PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #61

March 31, 1990

Comprehensive Examination for Spring 1990

PART I

General Instructions

This Comprehensive Examination for Spring 1990 (#61) consists of eight problems of equal weight (20 points each). It has two parts. The first part (Problems 1-4) is handed out at 9:00 am (duration: 3 hours) and the second part (Problems 5-8) at 1:00 pm (duration 3 hours).

Work carefully, indicate your reasoning briefly, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers, except the exam and bluebook, on the floor. Please return the bluebooks at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Physics Department Comprehensive Exam #61, March 31, 1990

PART I

Problem #1

Calculate the largest possible number of pions that could be produced by p-p scattering using a 500 MeV proton beam and a fixed target. Take the pion mass to be 140 MeV and the proton mass as 938 MeV.

If the same proton beam were used to fill a storage ring and the two beams (one is the primary proton beam and the other is from the storage ring) were allowed to collide head-on how many pions could be produced per collision?

Problem #2

A spinless particle of mass m moves nonrelativistically in two dimensions in the potential

$$U(x,y) = -U_0 e^{-(x^2+y^2)/R^2} > 0.$$

(a) Which of the following quantities are conserved? Energy E, momentum components

$$p_x\ ,\ p_y\ ,\ L_z \equiv x\, p_y - y\, p_x\ ,\ \text{parity } \Pi\ ?$$

(b) Find a Complete Set of Commuting Observables (C.S.C.O.) which includes the Hamiltonian. Define the corresponding basis of eigenstates of the C.S.C.O. What are the eigenvalues? What are the orthonormality and completeness relations?

(c) A beam of particles with energy E, moving along the x-axis, is incident on the above potential. The scattering wave function has the asymptotic form

$$\phi_k(x,y) = e^{ikx} + \frac{f(\theta)}{(x^2+y^2)^\alpha}\, e^{ik\sqrt{x^2+y^2}} \quad \text{for } x^2+y^2 \gg R^2$$

where $\tan\theta = y/x$. Find the value of α.

Problem #3

A heat engine takes an amount of heat Q_H from a reservoir at a high temperature T_H and deposits an amount of heat Q_L in a reservoir at a low temperature T_L. The entropy taken from the high temperature reservoir is S_H and the entropy generated in the engine is $S_H(T_H-T_L)/(T_H+T_L)$. Not all heat is available for work, and amount of $A(T_H-T_L)Q_H$ is transferred directly to the low temperature reservoir (A is a constant).

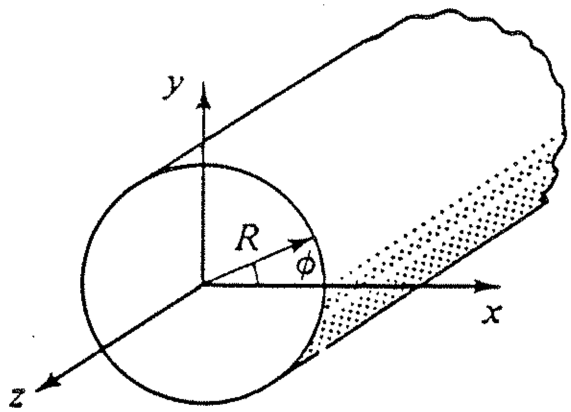
(a) Calculate the efficiency of this heat engine.

(b) What is the maximum value of T_H for which this engine will work (i.e. produce a positive amount of work).

(c) What is the maximum value of T_L for which this engine will work.

(d) Will this engine always work for $T_L<T_H$?

Problem #4



Charge density $\sigma = \sigma_o \sin\phi$ is "glued" over the surface of the infinite cylindrical shell of radius R shown in the figure.

(a) Find the electrostatic potential inside and outside of the infinite cylindrical surface.

(b) The region inside the cylinder is now filled with a homogeneous linear isotropic dielectric of susceptibility χ and permitivity ϵ. Determine the electric field within the dielectric and the polarization charge density on the surface of the dielectric.

