PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #62

January 7 and 8, 1991

Comprehensive examination for Winter 1991

PART I

General Instructions

This Comprehensive Examination for Winter 1990 (#62) consists of eight problems of equal weight (20 points each). It has two parts. The first part (Problems 1-4) is handed out at 3:00 pm on Monday, January 7 and lasts three hours. The second part (Problems 5-8) will be handed out at 3:00 pm on Tuesday, January 8 and will also last three hours.

Work carefully, indicate your reasoning briefly, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit – especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet. **Be sure to make note of your assigned student letter for use in Part II on January 8.**

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Physics Department Comprehensive Exam #62, January 7-8, 1991

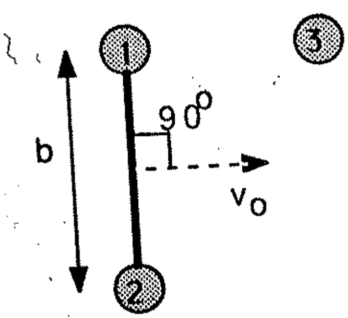
PART I

Problem #1

This is a problem in non-relativistic, classical mechanics. The first part is independent of the last parts.

(a) A hydrogen atom H collides with a hydrogen molecule H_2 at rest and the molecule is dissociated into two atoms. If the binding energy of H_2 is B, determine the minimum kinetic energy of the incident H atom required for dissociation.

(b) Two identical masses #1 and #2 (of mass m) are held at a fixed separation b by a rigid massless rod, thus forming a dumbbell, as shown in the figure at right. The dumbbell travels sideways at a velocity v_0 until mass #2 collides head-on in a perfectly elastic collision with a third identical mass #3 in the manner shown. Determine the final state of the system, in terms of m and v_0.



(c) The collision described in part (b) is now perfectly inelastic, i.e. masses #2 and #3 stick together following the collision. Calculate the loss of mechanical energy in terms of m and v_0.

Problem #2

(a) Write down Maxwell's equations (in differential form) for the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, in an uncharged nonmagnetic conducting medium. Assume charge density $\rho = 0$ and current density $\mathbf{J} = \sigma \mathbf{E}$, where σ is the conductivity.

(b) Derive the wave equation for propagation of the electric field $\mathbf{E}$ in the medium.

(c) Consider monochromatic plane waves traveling in the z - direction, for which $E(z, t) = E_0 \hat{x}\, e^{i(\gamma z - \omega t)}$. Show that energy is absorbed by the medium (i.e. that γ has an imaginary part) and find an expression for the skin depth δ in the low-frequency limit.

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Problem #3

Here is a simplified model of an electron in a vacancy in a cold (T =0) material. You need not evaluate any integrals appearing in your answers to this problem.

A spinless particle of mass m and charge q moves nonrelativistically in three dimensions in a harmonic potential $V(r) = \frac{1}{2}m\,\omega^2\,r^2$. An additional constant electric field of magnitude E_0 is applied in the x-direction.

(a) After waiting a long enough time, we can be sure that the charged particle will eventually be in its ground state. Why? What is its energy E_{gs} then?

(b) In case (a), what is the probability that a measurement of the particle's potential energy will yield a value greater than E_{gs} ?

(c) After a very long time, the electric field is suddenly removed at time t_0. What values of the energy may then be found?

(d) Use Ehrenfest's theorem to find equations for the mean values <R> and <P> of the particle's position and momentum as functions of time, for times later than t_0. Solve these equations with the initial conditions corresponding to part (c)..

(e) Explain the relation between your answers to parts (c) and (d).

Problem #4

Consider a system of N three-dimensional harmonic oscillators of frequency ω and mass m . For a classical canonical ensemble:

(a) Calculate the partition function.
(b) Calculate the Helmholtz free energy and the internal energy.

For a classical canonical ensemble, and assuming ω is inversely proportional to the volume:

(c) Calculate the specific heat at constant volume.
(d) Calculate the pressure on the system.
(e) Calculate the specific heat at constant pressure.

For a quantum-mechanical canonical ensemble,

(f) Calculate the partition function.
(g) Calculate the Helmholtz free energy.

