PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #63

March 26 and 27, 1991

Comprehensive examination for spring 1991

PART I

General Instructions

This Comprehensive Examination for Spring 1991 (#63) consists of eight problems of equal weight (20 points each). It has two parts. The first part (Problems 1-4) is handed out at 3:00 pm on Tuesday, March 26 and lasts three hours. The second part (Problems 5-8) will be handed out at 3:00 pm on Wednesday, March 27 and will also last three hours.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit – especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet. Be sure to make note of your student letter for use in Part II on March 27.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Physics Department Comprehensive Exam # 62, March 26-27, 1991

PART I

Problem #1

An atom with an even number of electrons, protons and neutrons is prepared in a stationary state, and measurements are then made of one component of its angular momentum, $\boldsymbol{J}_z$, and of the total squared angular momentum $\boldsymbol{J}^2$.

(a) What values are possible for $\boldsymbol{J}^2$?

(b) If the next-smallest possible value is found for $\boldsymbol{J}^2$, what values may be found for $\boldsymbol{J}_z$?

(c) After the measurements in part (b), another component J_x is measured. What values may be found?

(d) What will be the average value $\langle J_x \rangle$ of the measurements in part (c)?

(e) Suppose J_x^2 had been measured in part (c) instead of J_x. What values might have been found?

(f) The mean deviation ΔJ_x is defined as $\Delta J_x \equiv [\langle J_x^2 \rangle - \langle J_x \rangle^2]^{1/2}$. What is the maximum value of ΔJ_x that can be found consistent with a given measurement of $\boldsymbol{J}^2$ and J_z?

Problem #2

A nonrelativistic particle of mass m, positive charge q, and initial velocity v_0, makes a head-on (one-dimensional) collision with a fixed nucleus of charge Ze. The presence of electrons in the vicinity of the nucleus may be neglected, so that the interaction is described by the Coulomb potential of a point charge. Using classical radiation theory, calculate the total energy W radiated during this collision ($-\infty < t < \infty$), under the assumption that $W \ll mv_0^2$. You may use the Larmor power formula for the power radiated per unit solid angle: $dP/d\Omega = (q^2a^2/4\pi c^2)\sin^2\theta$ (in Gaussian units), where $\boldsymbol{a}$ is the acceleration of the particle and θ is the angle between the acceleration and the observation direction.

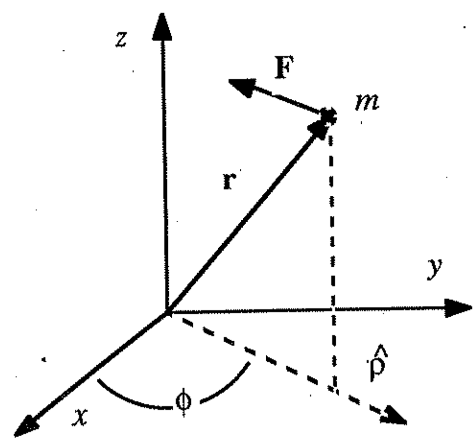
GO ON TO THE NEXT PAGE

Problem #3

A nonrelativistic particle of mass m is acted on by a force which is proportional to the distance from the z-axis, and directed toward the z-axis, as shown in the figure. The force is given by

$$\mathbf{F} = -K\vec{\rho} = -K\rho\hat{\rho} ,$$

where $\vec{\rho}$ is the radial coordinate in a cylindrical coordinate system about the z-axis, and where K is a positive constant.



(a) Find three conserved quantities, and prove that these are conserved.

(b) Assume the particle described in part (a) moves only in the x-y plane, and that it is charged, with a positive charge q. In addition to the force described in part (a), the particle is also subjected to a uniform magnetic field applied in the z-direction, i.e. $\mathbf{B} = B_0\hat{\mathbf{z}}$. Give a quantitiative description of the motion of the partice for initial position ρ_0 and velocity v_0, both in the x-y plane.

Problem #4

The Hamiltonian for a system of N identical nonrelativistic classical particles of mass m is given by

$$H = \sum_{i=1}^{N} \{\frac{p_i^2}{2m} + \alpha|r_i|\}$$ where α is a positive constant.

(a) Find the entropy $S(T)$ and the energy $E(T)$ for equilibrium at temperature T.

(b) Show that there is no thermodynamic limit as $N \to \infty$, if α is independent of N .

