = \int dy\ \ y \sin y \sin ny$

result $c_{n1} = \dfrac{32\alpha m a^3}{\pi^2\hbar^2}\,\dfrac{n}{(n^2-1)^2}$, $\delta\psi_1(x) = \dfrac{32\alpha m a^3}{\pi^2\hbar^2}\displaystyle\sum_{n\ \text{even}} \frac{n}{(n^2-1)^2}\psi_n(x)$

(e) Probability $= \int_0^a dx\ \ (\psi_1+\delta\psi_1) \approx 1 + 2\int_0^a dx\ \ \psi_1(x)\ \delta\psi_1(x)$

$$= 1 + \frac{64\alpha m a^3}{\pi^2\hbar^2}\sum_{n\ \text{even}} \frac{n}{(n^2-1)^2}\ \int_0^a dx\ \ \psi_1(x)\ \psi_n(x)$$

largest term $n{=}2$ has both ψ's >0 for $x < a \Rightarrow$ probability increased
interpretation: the particle spends more time where the potential is most attractive.

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SOLUTIONS

Problem #7

Mean Energy E determines temperature T

$$E = 3 \sum_k \hbar\omega_\pi(k) \frac{1}{e^{\beta\hbar\omega_\pi(k)} - 1}$$

↗ pion degeneracy

pions are Bosons without number conservation i.e. $\mu_\pi = 0$

(a) → $$E = 3 \frac{V}{(2\pi)^3} (4\pi) \int_0^\infty k^2 dk \, \hbar\omega_\pi(k) \frac{1}{e^{\beta\hbar\omega_\pi(k)} - 1}$$

$$= 3 \frac{V}{(2\pi)^3} (4\pi) \left(\frac{1}{c}\right)^3 \hbar \int_0^\infty \omega^3 \left(\frac{1}{e^{\beta\hbar\omega} - 1}\right) d\omega$$

$$= 3 \frac{V}{(2\pi)^3} (4\pi) \left(\frac{1}{c}\right)^3 \hbar \frac{1}{(\beta\hbar)^4} \underbrace{\int_0^\infty dz \, z^3 \left(\frac{1}{e^z - 1}\right)}_{3!\,\zeta(4)}$$

Thus $$\frac{1}{\beta^4} = (k_B T)^4 = \left(\frac{E}{V}\right) \frac{1}{3} \frac{(\hbar c)^3}{3!\,\zeta(4)} (2\pi^2)$$

(b) $$N_\pi = 3 \sum_k \frac{1}{e^{\beta\hbar\omega_\pi(k)} - 1}$$

$$= 3 \cdot 4\pi \frac{V}{(2\pi)^3} \int_0^\infty k^2 dk \frac{1}{e^{\beta\hbar\omega_\pi(k)} - 1}$$

$$= 3 \cdot 4\pi \frac{V}{(2\pi)^3} \frac{1}{c^3} \int_0^\infty \omega^2 \frac{1}{e^{\beta\hbar\omega} - 1} d\omega$$

$$= 3 \cdot 4\pi \frac{V}{(2\pi)^3} \frac{1}{c^3} \left(\frac{1}{\hbar\beta}\right)^3 2!\, \zeta(3)$$

$$= 3 \cdot \frac{V}{2\pi^2} \left(\frac{1}{\hbar c}\right)^3 \frac{2!\, \zeta(3)}{\beta^3}$$

$$= 3 \cdot \frac{V}{2\pi^2} \left(\frac{1}{\hbar c}\right)^3 2!\, \zeta(3) \times \left[\left(\frac{E}{V}\right)\left(\frac{1}{3}\right) \frac{(\hbar c)^3\, 2\pi^2}{3!\, \zeta(4)}\right]^{-3/4}$$

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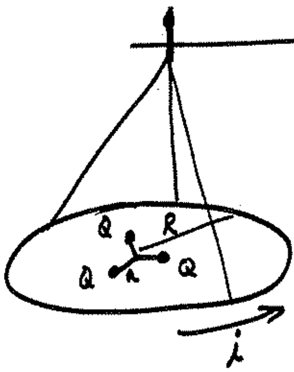
SOLUTIONS

Problem #8



a) At center of loop, $B = \frac{\mu_0}{4\pi}\frac{i \cdot 2\pi R}{R^2} = \frac{\mu_0 i}{2R}$

$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt} \Longrightarrow E \cdot 2\pi a = \frac{\mu_0}{2R}\frac{di}{dt}\cdot \pi a^2$

$\hookrightarrow E = \frac{\mu_0 a}{4R}\frac{di}{dt}$

Torque: $\tau = 3QE \cdot a$

$= \frac{3\mu_0 Q a^2}{4R}\frac{di}{dt}$

$\tau = I\frac{d\omega}{dt} \Longrightarrow \omega = \frac{1}{I}\int \tau\, dt = \frac{3\mu_0 Q a^2 i}{4IR}$ Counterclockwise, viewed from above.

b) $\vec{L} = \vec{r} \times (3Q)\vec{A}$ angular momentum associated with field. Direction: upward.

$L = 3QaA = 3Qa\cdot\left(\frac{1}{2}Ba\right) = \frac{3}{2}Qa^2 \cdot \frac{\mu_0 i}{2R} = \frac{3\mu_0 Q a^2 i}{4R}$

$\vec{L}_{initial} = \vec{L}_{field}$; $\vec{L}_{final} = \vec{L}_{mechanical}$; $\vec{L}_{final} = \vec{L}_{initial}$

and angular momentum is conserved.