END OF PART I

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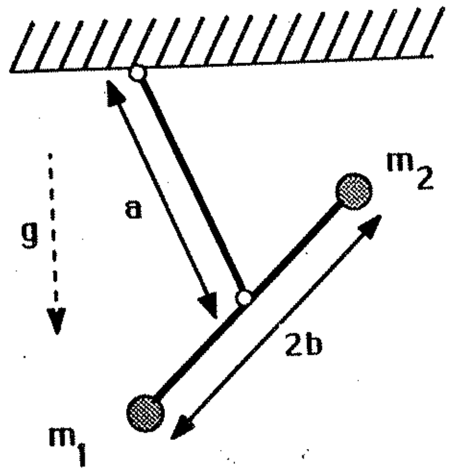
Physics Department Comprehensive Exam #62, January 7-8, 1991

PART II

Problem # 5



Shown in the figure are two massless rods of length a and $2b$ respectively, with two particles of mass m_1 and m_2 attached to the ends of the second rod. The second rod is free to pivot at its center about one end of the first rod and the other end of the first rod is fixed but pivots freely. The entire system is constrained to move in a vertical plane and the only external force is the earth's constant gravitational field. The rods are slightly offset horizontally, so that they do not hit one another. Assume $b < a$.

(a) Derive one of Lagrange's equations for one generalized coordinate. Hint: the equations of motion are coupled.

(b) Derive Lagrange's equations for the special case $m_1 = m_2 = m$.

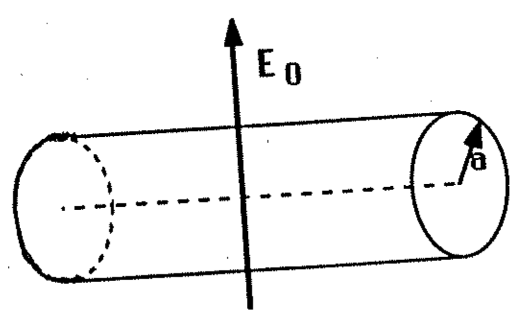
(c) Solve the equations of part (b) for small oscillations.

Problem #6

An infinitely long cylindrical conductor of radius a bearing no net charge is placed in an initially uniform electric field E_0 . The direction of E_0 is perpendicular to the axis of the cylinder.

(a) Find the electrostatic potential ϕ at all points exterior to the cylinder. Let $\phi = 0$ on the cylinder.

(b) Find the charge density σ on the cylindrical surface.



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Problem #7

A neutron and a proton, each of mass m and spin $\hbar/2$, move non-relativistically and independently in three dimensions in a potential $V = \frac{1}{2} m\omega^2 r^2 + C\,\mathbf{L}\cdot\mathbf{S}$. Their orbital motion is such that their radial wave functions both have the same number of nodes n_r and orbital angular momentum $\mathbf{L}^2 = l(l+1)\,\hbar^2$. Their angular momenta $\mathbf{J}_n^2$ and $\mathbf{J}_p^2$ are measured for the neutron and proton respectively and found to be given by $j(j+1)\,\hbar^2$, with both the neutron and proton having the same value of j. After these measurements, the square of the total angular momentum $\mathbf{J}_t \equiv \mathbf{J}_n + \mathbf{J}_p$ is measured and its value is found to be $J(J+1)\,\hbar^2$.

(a) What values might J have?

(b) After the above measurements, the z component of $\mathbf{J}_t$ is measured. What values $M\hbar$ might it have, consistent with the previous results?

(c) Now suppose that the largest possible value was found for M, consistent with the measurement of j. Write the state vector in terms of the individual-particle states $|\, n_r\, l\, j_p\, m_p\rangle$ and $|\, n_r\, l\, j_n\, m_n\rangle$.

(d) Suppose now that both particles were neutrons. How would your answer to part (c) change?

Problem #8

The chemical potential for an ideal gas of classical point particles of mass m at temperature T is given by

$$\mu = k_B T \ln\left(\frac{n}{n_G}\right)$$

where the density is $n = N/V$ and $n_G \equiv \left(\dfrac{m\, k_B\, T}{2\pi\hbar^2}\right)^{3/2}$.

This gas is put in a uniform gravitational field with constant acceleration g. Calculate the pressure in this isothermal gas as a function of height.

END OF PART II

undergrad mechanics #1.

