Physics Department Comprehensive Examination #65.

March 31 and April 1, 1992.

Comprehensive examination for Spring 1992.

Part I.

General Instructions.

This Comprehensive Examination for Spring 1992 (#65) consists of eight problems of equal weight (20 points each). It has two parts. The first part (Problems 1-4) is handed out at 1:00 pm on Tuesday, March 31, and that part of the exam lasts three hours. The second part (Problems 5-8) will be handed out at 1:00 pm on Wednesday, April 1, and that part also lasts three hours. Be sure to make a note of your assigned student letter for use in the second part of the examination.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit - especially if your understanding is manifest. Use no scratch paper, do all work in the bluebooks. Use one bluebook per problem, and be certain that your assigned student letter (but NOT your name) is on every booklet.

If something is omitted from the statement of the problem or if you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. During the exam, put all materials, books, and papers on the floor, except the exam, bluebooks, and the collection of formulas and data distributed with the exam. Calculators are NOT allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work, separated by at least one empty page from your solutions. This scratch-work will not be graded.

Physics Department Comprehensive Examination #65.

Part 1. Tuesday 31 March 1992, 1-4 pm.

Problem 1.

Consider the time-independent Schrödinger equation in D spatial dimensions for a spinless particle in a central potential.

A. How many quantum numbers are needed to label the quantum states?

The central potential in parts B through F of this problem is an isotropic harmonic oscillator potential in D dimensions: $V(x_1,..,x_D)=\frac{1}{2}\mu\omega^2\sum_i x_i^2$, where μ is the mass of the particle.

B. Show that the energy levels are given by $(n+\frac{1}{2}D)\hbar\omega$ with n=0,1,2,...

C. The degeneracy of the n^{th} level of the D dimensional harmonic oscillator is g(D,n). Calculate g(D=1,n) as a function of n.

D. Calculate g(D,n=0) and g(D,n=1) as functions of D.

E. Derive a formula for g(D,n) in terms of g(D − 1,k) for all k.

F. Calculate g(D=3,n) as a function of n.

Problem 2.

Consider a gas of N molecules of CO at temperature T confined to a volume V. Each molecule has a moment of inertia I.

A. Show that the partition function for the rotational motion is given by

$$\mathfrak{Z}_{rot} = \left\{ \sum_{l=0}^{\infty} (2l+1)\exp\left[-\frac{\Theta_{rot}\, l(l+1)}{T}\right] \right\}^N .$$

What is Θ_{rot}?

B. Show that for high temperatures $\mathfrak{Z}_{rot} \approx \left(\frac{T}{\Theta_{rot}}\right)^N$. How large does T have to be for this expression to be valid? (Give a condition of the form $T \gg$ something).

C. Give a general expression for the rotational contribution to the heat capacity C_V, as a function of $\mathfrak{Z}_{rot}$ and T. Evaluate this expression for large T.

D. What is the rotational contribution to the pressure?

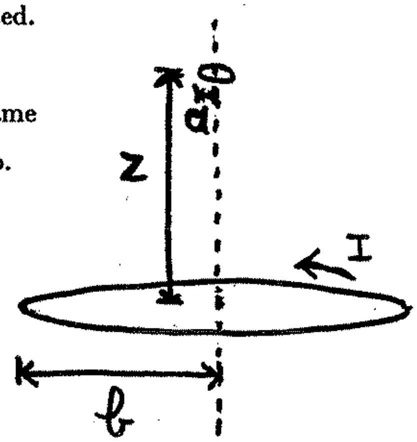
E. How would the answer to part A change if the molecules were $^{16}O_2$, a diatomic molecule of ^{16}O where each ^{16}O atom is a spinless boson?

Problem 3.

A small circular loop of wire has a radius a. The center of this loop lies a distance z above the center of a large circular loop of wire. The large loop has a radius b. The planes of the loops are parallel and are perpendicular to the common axis. We are given that $a \ll b$ and $a \ll z$.

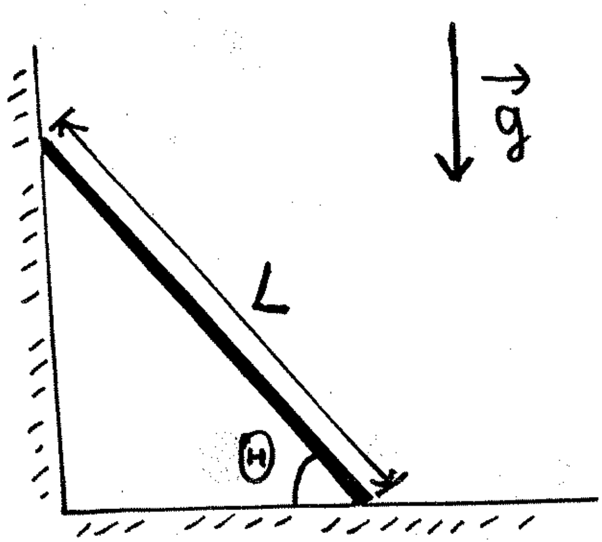
A. Suppose a current I flows in the big loop in the direction indicated. Calculate the flux through the little loop.

B. Suppose, instead, that a current I flows in the small loop in the same direction as the current in part A. Calculate the flux through the big loop.