(c) All of the dielectric, and the lower half of the cylindrical shell, are now removed. The upper half of the cylindrical shell is <u>replaced</u> by an identical conducting shell and the x-z plane is covered with an infinite conducting sheet. The half cylindrical conducting shell is held at a constant potential, V_o, and the conducting sheet is grounded. Determine the electrostatic potential everywhere outside the shell and above the x-z plane.

PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #61

March 31, 1990

Comprehensive Examination for Spring 1990

PART II

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PART II

Problem #5

N identical fermions (mass M) are free to move in a cubical box of volume $V=L^3$. The orbitals are designated by quantum numbers n_x, n_y, and n_z.

(a) Give an expression for the energy of the orbitals.

(b) Calculate the energy of this system of fermions at $T=0$. (N is very large)

(c) Calculate the pressure this system exerts on the walls at $T=0$.

(d) The gravitational self energy of this system is $-GN^2M^2/L$ where G is a constant. Calculate the pressure on the walls at $T=0$ including gravitational effects. What happens to this system when the walls are removed?

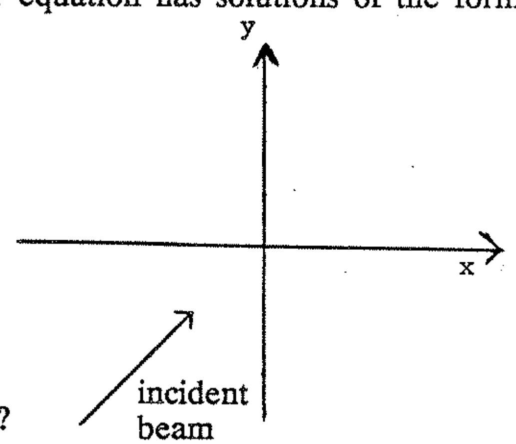
Problem #6

A spinless particle of mass *m* moves non-relativistically in two dimensions in the potential

$$V(x,y) = \begin{cases} -V_0 < 0 & \text{when } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Which of the following quantities are conserved? Energy *E*, momentum components p_x , p_y , $L_z \equiv x\,p_y - y\,p_x$, parity Π .

(b) Prove that the time-independent Schrödinger equation has solutions of the form $\phi(x,y) = f(x)\,g(y)$. What equations do f(x) and g(y) satisfy?



(c) A beam of particles of energy $E>0$ is incident from the left at an angle of 45°. What fraction of the incident particles are reflected? In what direction do the reflected particles move?

(d) For the incident beam of part c, what fraction of the incident particles are transmitted? In what direction do they move?

(e) The incident beam contains 1 particle per second per square centimeter. How many particles per cubic centimeter will be found at $x = x_0$, when (i) $x_0>0$? (ii) $x_0<0$?

Problem #7

An infinitely long solid cylinder of radius R is placed in a uniform static applied magnetic field, $\mathbf{B}_a$, with the axis of the cylinder parallel to $\mathbf{B}_a$ (Take this axis to be the z-axis of your coordinate system.). The cylinder is constructed of a homogeneous linear isotropic magnetic material of susceptibility χ_m and permeability μ.

(a) Find the magnetic field inside and outside the cylinder, find the magnetization within the cylinder, and determine the surface current density.

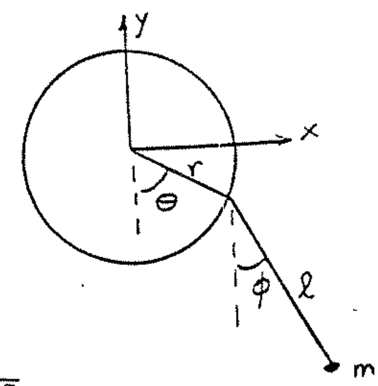
(b) Calculate the vector potential outside the cylinder.

(c) We now allow $\mathbf{B}_a$ to oscillate, so that $\mathbf{B}_a = B_0\,\hat{z}\sin\omega t$. The material of the cylinder remains linear. Ignoring the Maxwell displacement current (MDC) and retardation effects, calculate the Poynting vector, and its time average, outside the cylinder.