(c) Suppose α is a function of N. How must $\alpha(N)$ depend on N , in the limit as $N \to \infty$, for a thermodynamic limit to exist.

(d) Define the volume of the system by $V \equiv \langle\, |\mathbf{r}|\, \rangle^3$. Calculate $\frac{V}{N}$ in the limit $N \to \infty$, using the result of part (c).

END OF PART I

PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #63

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PART II

General Instructions

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PART II

Problem #5

A spinless particle of mass m moves nonrelativsitically in one dimension in the potential $V(x) = a + b\ x^4$ where a and b are positive constants.

(a) choosing a suitable set of trial functions, apply the variational method to estimate the energy E_0 of the ground state.

(b) Choosing a suitable set of trial functions, apply the variational method to estimate the energy E_1 of the first excited state.

(c) For each of the above cases, find the probability that a measurement of x will yield a positive number.

(d) For the ground state (case (a)) estimate the probability that a measurement of V will give a result greater that E_0.

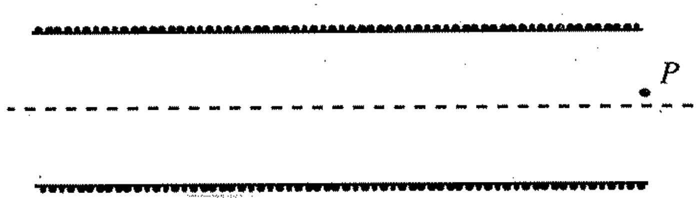
Problem #6

A long straight solenoid of length l and radius a ($a \ll l$), uniformly wound with N turns of fine wire, carries a slowly varying current $I(t)$. Cylindrical coordinates (ρ, ϕ, z) are defined in which the z-axis coincides with the axis of the solenoid. Calculate each of the following quantities at a point P located at the end of the solenoid and a small distance ρ from the axis ($\rho \ll a$), to leading order in ρ/a :

(a) The axial component of the magnetic field, B_z.

(b) The radial component of the magnetic field, B_ρ.

(c) The azimuthal component of the induced electric field, E_ϕ.



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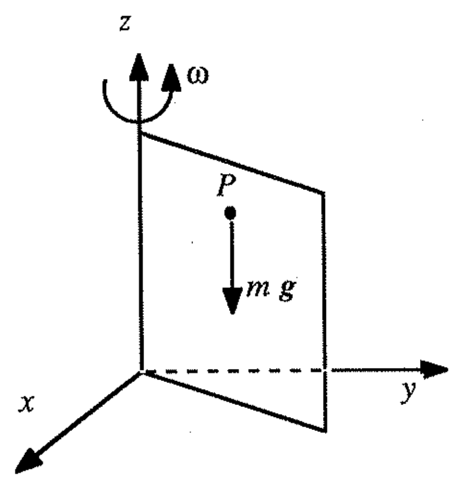
Physics Department Comprehensive Exam # 62, March 26-27, 1991

Problem #7

The vertical plane shown in the figure rotates about the z-axis with constant angular velocity ω. At point P a mass m moves on the plane without friction in a uniform gravitational field, $\boldsymbol{g} = -g\hat{z}$.

(a) Determine the time dependence of the force exerted on the particle by the plane.

(b) Suppose the gravitational field described in part (a) is replaced with a force directed toward the z-axis, and given by $\mathbf{F} = -K\vec{\rho} = -K\rho\hat{\rho}$, where $\hat{\rho}$ is the unit vector in the radial direction in cylindrical coordinates about the z-axis. Everything else described in part (a) remains the same. Give a quantitative description of the motion in the radial direction.



Problem #8

The equation of state of an imperfect gas is given by $P = \frac{N}{V} k\,T\,[1 + B(T)\frac{N}{V}]$, where P is the pressure, V the volume, N the number of particles, T the temperature, and B(T) is some function of T. This gas undergoes a small isothermal expansion ΔV at constant N. This expansion is caused by a change in pressure ΔP. The amount of heat going into the system during this expansion is ΔQ.

(a) Relate ΔQ to the change in pressure ΔV, keeping only linear terms.

(b) Construct the thermodynamic potential X which is appropriate for a system at constant V, T, and N.

(c) Give the expressions for the pressure P, the entropy S, and the chemical potential μ in terms of the thermodynamic potential X.

(d) Construct three Maxwell relations based on X.

(e) Use one of these Maxwell relations, together with the equation of state, to evaluate the partial derivatives in part (a).