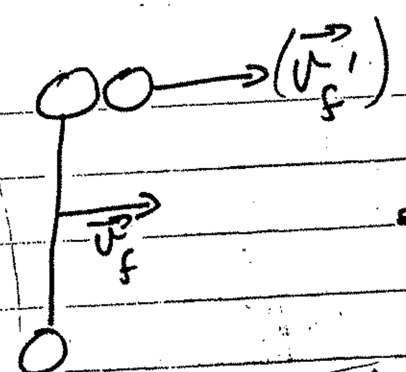
a) $m v_0 = 3 m v_f$

$\frac{1}{2} m v_0^2 = \frac{1}{2}(3m)\left(\frac{v_0}{3}\right)^2 + B$

$\frac{1}{2} m v_0^2 - \frac{1}{3}\left(\frac{1}{2} m v_0^2\right) = B$

$E_K\left(1 - \frac{1}{3}\right) = B, \quad E_K = \frac{3}{2} B$

b)



Conserve momentum:

$2 m v_0 = 2 m v_f + m v_f'$.

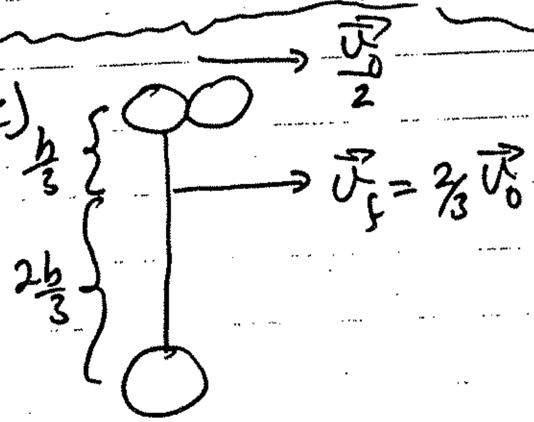
Since the collision is elastic

$v_f' = v_0 ; \therefore v_f = v_0/2$

Conserve energy: $\frac{1}{2}(2m) v_0^2 = \frac{1}{2} m v_0^2 + \frac{1}{2}(2m)\left(\frac{v_0}{2}\right)^2 + \frac{1}{2} I \omega^2$

$m v_0^2\left(1 - \frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}(2m)\left(\frac{b}{2}\right)^2 \omega^2, \quad \frac{m v_0^2}{4} = \frac{m b^2 \omega^2}{4}, \quad \omega = \frac{v_0}{b}$

c)



Conserve momentum:

$2 m v_0 = 3 m v_f, \quad v_f = \frac{2}{3} v_0$

$E_{rotational} = \frac{1}{2}(2m)\left(\frac{v_0}{6}\right)^2 + \frac{1}{2} m \left(\frac{v_0}{3}\right)^2$

$= m v_0^2 / 12$

$\Delta E = \frac{1}{2}(2m) v_0^2 - \frac{1}{2}(3m)\left(\frac{2 v_0}{3}\right)^2 - \frac{m v_0^2}{12}$

$= m v_0^2 / 4$

2.1

(a) $\vec{\nabla}\cdot\vec{E} = 0$; $\vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$; $\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\sigma\vec{E}$

(b) $$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{E})}_{\text{zero}} - \nabla^2\vec{E}$$

$$\therefore \nabla^2\vec{E} = -\nabla\times\left(-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right) = \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B}$$

$$\therefore \boxed{\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} + \frac{4\pi\sigma}{c^2}\frac{\partial\vec{E}}{\partial t}}$$ in Gaussian units.

(c) If $\vec{E}(z,t) = E_0\hat{x}\, e^{i(\gamma z-\omega t)}$, then $\begin{cases} \frac{\partial\vec{E}}{\partial t} = -i\omega\vec{E} \\ \frac{\partial^2\vec{E}}{\partial t^2} = -\omega^2\vec{E}. \end{cases}$

Also $\nabla^2\vec{E} = -\gamma^2\vec{E}$.

$$\therefore -\gamma^2\vec{E} = \frac{1}{c^2}(-\omega^2\vec{E}) + \frac{4\pi\sigma}{c^2}(-i\omega\vec{E}).$$

$$\gamma^2 = \frac{\omega^2}{c^2} + i\,\frac{4\pi\sigma\omega}{c^2}$$

Write $\gamma = k + i\beta$. Then $\gamma^2 = k^2 - \beta^2 + 2ik\beta$.

$$\Rightarrow k^2 - \beta^2 = \frac{\omega^2}{c^2} \quad \text{and} \quad 2k\beta = \frac{4\pi\sigma\omega}{c^2}.$$

In the low-frequency limit the ω^2-term drops out.