(d) Use the MDC to examine a possible first correction to the results of part (c). Do this calculation for the limit of small R. Thus, the only part played by the cylinder in this part of the problem is to establish symmetry.

Problem #8

A low-flying airplane carries a simple pendulum of mass m and length l (in order to measure the acceleration of gravity). To avoid a hill, the pilot flies the plane at constant speed v in a vertical circle of radius r. Assume the pendulum oscillates in the plane of the circle.



(a) Write the Lagrangian and derive the exact equation of motion for the angle ϕ between the pendulum and the vertical.

(b) Approximate the equation of motion for small oscillations. Can it be solved?

(c) Impose and interpret the additional restriction $\frac{v}{r} \ll \sqrt{\frac{g}{l}}$ and use it to further simplify the equation of motion. Now find the angular frequency for small oscillations. Interpret it in a simple physical way. Note: the answers to this part may depend on time.

Calculate the largest possible number of pions that could be produced by p-p scattering using a 500 MeV proton beam and a fixed target.

#1

1. $p + p \rightarrow p + p + N\pi$

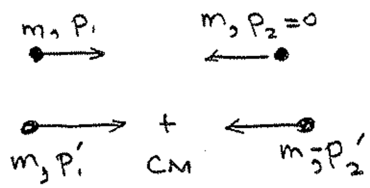
m = proton rest mass

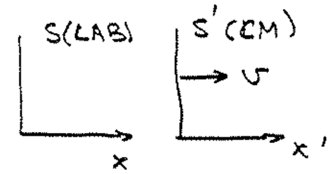
$mc^2 = 938$ MeV

conservation of momentum requires $p_1' = -p_2' \equiv p$

In LAB-frame S

IN CM-frame S'





v = velocity of S' as measured from S

The Lorentz Transformations are

$$p_x' = \gamma\left(p_x - \frac{v}{c^2}E\right)$$

$$E' = \gamma\left(E - v\,p_x\right)$$

In Zero Momentum Frame S'

$$p_1' = -p_2'$$

$$\gamma\left(p_1 - \frac{v}{c^2}E_1\right) = \gamma\frac{v}{c^2}E_2$$

$$p_1 - \frac{vE_1}{c^2} = \frac{v}{c^2}mc^2$$

$$p_1c - \beta E_1 = \beta mc^2 \quad : E_1 = 500 + 938$$

$$\beta = \frac{p_1c}{E_1 + mc^2} = \frac{1.09}{1.438 + .938} = 0.459$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 1.126$$

<u>for proton 1</u>

$$p_1' = \gamma\left(p_1 - \frac{v}{c^2}E_1\right) \quad ; \quad E_1' = \gamma\left(E_1 - \beta p_1 c\right)$$

$$p_1c = \sqrt{E_1^2 - mc^2}$$

$$E_1 = 1438$$

$$p_1c = \sqrt{(1.438)^2 - (.938)^2} = 1.09 \text{ GeV}$$

<u>for proton 2</u>

$$p_2' = \gamma\left(p_2 - \frac{v}{c^2}E_2\right) \quad ; \quad E_2' = \gamma\left(E_2 - \beta p_2 c\right)$$

$$p_2'c = -\gamma\beta mc^2 \quad \therefore \quad E_2' = \gamma mc^2$$

The total Energy in the CM-frame

$$E = E_1' + E_2' = 2E_2' = 2\gamma mc^2 = 2(1.126)(938) = 2{,}112 \text{ MeV}$$

max Energy avilable to create pions $= E - 2(mc^2) = 2112 - (2)(938) = 236$ MeV

since the pion rest energy is 140 MeV. Only <u>one pion can be produced</u>

<u>Another Method</u> is to consider the energy $E_{LAB} = 500 + (2)(938) = 2{,}376$ MeV to be avilable in the S-frame. In the S'-frame, $E_{CM} = \frac{E_{LAB}}{\gamma} = \frac{2376}{1.126} = 2{,}110$

the Energy availabe for the Pion system is $E_{CM} - 2(mc^2) = 2110 - 2(938) = 234$ Me

#6.

