PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #60

January 13, 1990

Comprehensive Examination for Winter 1990

PART I

General Instructions

This Comprehensive Examination for Winter 1990 (#60) consists of eight problems of equal weight (20 points each). It has two parts. The first part (Problems 1-4) is handed out at 9:00 am (duration: 3 hours) and the second part (Problems 5-8) at 1:00 pm (duration 3 hours).

Work carefully, indicate your reasoning briefly, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers, except the exam and bluebook, on the floor. Please return the bluebooks at the end of the exam.

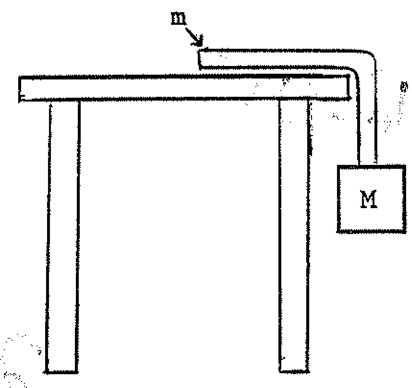
Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Physics Department Comprehensive Exam #60, January 13, 1990

PART I

Problem #1

An object of mass M is fastened to the end of a rope of length ℓ and mass m and hangs over the edge of a table as indicated schematically in the figure. The table is smooth and the rope is flexible.



(a) Choose a convenient coordinate, and obtain the equation of motion.

(b) Initially all of the rope is on the table and the system is at rest. How long does it take for the end of the rope to reach the edge of the table?

Problem #2

The heat capacity of a solid is $C = \alpha T^3$, where α is a constant. This solid is the low-temperature reservoir of a reversible refrigerator. The high-temperature reservoir is at room temperature. The solid is cooled from room temperature to absolute zero (approximately).

(a) Find an expression for the amount of work required to cool this solid.
(b) What is the decrease in entropy of the solid?
(c) What is the decrease in internal energy of the solid?
(d) What are the increases in entropy and internal energy of the high-temperature reservoir?

Problem #3

A spinless particle of mass M moves nonrelativistically in one dimension in the potential $V(x) = V_0$, where V_0 is a constant, for $-a < x < a$, and $V(x) = \infty$ elsewhere.

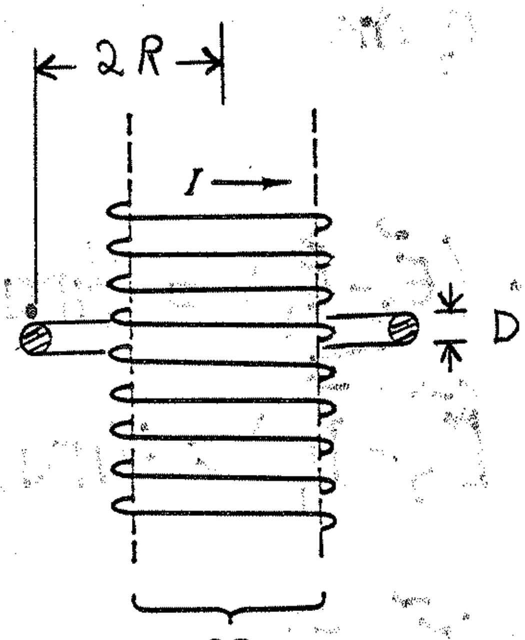
(a) The energy of the particle is measured. What is the least value that may be found?

(b) Suppose that the measurement of energy determines that the particle is in the ground state. Then its position is measured. What is the probability that the particle will be found in the following regions: (i) $x > a/2$? (ii) $x > a$?

(c) Suppose that, after the particle has been determined to be in the ground state, its momentum is measured instead of its position. What values may be found for the momentum, and with what probabilities?

Problem #4

A particular electrical transformer consists of a very long solenoid of radius R having N turns per unit length and a single turn coaxial secondary winding of radius 2R, as indicated in the figure. The secondary loop is closed and has a total resistance r. The diameter of the secondary wire is D, as shown. A sinusoidal current of frequency ν is established in the long solenoid. The magnitude of ν is small enough that equations from magnetostatics can be used. In other words, radiative effects can be ignored. In addition, assume the secondary to be purely resistive, i.e., ignore its self inductance.



(a) Determine the direction of the Poynting vector at the surface of the wire of the secondary turn. Discuss your results.

(b) Assume R >> D, and calculate the total magnetic field along the axis of the solenoid.

PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #60

January 13, 1990

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PART II

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PART II

Problem #5

Coexistence of a gas and a solid: The Helmholz free energy of a gas is given by

$$F = N k_B T \left\{ \ln \left[\frac{N}{VT^{3/2}C} \right] - 1 \right\},$$

where C is constant.

The same material as a solid has a partition function

$$Z = [1 - e^{-\hbar\omega/kT}]^{-N}$$

This gas and solid are in equilibrium. Find the vapor pressure curve for this model. In other words, find the relation between the vapor pressure and the temperature.

Problem #6

A point particle slides frictionlessly down a sphere, starting from rest at the top. Using the Lagrange undetermined multiplier method, obtain the equations of motion and the radial constraint force. In addition, find the polar angle at which the particle leaves the sphere.

Problem #7

In spherical coordinates, a distribution of free charge is described by

$$\rho_f(r) = q \left[\delta^3(r) - \frac{1}{\pi r_0^2} \frac{e^{-2r/r_0}}{r} \right].$$

This free charge is imbedded in an <u>inhomogeneous</u>, isotropic, linear dielectric of permittivity $\epsilon(r)$.

(a) Determine the electric field everywhere.

(b) Calculate the distribution of bound charge for two cases:
  (i) The permittivity, ϵ, is constant.
  (ii) The permittivity is not constant.

Instead,

$$\epsilon = \epsilon_0(1 + x_0 e^{-\alpha r}), \text{ where } x_0 \ll 1 .$$

Problem #8

A spin-1/2 particle of mass m moves non-relativistically in three dimensions in a potential given by

$$V = -\frac{e^2}{|\mathbf{R}|}$$

where $\mathbf{R}$ is the particle's position, and e is a real constant.

(a) What are the energy and degeneracy of the ground state and the first excited state?

(b) At time t = -1 second, the energy of the particle is measured and is found to be in the first excited state. Afterwards, at time t = 0, the projection of the particle's total angular momentum is measured to have its maximum possible value along the **z** axis. Then, starting at t = 0, the following time-dependent external potential is applied:

$$\delta H(t) = G(t) \frac{(\mathbf{R} \times \mathbf{P}) \cdot \mathbf{S}}{m\ |\mathbf{R}|^2}$$

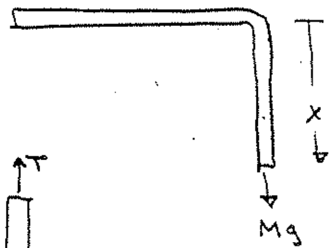
where $\mathbf{P}$ and $\mathbf{S}$ are the particle's momentum and spin respectively, and G(t) is given by $G(t) = G_0\, t\, e^{-\gamma t}$ for $t > 0$ where γ and G_0 are both very small constants. At the end of this experiment, the particle's orbital angular momentum $\mathbf{L}^2$ and its total angular momentum $\mathbf{J}^2$ are measured. What values may be found?

(c) Find an approximate expression for the probability that the particle is in an excited state after a long time $t >> 1/\gamma$. **You need not evaluate the radial integrals.**

(d) Explain the conditions for your answer to part (c) to be valid: Compared to what quantity G_{max} should G_0 be small, $G_0 << G_{max}$? Compared to what quantity γ_{max} should γ be small, $\gamma << \gamma_{max}$?

Comp. #60 #1

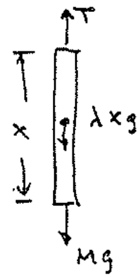
Newtonian Approach



$\lambda \equiv m/\ell$

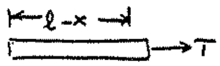
For the section of length x hanging over the edge of the table

$$Mg + \lambda x g - T = (\lambda x + \cancel{M})\ddot{x}$$



for the section still on the table

$$T = \lambda(\ell - x)\ddot{x}$$



illiminating T

$$Mg + \lambda x g - \lambda(\ell - x)\ddot{x} = \lambda x \ddot{x} + M\ddot{x}$$