Then $k \cong \beta$, so that $\beta^2 \cong \frac{2\pi\sigma\omega}{c^2}$.

$$\boxed{\delta = \frac{1}{\beta} \cong \frac{c}{\sqrt{2\pi\sigma\omega}}}$$ Skin depth

$\boxed{\sqrt{\frac{2}{\mu_0\sigma\omega}} \text{ in MKS}}$

$$\vec{E} = E_0\hat{x}\, e^{i(kz-\omega t)}\cdot e^{-\beta z}$$

Attenuation factor

Problem 3 (a) It will eventually radiate due to its coupling to the electromagnetic field.

Even with the external field, it is an oscillator,

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 \vec{r}^2 - q E_0 x$$

$$= \frac{p^2}{2m} + \frac{1}{2} m \omega^2 \left(\vec{r} - \hat{x} a\right)^2 - \frac{1}{2} m \omega^2 a^2$$

where a = equilibrium displacement $= \frac{q E_0}{m \omega^2}$

$$E_{g.s.} = \frac{3}{2} \hbar \omega - \frac{1}{2} m \omega^2 a^2$$

(b) local potential $V(r) = \frac{1}{2} m \omega^2 \left(\vec{r} - \hat{x} a\right)^2 - \frac{1}{2} m \omega^2 a^2$

$$V(\vec{r}) - E_{gs} = \frac{1}{2} m \omega^2 \left(\vec{r} - \hat{x} a\right)^2 - \frac{3}{2} \hbar \omega$$

Probability that $V - E > 0$ = probability of being in classically forbidden region

$$= \mathrm{Prob}\left[\left(\vec{r} - \hat{x} a\right)^2 > 3\hbar / m\omega\right]$$

$$= \int d^3r \left|\phi_{gs}(r)\right|^2 \quad \text{over region } \left|\vec{r} - \hat{x} a\right| > \sqrt{\frac{3\hbar}{m\omega}}$$

$$\phi_{gs}(r) = C e^{-(\vec{r} - \hat{x} a)^2 b^2} \quad \text{for a harmonic oscillator}$$

3(b) continued) Use schrödinger equation to find b:

$$(H - e_{gs})\phi = 0 = \left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2(\vec{r}-\hat{x}a)^2 - \frac{1}{2}m\omega^2 a^2 - e_{gs}\right)\phi$$

$$= \left[-\frac{\hbar^2}{2m}\left(-6Cb^2 + 4Cb^4(\vec{r}-\hat{x}a)^2\right)\right]\phi + \left[\frac{1}{2}m\omega^2(\vec{r}-\hat{x}a)^2 - \frac{3}{2}\hbar\omega\right]\phi$$

$$\Rightarrow \begin{cases} 4Cb^4\left(-\frac{\hbar^2}{2m}\right) + \frac{1}{2}m\omega^2 = 0 & \Rightarrow b^2 = \frac{m\omega}{2\hbar} \\ -6Cb^2\left(-\frac{\hbar^2}{2m}\right) - \frac{3}{2}\hbar\omega = 0 & \Rightarrow b^2 = \frac{m\omega}{2\hbar} \end{cases}$$ ✓

∴ classically forbidden region is $(r-\hat{x}a)^2 > 3/b^2$

define $\vec{r}' \equiv \vec{r} - \hat{x}a$

then $$\text{Probability} = \frac{4\pi\int_{\sqrt{3}/b}^{\infty} r'^2 dr'\, e^{-r'^2 b^2}}{4\pi\int_0^{\infty} r'^2 dr'\, e^{-r'^2 b^2}}$$

3(c) Now you have a new oscillator, undisplaced ⇒

$$E = \left(N + \frac{3}{2}\right)\hbar\omega, \quad N = 0, 1, 2, \ldots$$

actually only n_x can change but it can have any value so all values of $N = n_x + n_y + n_z$ are possible.