(f) Interpret the terms in the expression derived in part (e).

END OF EXAMINATION

Physics Department Comprehensive Exam # 62, March 26-27, 1991

SOLUTIONS

Problem #1

(a) $J^2 = J(J+1)\hbar^2$. Since there are an even number of fermion, J will be an integer, $J = 0,1,2...$

(b) $J = 1$ is next-smallest value. $J_z = m\hbar$, $m = -J, -J+1,..., J$. In this case, m = -1, 0, or +1.

(c) answers: same as part (b).

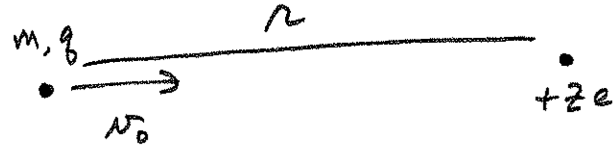
(d) Every eigenstate of J_z has $< J_x > = 0$. Proof: $<Jm \mid J_x \mid Jm> = \frac{1}{2} <Jm \mid J_+ + J_- \mid Jm>$ but J_+ and J_- only have matrix elements between states of different J.

(e) $J_x{}^2 = m^2 \hbar^2$.

(f) Note $J_x^2 + J_y^2 + J_z^2 = J^2$ so $J_x^2 \geq J^2 - J_z^2 - J_y^2 = \hbar^2(J(J+1)-m^2) - J_y^2 \geq \hbar^2(J(J+1)-m^2)$. Thus $\Delta J_x \geq \hbar \ (J(J+1)-m^2)^{1/2}$.

E & M Problem 2

Radiation problem



$$F_r = \frac{Zeq}{r^2} \quad ; \quad V(r) = \frac{Zeq}{r} \quad \text{Coulomb potential (repulsive).}$$

Conservation of energy: $\frac{1}{2} m v^2 + \frac{Zeq}{r} = \frac{1}{2} m v_0^2$

$$v = 0 \Rightarrow r = r_{min} = \frac{2Zeq}{m v_0^2} \quad \text{distance of closest approach.}$$

Power radiated into all angles:

$$P = \int_{(4\pi)} \frac{dP}{d\Omega} d\Omega = \frac{q^2 a^2}{4\pi c^3} \int_0^\pi \underbrace{\sin^2\theta}_{1-\cos^2\theta} \cdot 2\pi \underbrace{\sin\theta\, d\theta}_{-d(\cos\theta)}$$

$$= \frac{q^2 a^2}{2c^3} \int_{\theta=0}^{\pi} (\cos^2\theta - 1)\, d(\cos\theta) = \frac{q^2 a^2}{2c^3} \cdot \underbrace{\left[\frac{\cos^3\theta}{3} - \cos\theta\right]\Big|_0^\pi}_{-\frac{2}{3} - (-2) = \frac{4}{3}}$$

$$\therefore P = \frac{2q^2a^2}{3c^3}$$

The energy radiated is therefore $W = \int_{-\infty}^{\infty} P\, dt = \frac{2q^2}{3c^3} \int_{-\infty}^{\infty} a^2\, dt$

$$F = ma \Rightarrow a = \frac{F}{m} = \frac{Zeq}{mr^2} \qquad dt = \frac{dr}{v}$$

If $W \ll m v_0^2$, then we may calculate $\underline{a}$ in the absence of radiation

The integral is easiest if performed over v:

$$\int a^2 dt = \int a \frac{dv}{dt} dt = \int a\, dv$$

$$a = \frac{F}{m} = \frac{Zeq}{mr^2} \quad \text{where } \frac{Zeq}{r} = \frac{m}{2}(v_0^2 - v^2).$$

$$\therefore a = \frac{1}{mZeq}\left(\frac{Zeq}{r}\right)^2 = \frac{1}{mZeq} \cdot \frac{m^2}{4}(v_0^2 - v^2)^2$$

$$\therefore \int_{-\infty}^{\infty} a^2 dt = 2\int_0^{v_0} a\, dv = \frac{m}{2Zeq} \int_0^{v_0} (v_0^4 - 2v_0^2 v^2 + v^4)\, dv$$

(continued)

$$\therefore \int_{-\infty}^{\infty} a^2\, dt = \frac{m}{2 z e q}\left[v_0^5 - 2 v_0^2 \cdot \frac{v_0^3}{3} + \frac{v_0^5}{5} \right] = \frac{4 m v_0^5}{15 z e q}$$

$$1 - \frac{2}{3} + \frac{1}{5} = \frac{15 - 10 + 3}{15} = \frac{8}{15}$$

Finally,

$$W = \frac{2 q^2}{3 c^3} \cdot \frac{4 m v_0^5}{15 z e q}$$

$$\boxed{W = \frac{8 q m v_0^5}{45 Z e c^3}}$$

#3 a) $\frac{dL_z}{dt} = (\vec{r} \times \vec{F})_z = 0$ because $\vec{r} \times \vec{F}$ lies in the x-y plane; ∴, L_z = constant.