Problem 4.

A thin uniform rod, of length L and mass m, has one end touching the floor, which is horizontal, and the other end leaning against a vertical wall. Under the influence of gravity the rod slips downward, beginning from rest with an angle $\Theta = \Theta_0$ and remaining in a vertical plane perpendicular to the wall. If there is no friction, find the value of Θ when the upper end of the rod looses contact with the wall.



Physics Department Comprehensive Examination #65.

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Physics Department Comprehensive Examination #65.

Part 2. Wednesday 1 April 1992, 1-4 pm.

Problem 5.

The time independent Schrödinger equation in matrix form is $\sum_{m=1}^{\infty} H_{nm}\, c_m = \lambda\, c_n$ for n=1,2,... The solutions for the coefficients of the wave functions are normalized according to $\sum_{m=1}^{\infty} |\, c_m\, |^2=1$.

A. Show that the matrix equation above is equivalent to $c_n = \sum_{m=1}^{\infty} h_{nm}\, c_m$ with $h_{nm}=H_{nm}\,(\,1\, -\, \delta_{nm}\,)\,/\,(\,\lambda\, -\, H_{nn}\,)$.

In parts B through E of this problem we are only interested in a particular solution of the matrix equation for which we know that $|\, c_1\, |^2 \gg \sum_{m=2}^{\infty} |\, c_m\, |^2$.

B. Use an iterative procedure based on the matrix equation derived in part A and using the fact that c_m is small for m>1 to show how c_n can be found in terms of c_1 to any desired order.

C. Apply your procedure to the case n=1 to obtain an eigenvalue condition for λ.

D. Assume that $H_{nn}=n^2\epsilon$ and $H_{nm}=g$ for $n \neq m$, $|\, g\, | \ll \epsilon$. After multiplying the expression found in part C by $\lambda-\epsilon$, the expression looks like a perturbation expansion. Consider the terms proportional to g^n with $n \leq 2$. How do these terms differ from the corresponding expressions in standard Rayleigh-Schrödinger perturbation theory?

E. The same technique can be applied for a finite basis. Assume that we have only two coefficients, c_1 and c_2. We are interested in a state with $|\, c_1\, | \gg |\, c_2\, |$. The Hamiltonian is the same as in the previous part of the problem: $H_{11}=\epsilon$, $H_{22}=4\epsilon$, and $H_{12}=H_{21}=g$ with $|\, g\, | \ll \epsilon$. Show that the second-order expansion of part B gives the exact answer for the eigenvalue in this case.

Problem 6.

A uniform stretched string with fixed ends at x=0 and $x=L$ has tension τ, density ρ, and propagation velocity $\sqrt{\frac{\tau}{\rho}}$. At time t=0, when the string has no initial displacement, it is set into transverse motion by being struck by a small hammer of width $2s$, centered at $x=\frac{1}{2}L$. This section of the string is given an initial velocity v_0. The effects of gravity are negligible. Describe the subsequent wave motion of the string by developing a Fourier series for the transverse displacement y(x,t). Evaluate the Fourier amplitudes in terms of the given quantities.

Problem 7.

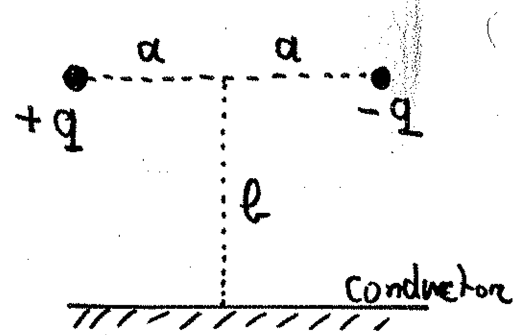
A steam engine works by evaporating water to steam, then extracting the mechanical energy of the steam's pressure to do mechanical work. Consider a steam engine pulling a frictionless train of cars holding water and fuel. The mass of the empty train is M_E, the mass of the water is M_W, and the mass of the fuel is M_F. The energy released when burning a unit mass of fuel is J_E. Assume that the engine uses all the energy released by burning the fuel to evaporate the water and bring the steam to a temperature of 200°C. The heat required for this process per mole of water is H_W. The engine then extracts the energy from the steam using a reversible, adiabatic expansion of the steam until the steam reaches a temperature of 100°C. The adiabatic index γ of water is $\frac{4}{3}$.

A. Calculate the fraction f of the fuel's energy which is available to do mechanical work.

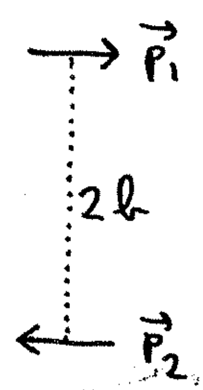
B. The train moves 10 km up a hill with a 1% grade. The amount of fuel used during this trip is m_F. Calculate m_F, assuming that $m_F \ll M_F$.

C. The amount of water used during this trip is m_W. Calculate m_W, assuming that $m_W \ll M_W$.

Problem 8.



A. Two electrical charges $+q$ and $-q$ lie a distance $2a$ apart and a distance b above an infinite plane conductor as shown on the right. Calculate the total energy stored in this configuration.