(d) $$i\hbar\frac{d}{dt}\langle\vec{R}\rangle = \langle[H,\vec{R}]\rangle = \left\langle\frac{\vec{P}}{m}\right\rangle i\hbar$$

$$i\hbar\frac{d}{dt}\langle\vec{P}\rangle = \langle[H,\vec{P}]\rangle = -m\omega^2\langle\vec{R}\rangle i\hbar$$

$$\frac{d^2\langle\vec{R}\rangle}{dt^2} = \frac{1}{m}\frac{d\langle\vec{P}\rangle}{dt} = -\omega^2\vec{R}$$

$$\vec{R} = \vec{\alpha}\sin\omega(t-t_0) + \vec{\beta}\cos\omega(t-t_0)$$

$$\vec{P} = m\frac{d\langle\vec{R}\rangle}{dt} = m\omega\alpha\cos\omega(t-t_0) - m\omega\vec{\beta}\sin\omega(t-t_0)$$

$$\langle \vec{R} \rangle_{t_0} = a\hat{x} = \vec{\beta}, \quad \langle \vec{P} \rangle_{t_0} = 0 = m\omega\vec{\alpha}$$

$$\text{so } \langle \vec{R} \rangle = a\hat{x} \cos \omega (t - t_0)$$

$$\langle \vec{P} \rangle = -m\omega a \hat{x} \sin \omega (t - t_0)$$

(e) Expectation values, indeed all matrix elements, of operators oscillate with the Bohr frequencies $\omega_{nm} = (E_n - E_m)/\hbar$. $\vec{R}$ and $\vec{P}$ only have matrix elements for $n - m = \pm 1$ ie $\omega_{nm} = \pm\omega$ (see formula sheet). This is just the frequency of the classical oscillator.

In general, Ehrenfest's equations give the classical equations of motion for narrow wave packets; for the harmonic oscillator, they are exact for all wave packets.

[redacted]

Problem 4:
$$Z_N(V,T) = \frac{1}{N!\, h^{3N}} \int d^{3N}p \, d^{3N}q \; e^{-\beta H}$$

$$H = \sum_{i=1}^{N} \frac{\bar{p}_i^2}{2m} + \frac{1}{2} m \omega^2 \sum_{i=1}^{N} (\bar{r}_i - \bar{r}_{0i})^2 .$$

Coordinate transformation:
$$\bar{q}_i = \sqrt{\frac{\beta}{2m}}\; \bar{p}_i$$
$$\bar{s}_i = \sqrt{\frac{\beta m \omega^2}{2}}\; (\bar{r}_i - \bar{r}_{0i})$$

$$\Rightarrow \quad Z_N(V,T) = \frac{1}{N!\, h^{3N}} \int d^{3N}q \, d^{3N}s \left(\frac{2m}{\beta}\right)^{3N/2} \left(\frac{2}{\beta m \omega^2}\right)^{\frac{3N}{2}} e^{-\sum_{i=1}^{N} \bar{q}_i^2 - \sum_{i=1}^{N} \bar{s}_i^2}$$

$$I = \int d^3x \; e^{-x^2}$$

$$\Rightarrow \quad Z_N(V,T) = \frac{1}{N!} \frac{1}{h^{3N}} \frac{2^{3N}}{\beta^{3N}} \frac{1}{\omega^{3N}} I^{2N}$$

(a) $F = -kT \ln Z = -kT \ln(N!) - 3NkT \ln\left(\frac{2kT}{h\omega}\right)$

$$- kT\, 2N \ln I$$

(b) $$U = -\frac{\partial}{\partial\beta}\ln Z$$

$$= \frac{3N}{\beta} = 3NkT$$

(c) $$C_v = \left(\frac{\partial U}{\partial T}\right)_V = 3Nk$$

(d) $$\omega = \alpha/V \qquad P = -\left(\frac{\partial F}{\partial V}\right)_T$$

$$\Rightarrow P = -\frac{\partial}{\partial V}\left\{3NkT\ln\omega\right\}$$

$$= 3NkT\,\frac{1}{\omega}\left(-\frac{\partial\omega}{\partial V}\right)$$

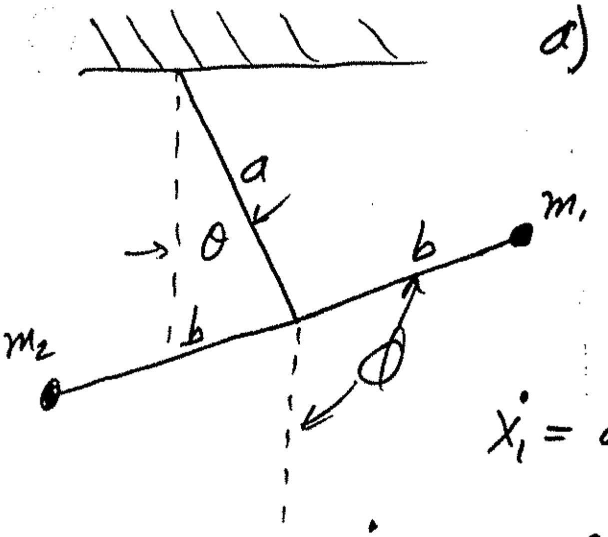
$$= 3NkT/V$$

(e) $$C_p = \left(\frac{\partial U}{\partial T}\right)_p + P\left(\frac{\partial V}{\partial T}\right)_p = C_v + P\left(\frac{\partial V}{\partial T}\right)_p$$