[illegible]

(a) E, L_z, Π

(b) H, L_z, $\quad H|Em\rangle = E|Em\rangle$

$L_z|Em\rangle = m\hbar|Em\rangle$

$0 \le E < \infty$

$m = 0, \pm 1, \pm 2, \cdots$

orthonorm $\langle Em|E'm'\rangle = \delta(E-E')\delta_{mm'}$

completeness : $\displaystyle\int_0^\infty dE \sum_{m=-\infty}^{\infty} |Em\rangle\langle Em| = \mathbb{1}$ = unit operator

(c) outgoing particles per second $= \displaystyle\oint_{\text{circle at } r} \vec{J}\cdot d\vec{A}_{rea}$

$\cancel{= v_{radial}\int_0^{2\pi} d\theta\, |f(\theta)|^2}$

$\simeq v_{radial}\displaystyle\int_0^{2\pi} d\theta\, \frac{|f(\theta)|^2}{(x^2+y^2)^{2\alpha}}$

+ terms that fall off faster in $r = \sqrt{x^2+y^2}$

$= \left(v_{radial}\displaystyle\int_0^{2\pi} |f(\theta)|^2\right)\left(\frac{r}{r^{2\alpha}}\right)$ independent of r

$\Rightarrow 2\alpha = 1 \quad \Rightarrow \alpha = 1/2$

Problem 3.

a $\quad W = Q_H - Q_L - A(T_H - T_L)Q_H$

$$\eta = \frac{W}{Q_H} = 1 - \frac{Q_L}{Q_H} - A(T_H - T_L)$$

$$\frac{Q_L}{Q_H} = \frac{T_L}{T_H}\frac{S_L}{S_H} = \frac{T_L}{T_H}\left[1 + \frac{T_H - T_L}{T_H + T_L}\right]$$

$$= \frac{2T_L}{T_H + T_L}$$

$$\Rightarrow \eta = 1 - \frac{2T_L}{T_H + T_L} - A(T_H - T_L)$$

$$= (T_H - T_L)\left\{\frac{1}{T_H + T_L} - A\right\}$$

b $\quad \left.\begin{matrix} T_L < T_H \\ \eta > 0 \end{matrix}\right\} \Rightarrow \frac{1}{T_H + T_L} > A \qquad T_H + T_L < \frac{1}{A}$

$$T_L > 0 \Rightarrow T_H < \frac{1}{A}$$

c $\quad 2T_L < T_L + T_H < \frac{1}{A} \qquad \Rightarrow T_L < \frac{1}{2A}$

d $\quad$ No. $\quad$ if $T_H = \frac{1}{2A} + x \qquad (x > 0)$

then $\quad T_L < \frac{1}{A} - T_H = \frac{1}{2A} - x$

#4

a) Label the potential function for $\rho \geq R$, $V_{out}(\rho, \phi)$, and that for $\rho \leq R$, $V_{in}(\rho, R)$. Since $\left.\frac{\partial V_{out}}{\partial \rho}\right|_{\rho=R} - \left.\frac{\partial V_{in}}{\partial \rho}\right|_{\rho=R} = -\frac{\sigma}{\epsilon_0}$,

we assume $V_{out} = \frac{RA_o}{\rho}\sin\phi$ and $V_{in} = \frac{A_i \rho}{R}\sin\phi$

$V_{out}(R,\phi) = V_{in}(R,\phi)$; $\therefore$ $A_o = A_i = A$.

Also, $-\frac{\sigma_0 \sin\phi}{\epsilon_0} = -\left.\frac{RA}{\rho^2}\right|_{\rho=R}\sin\phi - \frac{A}{R}\sin\phi$, or, $A = \frac{R\sigma_0}{\epsilon_0}$.