B. Calculate the interaction energy of two electric dipoles which are a distance $2b$ apart and whose dipole moments $\vec{p}_1$ and $\vec{p}_2$ are anti-parallel, equal in magnitude, and perpendicular to the line joining them, as shown on the right.

C. Compare the result of part B with the appropriate limit of the result in part A. Explain why there is a difference of a factor of two.

FORMULAS AND DATA FOR COMPREHENSIVE EXAMINATION

- Please Return After Exam -

CONSTANTS

$h = 6.63 \cdot 10^{-34}$ *joule*-sec $= 4.14 \cdot 10^{-15}$ *eV*-sec

$c = 3 \cdot 10^8$ *m*/sec

$$E_0 = \frac{m_e e^4}{2\hbar^2} = 13.6\ eV$$

Volume of Ideal Gas (STP) = 22.4 *l/mole*

$m_p c^2 = 938.28\ MeV$

$\hbar c = 197.3\ MeV\ Fermi$

$a_0 = \hbar^2 / m_e e^2 = 0.529$ Å

$k = 1.38 \cdot 10^{-23}$ *joule/K*

$N_A = 6.022 \cdot 10^{23}$ */mole*

$m_e c^2 = 0.511\ MeV$

$m_n c^2 = 939.57\ MeV$

$$\frac{e^2}{\hbar c} = \frac{1}{137}$$

$$\mu_B = \frac{e\hbar}{2m_e c} = 0.579 \times 10^{-14}\ MeV/gauss = 9.27 \cdot 10^{-21}\ erg/gauss$$

$1\ MeV = 1.602 \times 10^{-6}\ erg$

GEOMETRY

$$\vec{\nabla}(\varphi + \psi) = \vec{\nabla}\varphi + \vec{\nabla}\psi$$

$$\vec{\nabla}(\varphi\psi) = \varphi\vec{\nabla}\psi + \psi\vec{\nabla}\varphi$$

$$div(\vec{F} + \vec{G}) = div\ \vec{F} + div\ \vec{G}$$

$$c\vec{url}(\vec{F} + \vec{G}) = c\vec{url}\ \vec{F} + c\vec{url}\ \vec{G}$$

$$\vec{\nabla}(\vec{F}\cdot\vec{G}) = (\vec{F}\cdot\vec{\nabla})\vec{G} + (\vec{G}\cdot\vec{\nabla})\vec{F} + \vec{F}\times c\vec{url}\ \vec{G} + \vec{G}\times c\vec{url}\ \vec{F}$$

$$div\ \varphi\vec{F} = \varphi\ div\ \vec{F} + \vec{F}\cdot\vec{\nabla}\varphi$$

$$div(\vec{F}\times\vec{G}) = \vec{G}\cdot c\vec{url}\ \vec{F} - \vec{F}\cdot c\vec{url}\ \vec{G}$$

$$div\ c\vec{url}\ \vec{F} = 0$$

$$c\vec{url}\ (\varphi\vec{F}) = \varphi\ c\vec{url}\ \vec{F} + \vec{\nabla}\varphi\times\vec{F}$$

$$c\vec{url}(\vec{F}\times\vec{G}) = \vec{F}\ div\ \vec{G} - \vec{G}\ div\ \vec{F} + (\vec{G}\cdot\vec{\nabla})\vec{F} - (\vec{F}\cdot\vec{\nabla})\vec{G}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$c\vec{url}\ c\vec{url}\ \vec{F} = gr\vec{a}d\ div\ \vec{F} - \nabla^2\vec{F}$$

$$c\vec{url}\ \vec{\nabla}\varphi = 0$$

$$\oint_s \vec{F}\cdot\vec{n}\,da = \int_V div\ \vec{F}\,dv$$

$$\oint_c \vec{F}\cdot d\vec{l} = \int_s c\vec{url}\ \vec{F}\cdot\vec{n}\,da$$

$$\oint_s \varphi\vec{n}\,da = \int_V \vec{\nabla}\varphi\,dv$$

$$\oint_s \vec{F}(\vec{G}\cdot\vec{n})\,da = \int_V \vec{F}\ div\ \vec{G}\,dv + \int_V (\vec{G}\cdot\vec{\nabla})\vec{F}\,dv$$

$$\oint_s \vec{n}\times\vec{F}\,da = \int_V c\vec{url}\ \vec{F}\,dv$$

$$\oint_c \varphi\,d\vec{l} = \int_s \vec{n}\times\vec{\nabla}\varphi\,da$$

$$\int_V (\psi\nabla^2\varphi - \varphi\nabla^2\psi)\,dv = \oint_s (\psi\,gr\vec{a}d\,\varphi - \varphi\,gr\vec{a}d\,\psi)\cdot\vec{n}\,da$$

CYLINDRICAL COORDINATES

Coordinates (ρ, ϕ, z) Unit vectors $(\vec{i}_1, \vec{i}_2, \vec{i}_3)$

Gradient $$\vec{\nabla} f = \vec{i}_1 \frac{\partial f}{\partial \rho} + \vec{i}_2 \frac{1}{\rho}\frac{\partial f}{\partial \phi} + \vec{i}_3 \frac{\partial f}{\partial z}$$