$F_z = 0, \Rightarrow p_z$ = constant. $F_x = -k\rho\cos\phi = -kx$, and $F_y = -k\rho\sin\phi = -ky$.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -kx & -ky & 0 \end{vmatrix} = \left\{ \begin{array}{l} \hat{i}\left[0 + \frac{\partial}{\partial z}(ky)\right] + \\ + \hat{j}\left[\frac{\partial}{\partial z}(-kx) - 0\right] + \\ + \hat{k}\left[\frac{\partial}{\partial x}(-ky) + \frac{\partial}{\partial y}(kx)\right] \end{array} \right\};$$

∴, $\vec{\nabla} \times \vec{F} = 0$, which implies that $E = \frac{1}{2}m\vec{v}^2 + \frac{1}{2}k\rho^2$ = constant.

b) In the absence of the B-field the equations of motion are $\ddot{x} + \omega_0^2 x = 0$, $\ddot{y} + \omega_0^2 y = 0$, where $\omega_0^2 = \frac{k}{m}$.

$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & 0 \\ 0 & 0 & B_0 \end{vmatrix} = \hat{i} B_0 \dot{y} - \hat{j} B_0 \dot{x}.$$

Thus, the new equations of motion are

$\ddot{x} + \omega_0^2 x - \frac{qB_0}{m}\dot{y} = 0$, and $\ddot{y} + \omega_0^2 y + \frac{qB_0}{m}\dot{x} = 0$. Assume $x = x_0 e^{i\omega t}$ and $y = y_0 e^{i\omega t}$.

$-\omega^2 x + \omega_0^2 x - i\frac{\omega q B_0}{m} y = 0$, $-\omega^2 y + \omega_0^2 y + i\frac{\omega q B_0}{m} x = 0$

$x = \frac{i\omega q B_0}{m(\omega_0^2 - \omega^2)} y$, $(\omega_0^2 - \omega^2) y + \frac{i\omega q B_0}{m}\left[\frac{i\omega q B_0}{m(\omega_0^2 - \omega^2)}\right] y = 0$,

$(\omega_0^2 - \omega^2)^2 - \left(\frac{\omega q B_0}{m}\right)^2 = 0$, $\omega_0^2 - \omega^2 = \pm \frac{\omega q B_0}{m} = \pm \omega\omega_c$

Two equations: $\omega_+^2 + \omega_c\omega_+ - \omega_0^2 = 0$ and $\omega_-^2 - \omega_c\omega_- - \omega_0^2 = 0$, or,

$2\omega_+ = -\omega_c \pm \sqrt{\omega_c^2 + 4\omega_0^2}$, $2\omega_- = \omega_c \pm \sqrt{\omega_c^2 + 4\omega_0^2}$.

Take the + of ± in both cases. Thus, we have a linear superposition of simple harmonic motions, in the x-y plane, at angular frequencies ω_+ and ω_-.

Problem 4 :

a $$Z_C = \frac{1}{h^{3N}} \frac{1}{N!} \int d\vec{p}_1 \cdots d\vec{p}_N \, d\vec{r}_1 \cdots d\vec{r}_N \; e^{-\beta H}$$

$$= \frac{1}{h^{3N}} \frac{1}{N!} \left[\int d^3p \; e^{-p^2/2mkT} \right]^N \left[\int d^3r \; e^{-\alpha r/kT} \right]^N$$

$$= \frac{1}{h^{3N}} \frac{1}{N!} \left(2mkT\pi \right)^{\frac{3N}{2}} \left(\frac{kT}{\alpha} \right)^{3N} \left(8\pi \right)^N$$

$$A = -kT \log Z_C$$

$$= -NkT \log \left\{ \frac{(2mkT\pi)^{3/2}}{h^3} \left(\frac{kT}{\alpha} \right)^3 8\pi \right\} + kT \log N!$$