$$V = \frac{3NkT}{P} \Rightarrow \left(\frac{\partial V}{\partial T}\right)_p = \frac{3Nk}{P}$$

$$\Rightarrow C_p = C_v + 3Nk = 2C_v$$

Problem 5 grad mech.



a)
$$x_1 = a\sin\theta + b\sin\phi$$
$$y_1 = a(1-\cos\theta) - b\cos\phi$$
$$x_2 = a\sin\theta - b\sin\phi$$
$$y_2 = a(1-\cos\theta) + b\cos\phi$$

$$\dot x_1 = a\cos\theta\,\dot\theta + b\cos\phi\,\dot\phi,\quad \dot y_1 = a\sin\theta\,\dot\theta + b\sin\phi\,\dot\phi$$

$$\dot x_2 = a\cos\theta\,\dot\theta - b\cos\phi\,\dot\phi,\quad \dot y_2 = a\sin\theta\,\dot\theta - b\sin\phi\,\dot\phi$$

$$V = m_1 g y_1 + m_2 g y_2 .\quad \mathcal{L} = \tfrac12 m_1(\dot x_1^2+\dot y_1^2) + \tfrac12 m_2(\dot x_2^2+\dot y_2^2) + \\ - m_1 g y_1 - m_2 g y_2$$

$$\dot x_2^2 + \dot y_2^2 = a^2\dot\theta^2 + b^2\dot\phi^2 - 2ab\,\dot\theta\dot\phi\cos(\theta-\phi)$$
$$\dot x_1^2 + \dot y_1^2 = a^2\dot\theta^2 + b^2\dot\phi^2 + 2ab\,\dot\theta\dot\phi\cos(\theta-\phi)$$

$$\frac{\partial\mathcal{L}}{\partial\dot\theta} = m_1\left(a^2\dot\theta + ab\,\dot\phi\cos(\theta-\phi)\right) + m_2\left(a^2\dot\theta - ab\,\dot\phi\cos(\theta-\phi)\right)$$

$$\partial\mathcal{L}/\partial\theta = -m_1 ab\,\dot\theta^2\dot\phi\sin(\theta-\phi) + m_2 ab\,\dot\theta^2\dot\phi\sin(\theta-\phi) + \\ - m_1 g a\sin\theta - m_2 g a\sin\theta$$

$$\frac{d}{dt}\left\{m_1\left[a^2\dot\theta + ab\dot\phi\cos(\theta-\phi)\right] + m_2\left[a^2\dot\theta - ab\dot\phi\cos(\theta-\phi)\right]\right\} + \\ -(m_2-m_1)ab\,\dot\theta^2\dot\phi\sin(\theta-\phi) + (m_1+m_2)g a\sin\theta = 0$$