Divergence $$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Laplacian $$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \frac{\partial f}{\partial \rho}) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Curl $$\vec{\nabla}\times\vec{A} = \vec{i}_1\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \vec{i}_2\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \vec{i}_3\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right)$$

SPHERICAL COORDINATES

Coordinates (r, θ, ϕ) Unit vectors $(\vec{i}_1, \vec{i}_2, \vec{i}_3)$

Gradient $$\vec{\nabla} f = \vec{i}_1 \frac{\partial f}{\partial r} + \vec{i}_2 \frac{1}{r}\frac{\partial f}{\partial \theta} + \vec{i}_3 \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

Curl $$\vec{\nabla}\times\vec{A} = \vec{i}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi}\right]$$
$$+\vec{i}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]$$
$$+\vec{i}_3\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right]$$

Divergence $$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Laplacian $$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

OTHER MATH

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} (2l+1)\, i^l j_l(kr) P_l(\cos\theta) \qquad \int_{-\infty}^{\infty} e^{ixy}\, dy = 2\pi\,\delta(x)$$

$$j_0(z) = \frac{\sin z}{z}; \quad j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}; \quad n_0(z) = \frac{-\cos z}{z}; \quad n_1(z) = \frac{-\cos z}{z^2} - \frac{\sin z}{z}$$

$$P_0(x) = 1; \quad P_1(x) = x; \quad P_2(x) = \frac{1}{2}(3x^2-1); \quad P_3(x) = \frac{1}{2}(5x^3-3x); \quad P_l^m = (1-x^2)^{\frac{m}{2}} \frac{d^m P_l}{dx^m}$$

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}; \quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta; \quad Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\, e^{\pm i\phi}\sin\theta;$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1); \quad Y_2^{\pm1} = \mp\sqrt{\frac{15}{8\pi}}\, e^{\pm i\theta}\cos\theta\sin\theta; \quad Y_2^{\pm2} = \sqrt{\frac{15}{32\pi}}\, e^{\pm 2i\phi}\sin^2\theta$$

$$\frac{1}{\sqrt{1-2tz+t^2}} = \sum_{l=0} P_l(z)\, t^l$$

$$(1+x)^n = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} x^k$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad |x|<1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n \qquad |x|<1$$

<u>OTHER MATH</u> (continued)

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \qquad n=integer \qquad a>0$$

$$\int \sin^2 x dx = \frac{1}{2}(x - \cos x \sin x)$$

$$\int_0^\infty x^{2n} e^{-a^2x^2} dx = \frac{1,3,5\ldots(2n-1)\sqrt{\pi}}{2^{n+1} a^{2n+1}} \qquad n=integer$$

$$\ln n! \approx \frac{1}{2}\ln 2\pi n + n \ln n - n$$

$$\int_0^{2\pi} \sin nx \sin mx\, dx = \pi\delta_{nm} \quad n,m \geq 1$$

$$\int_0^{2\pi} \cos nx \cos mx\, dx = \pi\delta_{nm} \quad n,m \geq 0$$

$$\sin n\alpha = 2\sin(n-1)\alpha \cdot \cos\alpha - \sin(n-2)\alpha$$

$$\cos n\alpha = 2\cos(n-1)\alpha \cdot \cos a - \cos(n-2)\alpha$$

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta}$$

$$\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta) \cdot \cos\frac{1}{2}(\alpha-\beta)$$

$$\sin\alpha - \sin\beta = 2\cos\frac{1}{2}(\alpha+\beta) \cdot \sin\frac{1}{2}(\alpha-\beta)$$

$$\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta) \cdot \cos\frac{1}{2}(\alpha-\beta)$$

$$\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha+\beta) \cdot \sin\frac{1}{2}(\alpha-\beta)$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

QUANTUM MECHANICS

$$|jm> = \sum_{\substack{m_1\\m_2}} |m_1 m_2><m_1 m_2|jm>$$

$$J^{\pm}|jm> = \hbar\sqrt{(j\mp m)(j\pm m+1)}\ |m\pm 1>$$

$$\psi_{100} = \pi^{-1/2} a_0^{-3/2} \exp(-r/a_0)$$

$$E_n' = E_n + <n|V|n> + \sum_{m\neq n} \frac{|<n|V|m>|^2}{E_n - E_m} + \ldots$$

$$<n|x|n'> = \sqrt{\frac{\hbar}{m\omega}}\sqrt{\frac{n+1}{2}}\,\delta_{n+1,n'} + \sqrt{\frac{\hbar}{mw}}\sqrt{\frac{n}{2}}\,\delta_{n-1,n'}$$

$$<n|p|n'> = -i\sqrt{m\omega\hbar}\sqrt{\frac{n}{2}}\,\delta_{n+1,n'} + i\sqrt{m\hbar\omega}\sqrt{\frac{n}{2}}\,\delta_{n-1,n'}$$

$$E_{rotation} = \frac{J(J+1)\hbar^2}{2I}$$

ELECTRICITY AND MAGNETISM

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{-\vec{m}}{r^3} + \frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5}\right]$$

$$U = \sum_{\rho=1} A_\rho r^{-\rho} e^{i\rho\phi} + \sum_{\rho=0} B_\rho r^{\rho} e^{i\rho\phi} + A_0 \ln\rho$$

$$U = \sum_{\rho=0}\left[A_\rho r^{\rho} P_\rho(\cos\phi) + B_\rho r^{-(\rho+1)} P_\rho(\cos\phi)\right]$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left[\frac{\vec{M}\times(\vec{r}-\vec{r}')}{|r-r'|^3}\right] d^3r'$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\times(\vec{r}'-\vec{r})}{|r'-r|^3}\, d^3r'$$

$$\vec{B} = \frac{\mu_0 I}{2\pi a}\vec{k}$$

Problem 1:

A. We need D quantum numbers.

B. The Schrödinger equation is $\sum_{i=1}^{D}\left\{-\frac{\hbar^2}{2m}\Delta_i+\frac{1}{2}m\,\omega^2 x_i^2\right\}\psi(x_1,\ldots,x_D)$. This separates into D one-dimensional equations with energy eigenvalues $(n_i+\frac{1}{2})\hbar\omega$. Hence the energy eigenvalues of the D dimensional oscillator are $(n+\frac{1}{2}D)\hbar\omega$ with $n=\sum_{i=1}^{D} n_i$.

C. If D=1, $n=n_1$, and there is no degeneracy. g(1,n)=1.

D. If n=0, all n_i have to be zero, and g(D,0)=1. If n=1, one of the n_i has to be one, the others zero, and g(D,1)=D.

E. The degeneracy is the total number of ways D non-negative integers can sum up to n. If n_D=m, the remaining D−1 quantum numbers have to sum to n−m. Therefore $g(D,n)=\sum_{m=0}^{n} g(D-1,n-m)=\sum_{m=0}^{n} g(D-1,m)$.

F. $g(2,n)=\sum_{m=0}^{n} g(1,m)=n+1$. $g(3,n)=\sum_{m=0}^{n}(n+1)=\frac{1}{2}(n+1)(n+2)$.

Problem 2:

SOLUTION

(a) The rotational states have degeneracies $2l+1$ and energies $\hbar^2 l(l+1)/2I$, so $\theta_{rot}=\hbar^2/2Ik$.

(b) If $T \gg \theta_{rot}$, we can approximate the sum by an integral,

$$\left(\sum_{l=0}^{\infty}(2l+1)\exp[-\theta_{rot}l(l+1)/T] \approx \left(\int_0^{\infty} dl\,(2l+1)\exp[-\theta_{rot}l(l+1)/T] = T/\theta_{rot} \text{ using } y\equiv l(l+1)\right.\right.$$

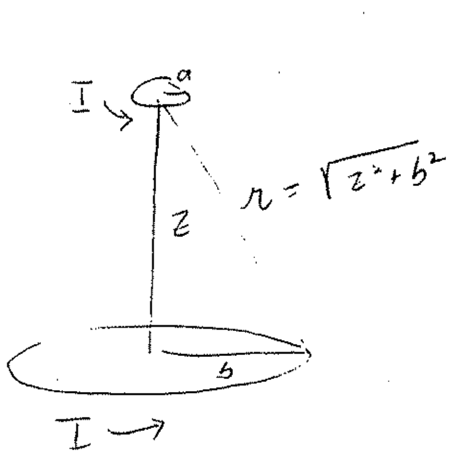
(c) $F=-kT\ln Z,\ S=-\left.\frac{\partial F}{\partial T}\right)_V$

$$C_V=\left.\frac{\partial U}{\partial T}\right)_V=\left.\frac{\partial(F+TS)}{\partial T}\right)_V=\left.\frac{\partial F}{\partial T}\right)_V+S+T\left.\frac{\partial S}{\partial T}\right)_V=T\left.\frac{\partial S}{\partial T}\right)_V=-T\left.\frac{\partial^2 F}{\partial T^2}\right)_V$$

$$=kT\frac{\partial}{\partial T}\left(\ln Z+T\left.\frac{\partial \ln Z}{\partial T}\right)_V\right.=2kT\left.\frac{\partial \ln Z}{\partial T}\right)_V+kT^2\left.\frac{\partial^2 \ln Z}{\partial T^2}\right)_V=2kTN\frac{1}{T}-kT^2N\frac{1}{T^2}=Nk$$

(d) 0, Z is independent of V → F is independent of N → $P_{rot}=-\left.\frac{\partial F}{\partial V}\right)_T=0$

(e) Boson wave functions must not change when the coordinates of the atoms are interchanged, but the parity of the rotational wave function Y_{lm} is $(-)^l$, so only even l are allowed in the sum.