$$\log N! \approx N \log N - N$$

$$\Rightarrow A = -NkT \log \left[\left\{ \frac{\sqrt{2mkT\pi}}{h} \frac{kT}{\alpha} \right\}^3 \frac{8\pi}{N} \right] - NkT$$

$$S = -\left(\frac{\partial A}{\partial T} \right)_{N,V} = -Nk \log [\ldots] - Nk + \frac{9}{2} Nk$$

$$E = A + TS = \frac{9}{2} NkT$$

b) $\lim_{N\to\infty} \frac{E}{N} = \frac{9}{2} kT$

$\lim_{N\to\infty} \frac{S}{N} = \lim_{N\to\infty} kT \log N = \infty$ hence no TD. limit.

c) We need $\lim_{N\to\infty} \alpha^3(N)\, N = C^3$

or $\alpha(N) = C / N^{1/3}$

d)

Identical particles $\langle |\bar{r}| \rangle = \langle |\bar{r}_i| \rangle$

$$= \frac{1}{Z_c} \frac{1}{h^{3N}} \frac{1}{N!} \int d^3\bar{p}_1 \cdots d\bar{r}_N \; |\bar{r}_1| \, e^{-\beta H}$$

$$= \int d^3r \; r \; e^{-\alpha r / kT} \Big/ \int d^3r \; e^{-\alpha r / kT} = \frac{3kT}{\alpha}$$

or $V = \frac{(3kT)^3}{\alpha^3}$ $\qquad \frac{V}{N} = \frac{(3kT)^3}{\alpha^3 N} \to \left(\frac{3kT}{C}\right)^3$

indeed $< \infty$.

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SOLUTIONS

Problem #5

(a) Choose as a trial wave function nodeless symmetric function peaked at $x = 0$. A simple example is the oscillater type function, $\phi(x) = \alpha \exp(-\beta^2 x^2)$.

Normalize: $1 = |\alpha|^2 \int_{-\infty}^{\infty} dx\ \exp(-2\beta^2 x^2) = |\alpha|^2 \sqrt{\pi/2}/\beta \Rightarrow \phi(x) = \dfrac{\beta^{1/2}}{(\pi/2)^{1/4}} \exp(-\beta^2 x^2)$

$$\langle\text{Kinetic energy}\rangle = \langle\phi|\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right\}|\phi\rangle = |\alpha|^2 \int_{-\infty}^{\infty} dx\ \exp(-\beta^2 x^2)\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right\}\exp(-\beta^2 x^2)$$

$$= |\alpha|^2 \int_{-\infty}^{\infty} dx\ \frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\exp(-\beta^2 x^2)\right)^2$$

$$= |\alpha|^2 \frac{\hbar^2}{2m}(-2\beta^2)^2 \int_{-\infty}^{\infty} dx\ x^2 \exp(-2\beta^2 x^2)$$

$$= \frac{\beta}{\sqrt{\pi/2}}(4\beta^4)\, 2\frac{\sqrt{\pi}}{4(\sqrt{2}\beta)^3}\frac{\hbar^2}{2m} = \frac{\hbar^2}{2m}\beta^2$$

$$\langle\text{ potential energy}\rangle = \langle\phi|V|\phi\rangle = a + b\ |\alpha|^2 \int_{-\infty}^{\infty} dx\ \exp(-2\beta^2 x^2)\ x^4$$

$$= a + b\frac{\beta}{\sqrt{\pi/2}} 2\frac{1\cdot 3\sqrt{\pi}}{2^3(\sqrt{2}\beta)^5} = a + b\ \frac{3}{16\beta^4}$$

vary the total energy: $0 = \dfrac{\partial}{\partial\beta}(\langle\text{k.e.}\rangle + \langle V\rangle) = \dfrac{\hbar^2}{m}\beta - b\ \dfrac{3}{4\beta^5}$

$$\beta^6 = \frac{3mb}{4\hbar^2} \Rightarrow E_0 = a + \frac{\hbar^2}{2m}\beta^2 + b\ \frac{3}{16\beta^4}\ .$$

Problem #5 (continued)

(b) first excited state $\Rightarrow$ 1 node, odd parity $\Rightarrow$ trial function for example $\phi(x) = \alpha\, x \exp(-\beta^2 x^2)$.