Thus, $V_o = \frac{R^2\sigma_0}{\rho\,\epsilon_0}\sin\phi$ and $V_{in} = \frac{\sigma_0 \rho}{\epsilon_0}\sin\phi$

b) $\vec{\nabla} V_{in} = \frac{\sigma_0}{\epsilon_0}\sin\phi\,\hat{\rho} + \frac{\sigma_0}{\epsilon_0}\cos\phi\,\hat{\phi} = \frac{\sigma_0}{\epsilon_0}\hat{y}$; $\therefore$,

$\vec{E}_{in} = -\frac{\sigma_0}{\epsilon_0}\hat{y}$. This uniform applied field polarizes the dielectric uniformly. $\vec{P} = \chi\vec{E}$, where $\vec{E} = \frac{\vec{D}}{\epsilon} = \frac{\epsilon_0 \vec{E}_{in}}{\epsilon}$

$\sigma_p = \vec{P}\cdot\hat{\rho} = \frac{\chi}{1+\chi}\vec{E}_{in}\cdot\hat{\rho} = -\frac{\chi}{1+\chi}\frac{\sigma_0}{\epsilon_0}\sin\phi$

c) $V(\rho,\phi) = \sum_{m=1}^{\infty} A_m \left(\frac{a}{\rho}\right)^m \sin m\phi$ satisfies $V(\rho,0) = V(\rho,\pi) = 0$

Since we want only maxima at $\phi = \frac{\pi}{2}$, we limit m to the odd numbers 1, 5, 9, ..., or $4n+1$, where $n = 0, 1, 2, \ldots$

It is easy to show that the functions $\sin m\phi$ are orthogonal for $0 \leq \phi \leq \pi$; $\therefore$, $V_0 \int_0^{\pi} \sin m\phi\, d\phi = A_m \left(\frac{a}{R}\right)^m \int_0^{\pi} \sin^2 m\phi\, d\phi$,

or, $\frac{2V_0}{m} = A_m \left(\frac{a}{R}\right)^m \frac{\pi}{2}$, $A_m = \frac{4V_0}{m\pi}\left(\frac{R}{a}\right)^m$. Thus,

$$V(\rho,\phi) = 4\frac{V_0}{\pi}\sum_{m\text{ odd, selected}} \left(\frac{R}{\rho}\right)^m \frac{\sin m\phi}{m} = 4\frac{V_0}{\pi}\sum_{n=0}^{\infty}\left(\frac{R}{\rho}\right)^{4n+1}\frac{\sin[(4n+1)\phi]}{4n+1}$$

Problem 5.

a $\quad \psi = A \sin \frac{\pi n_x x}{L} \sin \frac{\pi n_y y}{L} \sin \frac{\pi n_z z}{L}$

$$\Rightarrow E = \frac{\hbar^2}{2M} \left(\frac{\pi}{L}\right)^2 (n_x^2 + n_y^2 + n_z^2) \qquad n_x, n_y, n_z > 0$$

b

$$U = \sum_{n^2 = n_x^2 + n_y^2 + n_z^2 \leq n_F^2} \frac{\hbar^2}{2M} \left(\frac{\pi}{L}\right)^2 (n_x^2 + n_y^2 + n_z^2)$$

$$N = \sum_{n < n_F} 1$$

N large $\Rightarrow \quad N = \frac{1}{8} \frac{4\pi}{3} n_F^3 = \frac{\pi}{6} n_F^3$

$$U = \frac{\hbar^2}{2M} \left(\frac{\pi}{L}\right)^2 \frac{1}{8} \int_0^{n_F} 4\pi n^2 dn \; n^2 = \frac{\hbar^2}{2M} \left(\frac{\pi}{L}\right)^2 \frac{\pi}{10} n_F^5$$

$$\varepsilon_F = \frac{\hbar^2}{2M} \left(\frac{\pi}{L}\right)^2 n_F^2 \qquad \left(\propto V^{-2/3}\right)$$

$$\Rightarrow U = \frac{3}{5} N \varepsilon_F$$

c $\quad F(T=0) = U$

$$p = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = -\frac{3}{5} N \left(\frac{\partial \varepsilon_F}{\partial V}\right)_{T,N} = \frac{2}{3} \frac{U}{V}$$

d $\quad F(T=0) = U - GN^2M^2V^{-\frac{1}{3}}$

$$\Rightarrow p = \frac{2}{3} \frac{U}{V} - \frac{1}{3} GN^2M^2V^{-4/3}$$

equilibrium : $p = 0 \Rightarrow V = V_0 \qquad V \lessgtr V_0 \Rightarrow p \gtrless 0$: V_0 is stable.