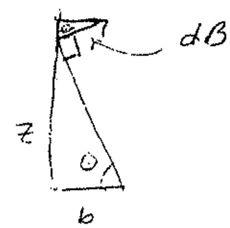


Problem 3

a) Assume a small enough that B is const. inside small loop

Biot-Savart

$$B = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{r}}{r^2}\, dl$$



$$dB = \frac{\mu_0}{4\pi} \frac{I\,dl}{(\sqrt{z^2+b^2})}$$ in indicated direction

but horizontal components cancel as go around loop

$$dB_{vert} = \frac{\mu_0}{4\pi} \frac{I \cos\theta\, dl}{(z^2+b^2)}$$

$$B = \frac{\mu_0}{4\pi} \frac{I \cos\theta}{(z^2+b^2)} \int dl$$

$$= \frac{\mu_0}{4\pi}\, 2\pi b\, \frac{I \cos\theta}{(z^2+b^2)}$$

but $\sqrt{z^2+b^2}\cos\theta = b$

~~Fluxes are equal~~

$$= \frac{\mu_0}{2} \frac{b^2 I}{(z^2+b^2)^{3/2}}$$

$$\text{Flux} = \int \bar{B} \cdot d\bar{a} = \frac{\mu_0 \pi}{2} \frac{I a^2 b^2}{(z^2+b^2)^{3/2}}$$

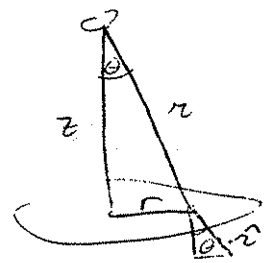
$$= BA = B\pi a^2$$

Problem 3 Continued.

b) Easy: Fluxes are equal mag. + opposite

Hard: Small loop is dipole $\vec{m} = I\pi a^2 \hat{z}$

Field of dipole (on crib sheet)



$\vec{m}\cdot\hat{r} = -m\cos$

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]$$

$\hat{r}\cdot d\vec{a} = \cos\theta\, da$

$$= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left(\frac{1}{(z^2+r^2)^{3/2}}\right)\left[-3m\cos\theta\,\hat{r} + m\hat{z}\right]$$

$\cancel{\cos\theta}$

$\cos\theta = \frac{z}{\sqrt{z^2+r^2}}$

$$\int \vec{B}\cdot da = \int_0^{2\pi}\int_0^{b}\frac{\mu_0}{4\pi}\frac{1}{(z^2+r^2)^{3/2}}\left[-3m\cos^2\theta + m\right] r\,dr\,d\phi$$

$$= \frac{\mu_0 m}{4}\int_0^{b}\left[\frac{-3z^2}{(z^2+r^2)^{5/2}} + \frac{1}{(z^2+r^2)^{3/2}}\right]2r\,dr$$

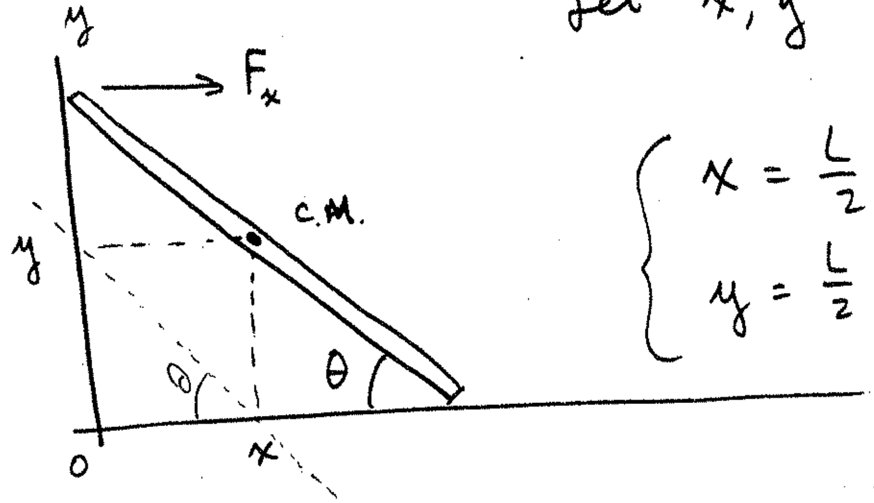
$$= \frac{\mu_0 m}{4}\left[+\frac{2}{\cancel{3}}\frac{\cancel{3}z^2}{(z^2+r^2)^{3/2}} - 2\frac{1}{(z^2+r^2)^{\frac{1}{2}}}\right]\Bigg|_0^{b}$$

$$= \frac{\mu_0 m}{2}\left[\frac{z^2}{(z^2+b^2)^{3/2}} - \cancel{\frac{z^2}{z^3}} - \frac{1}{(z^2+b^2)^{\frac{1}{2}}} + \cancel{\frac{1}{z}}\right]$$

$$= -\frac{\mu_0 m}{2}\frac{b}{(z^2+b^2)^{3/2}}$$

$$= -\frac{\mu_0 I\pi a^2 b^2}{2(z^2+b^2)^{3/2}}$$

Problem 4.



Let x, y = coordinates of center of mass.

$$\begin{cases} x = \frac{L}{2}\cos\theta \; ; & \dot{x} = -\frac{L}{2}\sin\theta\,\dot{\theta} \\ y = \frac{L}{2}\sin\theta \; ; & \dot{y} = \frac{L}{2}\cos\theta\,\dot{\theta} \end{cases}$$

$$I_{CM} = \frac{1}{12} mL^2$$

Conservation of energy: $T + U = T_0 + U_0$

$$\frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_{CM}\dot{\theta}^2 + mgy = 0 + mgy_0$$

$$\frac{m}{2}\cdot\frac{L^2}{4}\underbrace{(\sin^2\theta + \cos^2\theta)}_{1}\dot{\theta}^2 + \frac{1}{2}\cdot\frac{1}{12}mL^2\dot{\theta}^2 + mg\frac{L}{2}\sin\theta = mg\frac{L}{2}\sin\theta$$

$$\frac{1}{6}L\dot{\theta}^2 + \frac{1}{2}g\sin\theta = \frac{1}{2}g\sin\theta_0$$

$$\underline{\dot{\theta}^2 = \frac{3g}{L}(\sin\theta_0 - \sin\theta)}$$ general relation between $\dot{\theta}$ and θ