$$\text{Normalize: } 1 = |\alpha|^2 \int_{-\infty}^{\infty} dx\; x^2 \exp(-2\beta^2 x^2) = |\alpha|^2 \sqrt{\pi}/2(\sqrt{2}\beta)^3 \Rightarrow \phi(x) = \frac{\sqrt{8\beta^3}}{(2\pi)^{1/4}} x\; \exp(-\beta^2 x^2)$$

$$\langle\text{Kinetic energy}\rangle = \langle\phi|\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right\}|\phi\rangle = |\alpha|^2 \int_{-\infty}^{\infty} dx\, x \exp(-\beta^2 x^2)\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right\} x \exp(-\beta^2 x^2)$$

$$= |\alpha|^2 \int_{-\infty}^{\infty} dx\, \frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x} x \exp(-\beta^2 x^2)\right)^2 = |\alpha|^2 \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} dx\, [\,1 - 2\beta^2 x^2]^2 \exp(-2\beta^2 x^2)$$

$$= \frac{8\beta^3}{\sqrt{2\pi}}\frac{\hbar^2}{2m} 2 \left[\frac{\sqrt{\pi}}{2\sqrt{2}\beta} - 4\beta^2 \frac{\sqrt{\pi}}{4(\sqrt{2}\beta)^3} + 4\beta^4 \frac{3\sqrt{\pi}}{8(\sqrt{2}\beta)^5}\right]$$

$$= \frac{\hbar^2}{2m}\frac{\beta^2}{2}[\,8 - 8 + 6] = 3\frac{\hbar^2}{2m}\beta^2$$

$$\langle\text{ potential energy}\rangle = \langle\phi|V|\phi\rangle = a + b\; |\alpha|^2 \int_{-\infty}^{\infty} dx\; \exp(-2\beta^2 x^2)\; x^6$$

$$= a + b\frac{8\beta^3}{\sqrt{2\pi}} 2 \frac{3\cdot 5\sqrt{\pi}}{2^4(\sqrt{2}\beta)^7} = a + b\,\frac{15}{16\beta^4}$$

$$\text{vary the total energy: } 0 = \frac{\partial}{\partial\beta}(\langle\text{k.e.}\rangle + \langle V\rangle) = 6\frac{\hbar^2}{m}\beta - b\,\frac{15}{4\beta^5}$$

$$\beta^6 = \frac{5mb}{8\hbar^2} \Rightarrow E_0 = a + 3\frac{\hbar^2}{2m}\beta^2 + b\,\frac{15}{16\beta^4}.$$

(c) probability = 1/2 because both ϕ's are eigenstates of parity $\Rightarrow$ $|\phi|^2$ is an even function

(d) probability $= 2 \int_{\text{c.t.p.}}^{\infty} dx\, |\phi|^2$, c.t.p. (classical turning point) defined by $E_0 = V(\text{c.t.p.})$

E & M Problem 6

Solenoid problem

(a) The on-axis field is found immediately from the superposition principle: If two solenoids are placed end-to-end, the two end-fields add to give the interior field.

$$\therefore B_{end} = \frac{1}{2} B_{interior}$$

Ampere's law gives the interior field: $\oint \vec{B} \cdot d\vec{l} = \frac{4\pi N I}{c}$

or $B_{interior} \cdot l = \frac{4\pi N I}{c}$ $\quad \therefore \boxed{B_{z\,(end)} = \frac{2\pi N I}{c\, l}}$ $\quad$ MKS: $\frac{2\pi}{c} \to \frac{\mu_0}{2}$

(c) Faraday's law: $\oint \vec{E} \cdot d\vec{l} = -\frac{1}{c} \frac{d}{dt} \iint \vec{B} \cdot \hat{n}\, dA$

$$\therefore E_\phi \cdot 2\pi\rho = -\frac{1}{c} \frac{d}{dt}\left(B_z \cdot \pi \rho^2\right)$$

$$E_\phi = -\frac{\rho}{2c} \frac{dB_z}{dt}$$

or $\boxed{E_\phi = -\frac{\pi N \rho}{c^2 l} \frac{dI(t)}{dt}}$ $\quad$ MKS: $\frac{\pi}{c^2} \to \frac{\mu_0}{4}$

(b) It is probably easiest to find the radial field (at small) from $\vec{\nabla} \cdot \vec{B} = 0$.