#2

Undergrad quantum

(a) conserved: E, p_y

(b) $E\psi = H\psi = \left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + V(x,y)\right]\psi$

$Ef\,g = \left\{\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]f(x)\right\}g(y) + \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}g(y)\right\}f(x)$

okay if $\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]f(x) = E_x f(x)$

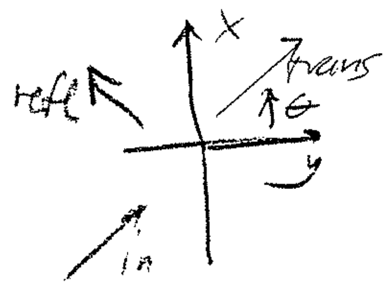
$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}\right) g(y) = E_y\, g(y)$ } eigenvalue equations

$E = E_x + E_y$

(c) direction: $45°$ back

$p_y|_{refl} = p_y|_{inc}$

$p_x|_{refl} = -p_x|_{inc}$



fraction:

$\psi_{inc}(x,y) = e^{ik_x x} e^{ik_y y}$ $\quad k_x^2 = k_y^2 = \frac{k^2}{2}$

where $\frac{\hbar^2 k^2}{2m} = E$

$\psi(x,y) = f(x)\, e^{ik_y y}$

$f(x) = \begin{cases} A e^{ik_x x} + B e^{-ik_x x} & x<0 \\ C e^{iK_x x} & x>0 \end{cases}$

$\frac{\hbar^2 K_x^2}{2m} = \frac{\hbar^2 k_x^2}{2m} - V_0$

$$f(x=0_+) = f(x \to 0_-)$$

$$A + B = C$$

$$\frac{\partial f}{\partial x}(x=0+) = \frac{\partial f}{\partial x}(x=0-)$$

$$ik_x(A-B) = iK_x C$$

$$ik_x(A-B) = iK_x(A+B)$$

$$(k_x - K_x)A = (K_x + k_x)B$$

$$B = \frac{k_x - K_x}{k_x + K_x} A$$

~~current~~ reflected fraction $= \frac{|B|^2}{A^2} = \frac{(k_x - K_x)^2}{(k_x + K_x)^2}$

d) Transmitted fraction $= 1 -$ Reflected, conservation of current

$$= 1 - \frac{(k_x - K_x)^2}{(k_x + K_x)^2}$$

direction $\tan\theta = \frac{K_x}{k_y} = \frac{K_x}{k_x}$ #

e) $\left(\text{incident current}\right)_{x\,\text{component}} = A^2 \frac{\hbar k_x}{m} = J_{ix} = \frac{1}{\sqrt{2}} J_i$

$\left(\text{transmitted current}\right)_{x\,\text{comp}} = C^2 \frac{\hbar K_x}{m} = J_{tx}$

probability density $(x_0 > 0) = |C|^2 = |A+B|^2$

$= |A|^2 \left(1 + \frac{k_x - K_x}{k_x + K_x} \right)^2$

$= J_{ix} \cdot \frac{m}{\hbar k_x} \left(\frac{2k_x}{k_x + K_x} \right)^2$

$= J_i \frac{4m}{\hbar\sqrt{2}} \frac{k_x}{(k_x + K_x)^2}$

for $x_0 < 0$, Prob density $= |Ae^{ikx} + Be^{-ikx}|^2$

$= |A|^2 + |B|^2 + 2AB\,\mathrm{Re}\,e^{2ikx}$

$= |A|^2 + |B|^2 + 2AB \cos kx$

$= |A^2| \left\{ 1 + \left(\frac{K_x - k_x}{k_x + K_x}\right)^2 \right\} + 2 \frac{k_x + K_x}{k_x + K_x} \cos x$