The rod loses contact with the wall when $F_x = 0$ or $\ddot{x} = 0$.

$$\rightarrow \frac{d}{dt}(\sin\theta\,\dot{\theta}) = 0 \quad \text{or} \quad \underline{\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2 = 0}$$

$$\rightarrow 2\dot{\theta}\ddot{\theta} = -\frac{3g}{L}\cos\theta\,\dot{\theta} \quad \text{or} \quad \underline{\ddot{\theta} = -\frac{3g}{2L}\cos\theta}$$

$$\therefore \sin\theta\left(-\frac{3g}{2L}\cancel{\cos\theta}\right) + \cancel{\cos\theta}\cdot\frac{3g}{L}(\sin\theta_0 - \sin\theta) = 0$$

$$-\frac{1}{2}\sin\theta + \sin\theta_0 - \sin\theta = 0 \; ; \quad \sin\theta = \frac{2}{3}\sin\theta_0 \quad \text{or} \quad \boxed{\theta = \sin^{-1}\left(\frac{2}{3}\sin\theta_0\right)}$$

Problem 5.

A. Schrödinger's equation gives $c_n = \sum_{m \neq n} \frac{H_{nm}}{\lambda - H_{nn}} c_m = \sum_{m=1}^{\infty} h_{nm} c_m$.

B. Hence $c_n = h_{n1} c_1 + \sum_{m=2}^{\infty} h_{nm} c_m$. The second term is much smaller than the first and we can replace c_m in that term by the same expression $c_n = h_{n1} c_1 + \sum_{m=2}^{\infty} h_{nm} (h_{m1} c_1 + \sum_{l=2}^{\infty} h_{ml} c_l)$.

Repeating this process gives $c_n = M_{n1}(\lambda) c_1$ with $M_{n1}(\lambda) = h_{n1} + \sum_{m=2}^{\infty} h_{nm} h_{m1} + \sum_{m=2}^{\infty} \sum_{l=2}^{\infty} h_{nm} h_{ml} h_{l1}$ + ... where h is a function of λ.

C. $M_{11}(\lambda)=1$

D. $h_{nm} = g\,(1 - \delta_{nm})/(\lambda - n^2)$, hence $M_{11}(\lambda) = 0 + g^2 \sum_{m=2}^{\infty} \frac{1}{(\lambda - \varepsilon)(\lambda - m^2 \varepsilon)}$, which leads to $\lambda = \varepsilon + g^2 \sum_{m=2}^{\infty} \frac{1}{(\lambda - m^2 \varepsilon)}$. The ordinary formula has a $1 - m^2$ in the denominator. This perturbation expansion gives an implicit relation for λ.

E. Need to solve $(\varepsilon - \lambda)(4\varepsilon - \lambda) = g^2$, which also follows from part D.

Problem 6. Wave equation: $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{v^2}\dfrac{\partial^2 y}{\partial t^2}$; $v = \sqrt{\dfrac{\tau}{\rho}}$

Standing waves must satisfy $L = n\dfrac{\lambda}{2}$ or $k_n = \dfrac{2\pi}{\lambda_n} = \dfrac{2\pi}{\left(\frac{2L}{n}\right)} = \dfrac{n\pi}{L}$

$n = 1, 2, 3, \cdots$

Normal frequencies: $\omega_n = k_n v = \dfrac{n\pi}{L}\sqrt{\dfrac{\tau}{\rho}} \equiv n\omega_1$

$y_n(x,t) \propto \sin k_n x \cdot \left\{\begin{matrix}\sin \\ \cos\end{matrix}\right\} \omega_n t$

$$\therefore\ y(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\omega_n t + B_n \sin\omega_n t\right)\sin\frac{n\pi x}{L}$$

Since $\displaystyle\int_0^L \sin\frac{n\pi x}{L}\sin\frac{n'\pi x}{L}\,dx = \frac{L}{2}\delta_{nn'}$, the Fourier amplitudes

are
$$\begin{cases} A_n = \dfrac{2}{L}\displaystyle\int_0^L y(x,0)\sin\frac{n\pi x}{L}\,dx \equiv 0 \quad \text{for all } n \\[2ex] B_n = \dfrac{2}{\omega_n L}\displaystyle\int_0^L \frac{\partial y}{\partial t}(x,0)\sin\frac{n\pi x}{L}\,dx \end{cases}$$

$$= \frac{2}{\omega_n L}\int_{\frac{L}{2}-s}^{\frac{L}{2}+s} v_0 \sin\frac{n\pi x}{L}\,dx = \frac{-2v_0}{\omega_n L}\cdot\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Bigg|_{\frac{L}{2}-s}^{\frac{L}{2}+s}$$

$$= \frac{-2v_0}{n^2\pi\omega_1}\left[\cos\frac{n\pi}{L}\left(\frac{L}{2}+s\right) - \cos\frac{n\pi}{L}\left(\frac{L}{2}-s\right)\right]$$

$$= \frac{-2v_0}{n^2\pi\omega_1}\left[\cancel{\cos\frac{n\pi}{2}\cos\frac{n\pi s}{L}} + \sin\frac{n\pi}{2}\sin\frac{n\pi s}{L} - \cancel{\cos\frac{n\pi}{2}\cos\frac{n\pi s}{L}} - \sin\frac{n\pi}{2}\sin\frac{n\pi s}{L}\right.$$

$\sin\dfrac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases} \implies B_n = 0$ for even n.