10 $\boxed{(M + \lambda\ell)\ddot{x} - \lambda g x = Mg}$

the solution of the homogeneous eq may be obtained by assuming $x = e^{rt}$, $\dot{x} = re^{rt}$, $\ddot{x} = r^2 e^{rt}$

this leads to the

$(M + \lambda \ell) r^2 - \lambda g = 0$ from which $r = \pm\sqrt{g/\ell}\sqrt{\frac{m}{m+M}} \equiv \pm\omega$

the general solution is

$$x = Ae^{rt} + Be^{-rt} + C$$

the particular solution C satisfies the equation of motion when $C = -\frac{M}{\lambda} = -\frac{M}{m}\ell$

the initial conditions require

$A + B + C = 0 \qquad C = -2A \quad \therefore A = \frac{M}{2\lambda} = \frac{M}{2m}\ell$

$Ar - Br = 0 \qquad A = B$

thus

$$x = \frac{M\ell}{2m}\left(e^{\omega t} + e^{-\omega t}\right) - \frac{M}{m}\ell$$

$$= \frac{M}{m}\ell \cosh \omega t - \frac{M}{m}\ell$$

6 $\boxed{x(t) = \frac{M}{m}\ell\left(\cosh \omega t - 1\right)}$

when $x = \ell$; $\frac{M}{m}\ell \cosh t = \ell + \frac{M}{m}\ell$

4 $\boxed{t = \frac{1}{\omega}\cosh^{-1}\left(1 + \frac{m}{M}\right)}$

when $\frac{m}{M} \to 0 \qquad t = 0$

$\frac{m}{M} = 1 \qquad t = 1.3\sqrt{\frac{\ell}{g}}\sqrt{\frac{m+M}{m}}$

#2

(a) $$\gamma = \frac{\text{heat extracted at low } T}{\text{energy consumed}} = \frac{T_{low}}{T_{high} - T_{low}} \quad (\text{Carnot})$$

$$\Delta W = \frac{1}{\gamma} \Delta U = -\frac{1}{\gamma} \frac{\partial U}{\partial T} \Delta T$$

$$\Rightarrow \quad \frac{dW}{dT} = -\frac{T_{Room} - T}{T} \alpha T^3$$

$$\Rightarrow \quad W = \int_{T_{Room}}^{0} \frac{T - T_{Room}}{T} \alpha T^3 \, dT$$

$$= \alpha \int_0^{T_R} (T_R - T) T^2 dT = \alpha \left\{ T_R \frac{1}{3} T_R^3 - \frac{1}{4} T_R^4 \right\}$$

$$= \frac{\alpha}{12} T_R^4$$

#2. (Cont.)

(b) $$\left(\frac{\partial S}{\partial T}\right)_V = \frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_V = \alpha T^2$$

$$\Rightarrow \quad \Delta S = \int_{T_R}^{0} \alpha T^2 dT = -\frac{\alpha}{3} T_R^3$$

(c) $$\Delta U = \int_{T_R}^{0} \alpha T^3 dT = -\frac{\alpha}{4} T_R^4$$

$$\cancel{\Delta U = W + T_R \Delta S}$$

(d) $$\Delta S = +\frac{\alpha}{3} T_R^3$$

$$\Delta U = \frac{\alpha}{4} T_R^4 + W = \frac{\alpha}{3} T_R^4 = T_R \Delta S$$

#3

12/89

QM1 a. $H|\psi\rangle = E|\psi\rangle$ stationary state Schr. eq.

$\langle x|\psi\rangle = \psi(x)$ coordinate representation

$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x)$ Schr. eq.

for $-a<x<a$, $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E-V_0)\psi \Rightarrow \psi_k = A_k \sin kx + B_k \cos kx$

otherwise, $\psi = 0$ $\qquad k = \sqrt{2m(E-V_0)/\hbar^2}$

H has even parity $\Rightarrow A_k = 0$ or $B_k = 0$

normalize $1 = \int_{-a}^{a} dx\, |\psi(x)|^2 = a\,(A_k^2 \text{ or } B_k^2) \Rightarrow A_k \text{ or } B_k = \frac{1}{\sqrt{a}}$

continuity $\psi_k(a) = 0 \Rightarrow ak_n = n\pi$ for $A_k \neq 0$

or $ak_n = (n+\frac{1}{2})\pi$ for $B_k \neq 0$

lowest $E \Rightarrow$ lowest $k_n \Rightarrow k_0 = \pi/2a$, $\boxed{E_{gs} = V_0 + \frac{\hbar^2}{2m}\left(\frac{\pi}{2a}\right)^2}$

b. probability density $= |\psi(x)|^2 = \left|\frac{1}{\sqrt{a}}\cos\frac{\pi x}{2a}\right|^2$ for $|x|<a$

i) $\text{Prob}(x>a/2) = \int_{a/2}^{\infty} dx\, |\psi(x)|^2 = \int_{a/2}^{a} dx \left|\frac{1}{\sqrt{a}}\cos\frac{\pi x}{2a}\right|^2 + \int_{a}^{\infty} dx\,($