$= J_i \frac{4m}{\hbar\sqrt{2}} \{ \qquad \}$

#7.

a) Place the axis of the cylinder along the z-axis, and use cylindrical coordinates. $\vec{H} = \vec{B}_a / \mu_0$, everywhere. Outside the cylinder, $\vec{B}_0 = \vec{B}_a$. Within the cylinder,

$$\vec{B}_{in} = \mu \vec{H} = \frac{\mu}{\mu_0}\vec{B}_a . \quad \vec{M}_{in} = \chi \vec{H} = \frac{\chi}{\mu_0}\vec{B}_a$$

$$\vec{K} = \vec{M}_{in}\times\hat{n} = \frac{\chi}{\mu_0}B_a\,\hat{z}\times\hat{\rho} = \frac{\chi}{\mu_0}B_a\hat{\phi}.$$

b) $\oint \vec{A}\cdot d\vec{l} = \iint \vec{B}\cdot d\vec{a}$. within the cylinder, $2\pi\rho A_\phi^{(in)} = B_{in}\pi\rho^2$, or, $\boxed{A_\phi^{(in)} = \frac{\mu B_a}{2\mu_0}\rho}$. Similarly, $A_\phi^{(out)} = \frac{B_{in}\pi R^2 + B_a\pi(\rho^2 - R^2)}{2\pi\rho}$,

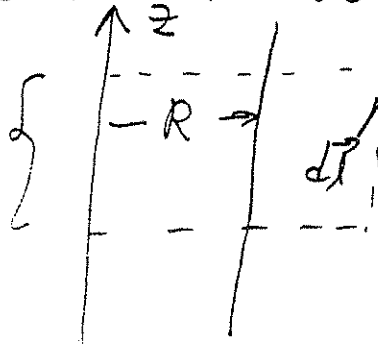
or, $A_\phi^{out} = \left[\left(\frac{\mu}{\mu_0} - 1\right)R^2 + \rho^2\right]\frac{B_a}{2\rho} = \frac{\chi R^2 + \rho^2}{2\rho}B_a$

c) $\vec{E}_{out} = -\frac{\partial \vec{A}_{out}}{\partial t}$; $E_\phi^{(out)} = -\frac{\chi R^2 + \rho^2}{2\rho}\dot{B}_a$

$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}$: $S_\rho = -\hat{\rho}\left(\frac{\chi R^2 + \rho^2}{2\rho}\right)^2 B_0^2\,\omega\sin\omega t\cos\omega t$

$\langle S_\rho(\rho)\rangle = 0$

d) The MDC, $\mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$, induces a correction to the $\vec{B}$-field: $\oint \vec{B}_c\cdot d\vec{l} = \mu_0\epsilon_0\iint\frac{\partial\vec{E}}{\partial t}\cdot d\vec{a}$



← Amperian loop.

Assume $\vec{B}_c(\rho = 0) = 0$. Then,

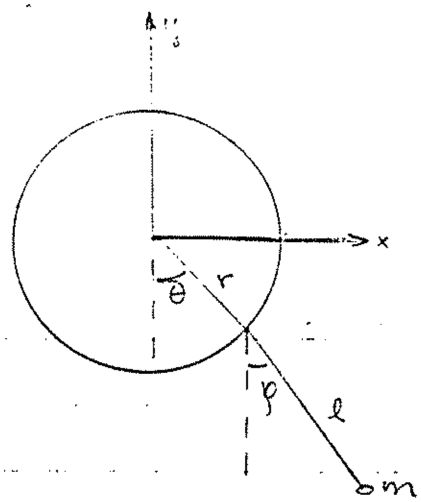
$$B_c l = \frac{\mu_0\epsilon_0\ddot{B}_a}{2}\int_0^\rho \rho\, l\, d\rho = l\frac{\ddot{B}_a}{4c^2}\rho^2$$

$$\vec{B}_c = -\hat{z}\frac{\rho^2}{4c^2}\omega^2 B_0\sin\omega t.$$

$$\vec{S}_c = \frac{\vec{E}\times\vec{B}_c}{\mu_0} = \hat{\rho}\frac{\omega^2 B_0^2\rho^3}{8\mu_0 c^2}\cos\omega t\sin\omega t.$$

Again, $\langle\vec{S}_c\rangle = 0$

#8.