For odd n, $B_n = +\dfrac{4v_0}{n^2\pi\omega_1}(-1)^{\frac{n-1}{2}}\sin\dfrac{n\pi s}{L}$; $\omega_1 = \dfrac{\pi}{L}\sqrt{\dfrac{\tau}{\rho}}$

$$\therefore\ \boxed{y(x,t) = \frac{4v_0 L}{\pi^2}\sqrt{\frac{\rho}{\tau}}\sum_{\substack{n=1\\ \text{(odd only)}}}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin\frac{n\pi s}{L}\sin\frac{n\pi x}{L}\sin\left(\frac{n\pi}{L}\sqrt{\frac{\tau}{\rho}}\,t\right)}$$

Physics Department Comprehensive Exam # 65, March 31-April 1, 1992

SOLUTIONS

Problem #7

(A) Air is nearly an ideal non-interacting gas. $C_P = C_V + R$, $C_P/C_V = \gamma = 4/3 \rightarrow C_V = 3$ R for water$\rightarrow U = 3RT$ per mole. Efficiency $f = \frac{\text{work in}}{\text{energy out}}$, H_W = energy in per mole of water. Adiabatic expansion$\rightarrow$energy out per mole of water = change in internal energy ($dU = -P\,dV + T\,dS$, $dS = 0$ for adiabat)$\rightarrow \Delta U = 3R\Delta T$ per mole, $f = \Delta U/H_W = 3R(100\text{ K})/H_W$. This part can also be done by integrating PV^γ = constant in the P-V diagram.

(B) Work done = potential energy change = $m_{tot}\, g\, \Delta h$,

$m_{tot} = M_E + M_W + M_F$, $\Delta h = 1\% \times 10$ km = 100 m.

work done = $(M_E + M_W + M_F) \times g \times 100$ m.

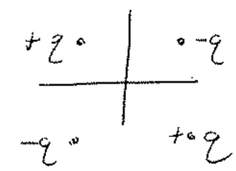
but work done = f x fuel energy = $f \times m_f \times J_E \rightarrow m_f = \frac{(M_E + M_W + M_F) \times 100\text{ m} \times g}{f \times J_E}$

(C) Work done = $3R\,\Delta T \times N_W$ where N_W = # moles water = $m_W/18$ gm since H_2O has molecular weight 18.

$$m_W = 18\text{ gm} \times N_W = \frac{(18\text{ gm})(\text{work done})}{3R(100\text{ K})} = \frac{18\text{ gm} \times (M_E + M_W + M_F) \times g \times 100\text{ m}}{3R \times 100\text{ K}}$$

Problem 8

a+c) Method of images says $\vec{E}$ for this problem above the conductor is same as for



but $\vec{E}$ below conductor is 0. Since $W \propto \int E^2 d^3x$ + symmetry the energy for this problem is $\frac{1}{2}$ the energy for the image problem

$$W = \frac{1}{2} W_{image} = \frac{1}{2} \left\{ \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{4} \sum_{\substack{j=1 \\ j>i}}^{4} \frac{q_i q_j}{r_{ij}} \right\}$$

$$= \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \left(-\frac{q^2}{2a} - \frac{q^2}{2a} - \frac{q^2}{2b} - \frac{q^2}{2b} + \frac{q^2}{2\sqrt{a^2+b^2}} + \frac{q^2}{2\sqrt{a^2+b^2}} \right)$$

$$= \frac{1}{2} \frac{q^2}{4\pi\varepsilon_0} \left(-\frac{1}{a} - \frac{1}{b} + \frac{1}{\sqrt{a^2+b^2}} \right)$$

b) $\vec{p} = 2aq\,\hat{y}$ $\qquad b \gg a \Rightarrow (a^2+b^2)^{-\frac{1}{2}} \approx \frac{1}{b}\left(1 - \frac{1}{2}\frac{a^2}{b^2}\right)$

$$W = \frac{q^2}{4\pi\varepsilon_0} \left(\underbrace{\cancel{-\frac{1}{a}}}_{\text{self energy}} \cancel{-\frac{1}{b}} + \frac{1}{b}\left(\cancel{1} - \frac{1}{2}\frac{a^2}{b^2}\right) \right)$$

$$= -\frac{1}{4\pi\varepsilon_0} \frac{p^2}{8b^3}$$

Same as using formula sheet $W = \vec{p} \cdot \vec{E}$ ← from formula sheet

with $\vec{p}_1 \perp \hat{r}$, $\vec{p}_2 \perp \hat{r}$, $\vec{p}_1 = -\vec{p}_2$, $r = 2b$