$= \int_{a/2}^{a} \frac{dx}{a}\cos^2\frac{\pi x}{2a}$, $y \equiv \frac{\pi x}{2a}$

$= \frac{2}{\pi}\int_{\pi/4}^{\pi/2} dy\, \cos^2 y = \frac{2}{\pi}\int_{\pi/4}^{\pi/2} dy\, \frac{1+\cos 2y}{2}$

$= \frac{2}{\pi}\int_{\pi/4}^{\pi/2} dy\left(\frac{1}{2}\right) + \frac{1}{\pi}\int_{\pi/4}^{\pi/2} dy\, \cos 2y$, $z \equiv 2y$

$= \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + \frac{1}{2\pi}\int_{\pi/2}^{\pi} dz\, \cos z$

$= \frac{1}{4} + \frac{1}{2\pi}\left(\sin\pi - \sin\frac{\pi}{2}\right) = \boxed{\frac{1}{4} - \frac{1}{2\pi}} \approx 0.$

ii) $\text{Prob}(x>a) = 0$

#3 (Cont.)

QM 1

c. Momentum probability density = $\bar{\Psi}(p) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \psi(x)$

all real values of p are possible.

Probability = $|\bar{\Psi}(p)|^2 \, dp$

$$\bar{\Psi}(p) = \int_{-a}^{a} \frac{dx}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \frac{1}{\sqrt{a}} \cos\frac{\pi x}{2a}$$

$$= \frac{1}{\sqrt{2\pi\hbar a}} \int_{-a}^{a} dx \, e^{ipx/\hbar} \frac{1}{2}\left(e^{i\pi x/2a} + e^{-i\pi x/2a}\right)$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi\hbar a}} \left[\frac{e^{i\left(\frac{p}{\hbar} + \frac{\pi}{2a}\right)x}}{i\left(\frac{p}{\hbar} + \frac{\pi}{2a}\right)} + \frac{e^{i\left(\frac{p}{\hbar} - \frac{\pi}{2a}\right)x}}{i\left(\frac{p}{\hbar} - \frac{\pi}{2a}\right)} \right]_{-a}^{a}$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi\hbar a}} \left[\frac{e^{ipa/\hbar} - e^{-ipa/\hbar}}{\frac{p}{\hbar} + \frac{\pi}{2a}} - \frac{e^{ipa/\hbar} - e^{-ipa/\hbar}}{\frac{p}{\hbar} - \frac{\pi}{2a}} \right]$$

$$= \frac{i}{\sqrt{2\pi\hbar a}} \sin\frac{pa}{\hbar} \left[\frac{1}{\frac{p}{\hbar} + \frac{\pi}{2a}} + \frac{1}{\frac{\pi}{2a} - \frac{p}{\hbar}} \right]$$

$$= \frac{i}{\sqrt{2\pi\hbar a}} \sin\frac{pa}{\hbar} \left[\frac{\pi/a}{\frac{\pi^2}{4a^2} - \frac{p^2}{\hbar^2}} \right]$$

$$= i\sqrt{\frac{\pi a}{2\hbar}} \frac{\sin(pa/\hbar)}{\pi^2/4 - p^2a^2/\hbar^2}$$

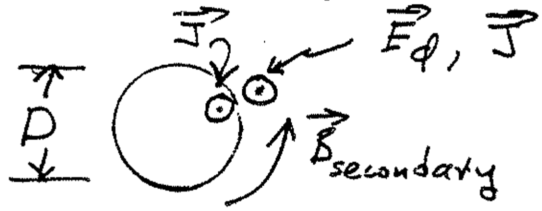
$$\boxed{\text{Probability} = \frac{\pi a}{2\hbar} \left| \frac{\sin pa/\hbar}{\pi^2/4 - (pa/\hbar)^2} \right|^2 dp}$$

alternative: use $\psi(x) = \psi(-x) \Rightarrow 0 = \int \cos\frac{\pi x}{2a} \sin\frac{px}{\hbar}$