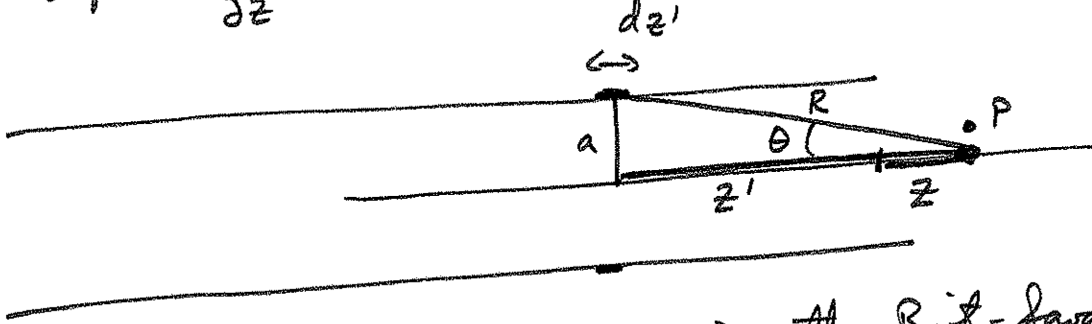
$$\therefore \underbrace{\frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho B_\rho)}_{\frac{B_\rho}{\rho} + \frac{\partial B_\rho}{\partial \rho}} + \frac{\partial B_z}{\partial z} + \underbrace{\frac{1}{\rho} \frac{\partial B_\phi}{\partial \phi}}_{\text{zero, by symmetry.}} = 0$$

These terms are equal for small ρ, since $B_\rho(\rho) = \overbrace{B_\rho(0)}^{\text{zero.}} + \frac{\partial B_\rho}{\partial \rho}\Big|_0 \rho +$

(b) (continued).

$$\therefore B_\rho(\rho) \cong \left.\frac{\partial B}{\partial \rho}\right|_{\rho=0} \rho = -\frac{1}{2}\left.\frac{\partial B_z}{\partial z}\right|_{\rho=0} \rho$$

To find $\frac{\partial B_z}{\partial z}$ we first find $B_z(z)$:



For a single circular loop of wire, the Biot-Savart law gives the on-axis field:

$$\vec{B} = \frac{I}{c}\int \frac{d\vec{l} \times \hat{R}}{R^2} \Rightarrow B_z = \frac{I}{c}\cdot\frac{2\pi a}{R^2}\sin\theta = \frac{2\pi I}{ca}\sin^3\theta$$

For a distribution of loops the total field is

$$B_z = \frac{2\pi}{ca}\int \sin^3\theta \, dI \qquad \text{where } dI = I\,dN = I\frac{dN}{dz'}dz'$$

$$= I\frac{N}{l}dz'$$

$$= I\frac{N}{l}d(a\cot\theta)$$

$$= -I\frac{N}{l}\cdot a\csc^2\theta\, d\theta$$

$$= \frac{2\pi}{ca}\left(-I\frac{N}{l}a\right)\int \underbrace{\sin^3\theta\csc^2\theta}_{\sin\theta}\, d\theta$$

$$B_z = \frac{2\pi}{c}\cdot\frac{NI}{l}\cdot\cos\theta\Big|_{\cos\theta=\frac{z}{\sqrt{z^2+a^2}}}^{\cos\theta\cong 1.} = \underbrace{\frac{2\pi NI}{cl}\left[1-\frac{z}{\sqrt{z^2+a^2}}\right]}_{\text{This again gives answer (a).}}$$

For small z,

$$\left.\frac{dB_z}{dz}\right|_{z=0} = \frac{2\pi NI}{cl}\cdot\left(-\frac{1}{a}\right)$$

$$\therefore B_\rho(\rho) \cong -\frac{1}{2}\left[-\frac{2\pi NI}{cla}\right]\cdot\rho \quad \text{or} \quad \boxed{B_\rho(\rho) \cong \frac{\pi NI\rho}{cla}} \text{ for } \rho \ll a.$$

MKS: $\frac{\pi}{c} \rightarrow \frac{\mu_0}{4}$

#7. a) $\frac{dL_z}{dt} = -\rho F_c = \frac{d}{dt}(m\rho^2\dot{\phi}) = 2m\omega\rho\dot{\rho}$, $F_c = -2m\omega\dot{\rho}$

$$\mathcal{L} = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - mgz$$

$$\frac{d}{dt}(m\dot{\rho}) - m\rho\omega^2 = 0, \quad \ddot{\rho} - \omega^2\rho = 0,$$

$$\rho = Ae^{\omega t} + Be^{-\omega t}, \quad \dot{\rho} = \omega[Ae^{\omega t} - Be^{-\omega t}]; \therefore$$