b) If $m_1 = m_2 = m$ the cross terms in the kinetic energies cancel, and

$$\mathcal{L} = m(a^2\dot{\theta}^2 + b^2\dot{\phi}^2) - mg(y_1+y_2)$$

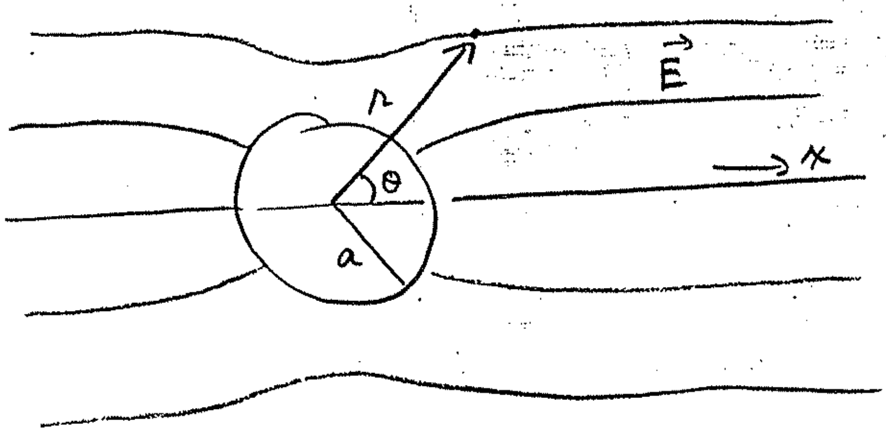
$$= m(a^2\dot{\theta}^2 + b^2\dot{\phi}^2) - 2mg(1-\cos\theta)a$$. Thus, $\mathcal{L}$ is cyclic in ϕ, and $\frac{d}{dt}(2mb^2\dot{\phi}) = 0$, or $L_\phi = \text{const}$

$$\frac{d}{dt}(2ma^2\dot{\theta}) + 2mga\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{a}\sin\theta = 0$$

c) For small oscillations, $\sin\theta \simeq \theta$, and $\theta \simeq \theta_0\cos(\omega t + \alpha)$, where $\omega = \sqrt{\frac{g}{a}}$ and θ_0 and α are determined by initial conditions.

6.



$\vec{E} = \vec{E}_0 = E_0\hat{x}$ at $r \to \infty$.

Take z-axis along cylinder axis.

Then $\frac{\partial^2 \phi}{\partial z^2} = \frac{\partial \phi}{\partial z} = 0$.

(a)

$$\nabla^2 \phi = 0 \implies \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Separation: Let $\phi = F(r)\cdot G(\theta)$.

Then $\frac{G}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right) + \frac{F}{r^2}\frac{d^2 G}{d\theta^2} = 0$

$$\therefore \frac{r}{F}\frac{d}{dr}\left(r\frac{dF}{dr}\right) = -\frac{1}{G}\frac{d^2 G}{d\theta^2} \equiv k^2 \quad \text{separation constant.}$$

Angular equation: $\frac{d^2 G}{d\theta^2} + k^2 G = 0 \implies G = \sin k\theta$ or $\cos k\theta$.

$k = 0, 1, 2, \ldots$ for continuity of $G(\theta)$.

Radial equation: $r\frac{d}{dr}\left(r\frac{dF}{dr}\right) = k^2 F$.

If $F(r) = r^\alpha$, then $r\frac{d}{dr}\left(\alpha r^\alpha\right) = k^2 r^\alpha$

or $\alpha^2 r^\alpha = k^2 r^\alpha \implies \alpha = \pm k$

If $k = 0$, then $F(r) = a_0 + a_1 \ln r$

$$\therefore \phi(r,\theta) = a_0 + a_1 \ln r + \sum_{k=1}^{\infty}\left[r^k\left(b_k \cos k\theta + c_k \sin k\theta\right) + r^{-k}\left(d_k \cos k\theta + e_k \sin k\theta\right)\right]$$

General solution form.

Boundary conditions: $\phi(a,\theta) = 0$ and $\lim_{r\to\infty} \vec{\nabla}\phi = -E_0\hat{x}$.

The condition $\phi(a,\theta) = 0$ implies $\begin{cases} a^k b_k + a^{-k} d_k = 0 \\ a^k c_k + a^{-k} e_k = 0 \end{cases}$

and also $a_0 + a_1 \ln a = 0$.

$$\therefore \phi(r,\theta) = a_1 \ln \frac{r}{a} + \sum_{k=1}^{\infty} \left[r^k - \left(\frac{a^2}{r}\right)^k \right] (b_k \cos k\theta + c_k \sin k\theta)$$

$\hat{x} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$.

The condition $\lim_{r\to\infty} \vec{\nabla}\phi = -E_0 \hat{x}$ implies $\begin{cases} ① \lim_{r\to\infty} \frac{\partial \phi}{\partial r} = -E_0 \cos\theta \\ ② \lim_{r\to\infty} \frac{1}{r}\frac{\partial \phi}{\partial \theta} = +E_0 \sin\theta . \end{cases}$

From equation ①:

$$\lim_{r\to\infty} \left[\frac{a_1}{r} + \sum_{k=1}^{\infty} k r^{k-1} (b_k \cos k\theta + c_k \sin k\theta) \right] = -E_0 \cos\theta$$

only the $k=1$ terms contribute: $b_1 = -E_0$; $b_{k>1} = 0$;

all $c_k = 0$.

$$\therefore \phi(r,\theta) = a_1 \ln \frac{r}{a} - E_0 \left(r - \frac{a^2}{r} \right) \cos\theta .$$

From equation ②:

$$\lim_{r\to\infty} \frac{1}{r} \left[-E_0 \left(r - \frac{a}{r}\right) (-\sin\theta) \right] = + E_0 \sin\theta$$ already satisfied.