+ get $\bar{\Psi}(p) = \int_{-a}^{a} \frac{dx}{\sqrt{2\pi\hbar}} \cos\frac{px}{\hbar} \frac{1}{\sqrt{a}} \cos\frac{\pi x}{2a}$

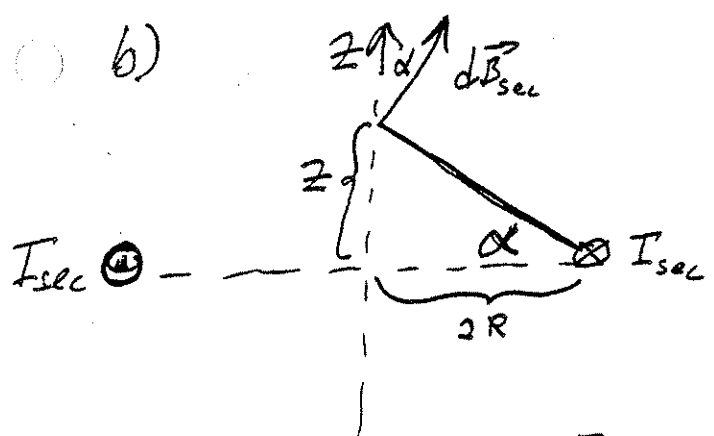
then use integral tables.

#4 a) $\oint \vec{A}\cdot d\vec{l} = \iint \vec{B}\cdot d\vec{A}$, where $\vec{B} = \mu_0 N I \hat{k}$ and $I = I_0 \sin\omega t$. By symmetry, $A_\phi \cdot 2\pi\rho = \mu_0 N I \cdot \pi R^2$. If $\rho = 2R$, $A_\phi = \frac{\mu_0 N R}{4} I_0 \sin\omega t$, $E_\phi = -\frac{\partial A_\phi}{\partial t} = -\omega A_m \cos\omega t$, where $A_m = \frac{\mu_0 N R I_0}{4}$. The current in the secondary follows E_ϕ. So, a segment of secondary wire appears in cross section as follows:



$\vec{E}_\phi \times \vec{B}_{secondary}$ is inward, allowing field energy to flow <u>into</u> the secondary winding to compensate for $I^2 r$ losses.

b)



By symmetry $\vec{B}_{sec}(0,0,z)$ is along the z-axis.

$$dB_{sec}\cos\alpha = \frac{\mu_0}{4\pi}\frac{I_{sec}(2R\,d\phi)}{z^2+4R^2}\frac{2R}{\sqrt{z^2+4R^2}}$$

$$B_{sec}(0,0,z) = \frac{2\mu_0 R^2}{(z^2+4R^2)^{3/2}} I_{sec}.$$

$I_{sec} = E_{sec}\cdot 2\pi(2R)/r$, where

$E_{sec} = E_\phi = -\omega\cos\omega t\,\frac{\mu_0 N R I_0}{4}$. Thus,

$$B(0,0,z) = B_{prim} + B_{sec} = \mu_0 N I_0\left[\sin\omega t - \frac{2\mu_0 R^2}{(z^2+4R^2)^{3/2}}\cdot\frac{4\pi R}{r}\cdot\frac{\omega R}{4}\cos\omega t\right]$$

$$B = \mu_0 N I\left[\sin\omega t - \frac{2\pi\mu_0\omega R^4}{r(z^2+4R^2)^{3/2}}\cos\omega t\right]$$

$$= \mu_0 N I\left[\sin\omega t + \frac{2\pi\mu_0\omega R^4}{r(z^2+4R^2)^{3/2}}\sin(\omega t - \tfrac{\pi}{2})\right]$$

$$B(0,0,0) = \mu_0 N I\left[\sin\omega t + \frac{\pi\mu_0\omega R}{4r}\sin(\omega t - \tfrac{\pi}{2})\right]$$

#5.