$F_c = -2m\omega^2[Ae^{\omega t} - Be^{-\omega t}]$, where A and B are determined by initial conditions.

b) $\mathcal{L} = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - \frac{1}{2}k\rho^2$

$$\frac{d}{dt}(m\dot{\rho}) - m\rho\omega^2 + k\rho = 0,$$

$$\ddot{\rho} + (\omega_0^2 - \omega^2)\rho = 0, \text{ where } \omega_0^2 = \frac{k}{m}$$

$\rho = \rho_0 \cos(\omega' t + \alpha)$, where $\omega' = \sqrt{\omega_0^2 - \omega^2}$

and ρ_0 and α are determined by initial conditions.

Problem 8:

a) $$\Delta Q = T \left(\frac{\partial S}{\partial V}\right)_{T,N} \Delta V$$

b) $$A(T, V, N) = U - TS$$

c) $$P = -\left(\frac{\partial A}{\partial V}\right)_{T,N} \qquad S = -\left(\frac{\partial A}{\partial T}\right)_{V,N} \qquad \mu = \left(\frac{\partial A}{\partial N}\right)_{T,V}$$

d) $$\frac{\partial^2 A}{\partial T \partial V} \Rightarrow \left(\frac{\partial P}{\partial T}\right)_{N,V} = \left(\frac{\partial S}{\partial V}\right)_{N,T}$$

$$\frac{\partial^2 A}{\partial T \partial N} \Rightarrow \left(\frac{\partial \mu}{\partial T}\right)_{N,V} = -\left(\frac{\partial S}{\partial N}\right)_{T,V}$$

$$\frac{\partial^2 A}{\partial V \partial N} \Rightarrow \left(\frac{\partial \mu}{\partial V}\right)_{N,T} = -\left(\frac{\partial P}{\partial N}\right)_{V,T}$$

e) $$\left(\frac{\partial S}{\partial V}\right)_{T,N} = \left(\frac{\partial P}{\partial T}\right)_{N,V} = \frac{P}{T} + \left(\frac{N}{V}\right)^2 k T \, B'(T)$$

$$\Rightarrow \Delta Q = P \Delta V + \left(\frac{N}{V}\right)^2 k T^2 B'(T) \Delta V$$

f) first term: work done on environment
second term: change in internal energy!

April 1, 1991

To: Faculty

From: K. S. Krane, Chairman

Subj: Comp Exams Spring 1991

There will be a faculty meeting on

Thursday, April 4
10:30 am
Weniger 305

to consider the results of the comprehensive exam.

lkb

29 March 1991

MEMORANDUM
TO: Physics faculty

FROM: Phil Siemens

SUBJECT: Comprehensive examination Spring 1991

The examination was administered to 20 students on Tuesday and Wednesday, March 26 and 27 from 3 to 6 pm. A copy of the examination and answer key is enclosed. In response to students' questions during the examination, the following clarifications and corrections were written on the blackboard during the examination:

<u>Wednesday March 26</u>
#1. electrons: even; protons: even; neutrons: even
#3. No gravitational force
    a) The motion of the particle is general

<u>Wednesday March 27</u>
Problem 8<u>a</u>   <u>volume</u> ΔV

23 January 91

MEMORANDUM

TO: Physics students and faculty

FROM: Phil Siemens, Chair, Comprehensive Examination Committee

In the past, appeals of grading on the comprehensive examination have been handled informally on an *ad hoc* basis. The committee feels that we should establish an orderly mechanism for dealing with such problems before they arise.

Here is our proposed policy regarding appeals of grading of the comp exam:

Each student will receive a xerocopy of his/her own examination including grading. Copies of the problems and their solutions will be made available in the reading room and for copying at a convenient copy service. Any student who has good reason to believe that a mistake was made in grading the examination may complain within two weeks after the results are announced by writing a letter to the chairman of the examination committee. The written complaint will be considered by the examination committee, which will recommend appropriate action or inaction. Any complaints, together with the committee's recommendations, will then be presented to the department faculty which may choose to revise its decision on the results of the examination. A record of the complaint and any resulting action will be incorporated into the student's file.