The logarithmic term is the potential due to the net charge.

$$q = 0 \Rightarrow a_1 = 0 .$$

$$\therefore \boxed{\phi(r,\theta) = E_0 \left(\frac{a^2}{r} - r \right) \cos\theta}$$ Electric potential.

(b) $$4\pi\sigma = E_r\Big|_{r=a} = -\frac{\partial \phi}{\partial r}\Big|_{r=a} = -E_0 \left(-\frac{a^2}{r^2} - 1 \right)\Big|_{r=a} \cos\theta$$

Gauss' law

$$\therefore \boxed{\sigma = \frac{E_0}{2\pi} \cos\theta}$$ Surface charge density (at $r=a$) in Gaussian units.

Problem 7

1. a. $2j,\ 2j-1,\ 0$ ~~...~~ triangle rule 

b. $M = -J, -J+1, \cdots J$ upto $2j$

c. $M_{max} = 2j$

$|n_r n_r \cancel{l} \ j_p j_n \ \cancel{J}, J = 2j, M = 2j\rangle$

$= |n_r l j_p j_p\rangle |n_r l j_p j_p\rangle$

d. now has to be antisymmetric $\Rightarrow$ state $2j$ is impossible

$J = 2j-1$, $M_{max} = J = 2j-1$

to get state, first get state $J = 2j$, $M = 2j-1$, ~~then fix~~

$$|J, J-1\rangle = \frac{1}{\hbar\sqrt{(J+M)(J-M+1)}} J_- |J, J\rangle \quad \text{for } M = J = 2j$$

$$= \frac{1}{\hbar\sqrt{4j}} \left((J_1 + J_2) |jj\rangle_1 |jj\rangle_2 \right)$$

$$= \frac{\hbar}{\hbar\sqrt{4j}} \sqrt{2j} \left(|j\,j-1\rangle_1 |jj\rangle_2 + |jj\rangle_1 |j\,j-1\rangle_2 \right)$$

$$= \frac{1}{\sqrt{2}} \left(\quad + \quad \right)$$

$$|J-1, J-1\rangle = \frac{1}{\sqrt{2}} \left(\quad - \quad \right)$$

$$= \frac{1}{\sqrt{2}} \left(|jj\rangle_1 |j\,j-1\rangle_2 - |j\,j-1\rangle_1 |jj\rangle_2 \right)$$

$\times$ arbitrary phase

Problem 8:

$$\mu = kT \ln(n/n_Q)$$

$$n = N/V \qquad n_Q = (m kT/2\pi\hbar^2)^{3/2}$$

With gravity: $$\mu = kT \ln(n(h)/n_Q) + mgh$$

Equilibrium:

$$\Rightarrow \quad kT \ln(n(0)/n_Q) = kT \ln(n(h)/n_Q) + mgh$$

$$\Rightarrow \quad \frac{n(0)}{n_Q} = \frac{n(h)}{n_Q} e^{mgh/kT}$$

$$\text{or} \quad n(h) = n(0)\, e^{-mgh/kT}$$

$$pV = NkT \qquad p = nkT$$

$$\Rightarrow \quad p(h) = p(0)\, e^{-mgh/kT}$$

(b) $$Z_N(V,T) = \sum_{\{n_1 \dots n_N\}} e^{-\beta \sum_{i=1}^{N} (n_i + \frac{1}{2}) \hbar\omega}$$

$$= \prod_{i=1}^{N} \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\hbar\omega} = \left\{ \frac{e^{-\beta\frac{1}{2}\hbar\omega}}{1 - e^{-\beta\hbar\omega}} \right\}^N$$

$$\Rightarrow \quad F = -NkT \ln \left\{ \frac{e^{-\frac{1}{2}\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} \right\}$$

$$= \frac{N}{2}\hbar\omega + NkT \ln \left\{ 1 - e^{-\beta\hbar\omega} \right\}$$