Gas

$$\mu_g = \left(\frac{\partial F}{\partial N}\right)_{T,V} = kT \ln \frac{N}{V T^{3/2} c}$$

$$P_{vap} = \frac{NkT}{V}$$

$$\Rightarrow \mu_g = kT \ln \frac{P_{vap}}{k T^{5/2} c}$$

Solid

$$F = -kT \ln Z$$

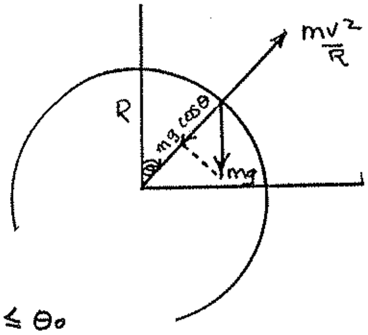
$$= NkT \ln \{ 1 - e^{-\hbar\omega/kT} \}$$

$$\mu_s = kT \ln \{ 1 - e^{-\hbar\omega/kT} \}$$

Equilibrium : $\mu_g = \mu_s$

$$\Rightarrow P_{vap} = kT^{5/2} c \{ 1 - e^{-\hbar\omega/kT} \}.$$

#6. Take the Lagrange multiplier approach and find the generalized force of constraint.



The particle will start to leave the sphere at the angle θ_0 where

$$\frac{mv^2}{R} = mg\cos\theta$$

the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$$

where r is constrained to be equal to R in the region $\theta \le \theta_0$ also $\dot{r} = 0$ in the same angular region. The equation of constraint is $f = r - R = 0$

the Lagrange eqs. for r and θ are

$$m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta - \lambda(\theta) = 0 \qquad (1)$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mgr\sin\theta = 0 \qquad (2)$$

where λ is expressed as a function of θ. λ represents the radial constraint force and as such is a function of time.

Using $r = R$, $\dot{r} = 0$ and $\ddot{r} = 0$ (1) and (2) become

$$-mR\dot{\theta}^2 + mg\cos\theta = \lambda(\theta) \qquad (3)$$

$$mR^2\ddot{\theta} - mgR\sin\theta = 0 \qquad (4)$$

Differentiating (1) with respect to time and using the fact that

$$\frac{d\lambda}{dt} = \frac{d\lambda}{d\theta}\frac{d\theta}{dt} \quad \text{we have}$$

$$-2mR\ddot{\theta} - mg\sin\theta = d\lambda/dt$$

or $$-mR^2\ddot{\theta} - \frac{mgR}{2}\sin\theta = \frac{R}{2}\frac{d\lambda}{dt}$$

adding this to (4) we get

$$\frac{d\lambda}{dt} = -3mg\sin\theta$$

Integrating, and using the initial condition $\lambda(0) = mg$, we find

$$\lambda(\theta) = 3mg\cos\theta - 2mg$$

Now, the particle will remain on the surface as long as the radial constraint force $\lambda(\theta)$ is positive.

$$\therefore \boxed{\theta_0 = \cos^{-1}\left(\frac{2}{3}\right)}$$

#7

a) $$\rho_f = q\left[\delta^3(\vec r) - \frac{1}{\pi r_0^2}\,\frac{e^{-2r/r_0}}{r}\right]$$

For a ~~linear~~ isotropic dielectric, $\rho_b = -\frac{\chi}{1+\chi}\rho_f$. To check this out we calculate $\vec D(\vec r)$, $\vec E(\vec r)$, $\vec P(\vec r)$, and $\vec\nabla\cdot\vec P$:

$$D_r(r)\cdot 4\pi r^2 = q\left[1 - \frac{4\pi}{\pi r_0^2}\int_0^r r\,e^{-2r/r_0}\,dr\right]$$

$$= q\left\{1 - \frac{4}{r_0^2}\left[-\frac{r r_0}{2}e^{-2r/r_0}\Big|_0^r + \frac{r_0}{2}\int_0^r e^{-2r/r_0}\,dr\right]\right\}$$

$$= q\left\{1 - \frac{4}{r_0^2}\left[-\frac{r_0}{2}r\,e^{-2r/r_0} - \frac{r_0^2}{4}\left(e^{-2r/r_0} - 1\right)\right]\right\}$$

$$= q\left[e^{-2r/r_0}\left(\frac{2r}{r_0} + 1\right)\right]$$, or, $\vec D = \frac{q}{4\pi r^2}[\;]\hat r$. Also,

$\boxed{\vec E = \frac{q}{4\pi\epsilon r^2}[\;]\hat r}$, and (b) $\vec P = \frac{\chi q}{4\pi(1+\chi)r^2}[\;]\hat r$.