The coordinates of the pendulum are

$x = r \sin \omega_0 t + l \sin \varphi$

$y = -r \cos \omega_0 t - l \cos \varphi$

$\therefore \begin{cases} \dot{x} = \omega_0 r \cos \omega_0 t + l \dot{\varphi} \cos \varphi \\ \dot{y} = r \omega_0 \sin \omega_0 t + l \dot{\varphi} \sin \varphi \end{cases}$

Thus

$\theta = \frac{v}{r} t = \omega_0 t$ $\qquad T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}\left[r^2 \omega_0^2 + l^2 \dot{\varphi}^2 \right.$

$\omega_0 \equiv \frac{v}{r}$ $\qquad \left. + 2 l r \omega_0 \dot{\varphi} (\cos\varphi \cos\omega_0 t + \sin\varphi \sin\omega_0 t) \right]$

$T = \frac{m}{2}\left[\omega_0^2 r^2 + l^2 \dot{\varphi}^2 + 2 l r \omega_0 \dot{\varphi} \cos(\varphi - \omega_0 t) \right]$

and $V = -mg\left[r \cos\omega_0 t + l \cos\varphi \right]$

the first term in T and V are functions of t only and may be discarded

$\therefore \quad L = \frac{m}{2}\left[l^2 \dot{\varphi}^2 + 2 l r \omega_0 \dot{\varphi} \cos(\varphi - \omega_0 t) \right] + mgl \cos\varphi$

Lagrange's equation yields

$$\frac{d}{dt}\left\{ m\left[l^2 \dot{\varphi} + l r \omega_0 \cos(\varphi - \omega_0 t) \right] \right\} + m l r \omega_0 \dot{\varphi} \sin(\varphi - \omega_0 t) + mgl \sin\varphi =$$

$$\therefore \quad l \ddot{\varphi} + r^2 \omega_0^2 \sin(\varphi - \omega_0 t) - r \omega_0 \dot{\varphi} \sin(\varphi - \omega_0 t) + r \omega_0 \dot{\varphi} \sin(\varphi - \omega_0 t) + g \sin\varphi = 0$$

$$\underbrace{l \ddot{\varphi} + g \sin\varphi}_{\text{simple pendulum}} + \underbrace{r \omega_0^2 \sin(\varphi - \omega_0 t)}_{\text{correction term}} = 0 \qquad \underline{\text{EXACT}}$$

b) $|\varphi| \ll 1 \Rightarrow \sin\varphi \simeq \varphi, \ \cos\varphi \simeq 1 \ ; \ \sin(\varphi - \omega_0 t) \simeq \varphi \cos\omega_0 t - \sin\omega_0 t$

$\therefore \ l \ddot{\varphi} + g \varphi + r^2 \omega_0^2 (\varphi \cos\omega_0 t - \sin\omega_0 t) = 0$ $\quad$ SHO Eq. with Driving Fo

c) If $\omega_0 \ll \sqrt{g/l}$, we can regard terms in $\omega_0 t$ as constant during one period

we expect φ to vary like $\sqrt{g/l}$, then the equation of motion is

$l \ddot{\varphi} + (g + r \omega_0^2 \cos\omega_0 t)\varphi = r \omega_0^2 \sin\omega_0 t \simeq 0$

which is the equation of a SHO with a constant driving term and the frequency of osc. is $\omega^2 = \frac{g + r \omega_0^2 \cos\omega_0 t}{l}$ = gravity + vertical centripetal