$$\rho_b(\vec r) = -\vec\nabla\cdot\vec P = -\frac{\chi q}{4\pi(1+\chi)}\left\{[\;]\,\vec\nabla\cdot\left(\frac{\hat r}{r^2}\right) + \frac{\hat r}{r^2}\cdot\vec\nabla[\;]\right\}$$

$$= -\frac{\chi}{1+\chi}\,q\left\{\delta^3(\vec r) + \frac{1}{4\pi r^2}\,\hat r\cdot\hat r\left[\frac{2}{r_0}e^{-2r/r_0} - \frac{2}{r_0}e^{-2r/r_0}\left(\frac{2r}{r_0}+1\right)\right]\right\}$$

$$= -\frac{\chi}{1+\chi}\,q\left[\delta^3(\vec r) - \frac{1}{\pi r_0^2}\,\frac{e^{-2r/r_0}}{r}\right]$$

If $\chi = \chi_0 e^{-\alpha r}$, $\rho_b \simeq -\chi_0 q\,\delta^3(\vec r) + \frac{q\chi_0}{4\pi r^2}\frac{\partial}{\partial r}\left[e^{-(\alpha+\frac{2}{r_0})r}\left(1+\frac{2r}{r_0}\right)\right]$

#8

QM2 a. This is the hydrogen atom

$$E_n = E_1/n^2, \quad n=1,2,\ldots, \quad E_1 = -\frac{me^4}{2\hbar^2}$$

g.s. $\boxed{E = -\frac{me^4}{2\hbar^2}, \quad \text{degen} = 2}$ for spin

1^{st} state $\boxed{E = -\frac{me^2}{8\hbar^2}, \quad \text{degen} = 8}$ $= \underset{spin}{2} \times (\underset{L=0}{1} + \underset{L=1}{3})$

b. $\delta H(t) = G(t)\,\frac{\vec{L}\cdot\vec{S}}{mR^2}$

$\Rightarrow L^2, S^2, J^2, J_z$ conserved

starts from $L=1$, $S=\frac{1}{2}$, $J=\frac{3}{2}$, $M_J = 3/2$

$\Rightarrow$ same values at end, $\boxed{L^2 = L(L+1)\hbar^2 = 2\hbar^2, \; J^2 = J(J+1)\hbar^2 = \frac{15}{4}\hbar^2}$

c. Transition amplitude $A_{fi} = -\frac{i}{\hbar}\int_0^\infty dt\, e^{i(E_f - E_i)t/\hbar} \langle f|\delta H(t)|i\rangle$

$$\langle f|\delta H(t)|i\rangle = G(t)\,\langle f|\frac{\vec{L}\cdot\vec{S}}{mR^2}|i\rangle$$

~~$A_{fi} = \ldots$~~ $\langle f|, |i\rangle$ both eigenstates of $\vec{L}\cdot\vec{S}$

eigenvalues $\frac{J(J+1) - L(L+1) - S(S+1)}{2}\hbar^2 = \frac{\hbar^2}{2}$

$$\langle f|\frac{\vec{L}\cdot\vec{S}}{mR^2}|i\rangle = \frac{\hbar^2}{2m}\langle f|\frac{1}{R^2}|i\rangle = \frac{\hbar^2}{2m} I_{radial}$$

where $\boxed{I_{radial} \equiv \int_0^\infty dr\, u_f(r)\frac{1}{r^2}u_i(r)}$

$$A_{fi} = -\frac{i\hbar}{2m} I_{radial}\, G_0 \int_0^\infty dt\, e^{i(E_f - E_i)t/\hbar}\, t\, e^{-\gamma t}$$

$$= -\frac{i\hbar}{2m} G_0 I_{radial} \frac{1}{(i(E_f - E_i)/\hbar - \gamma)^2}$$

$$\text{Probability}(t) = |A_{fi}|^2 = \left(\frac{\hbar G_0 I_{rad}}{2m}\right)^2 \left(\frac{1}{\gamma^2 + (E_f - E_i)^2/\hbar^2}\right)^2$$

#8 (Cont.)

if γ is small then it can be ignored $\gamma << |E_f - E_i|/\hbar$

$$\boxed{\text{Probability} = \sum_n \left[\frac{\hbar^3 G_0}{2m(E_n-E_1)^2} \int_0^\infty dr\, u_{n1}(r) \frac{1}{r^2} u_{11}(r)\right]^2}$$

d. Condition for γ : $\gamma << |E_f - E_i|/\hbar < \frac{E_1}{4\hbar}$

Condition for G_0 is Prob $<< 1$

$$\Rightarrow G_0 << \left[\sum_n \frac{\hbar^3 G_0}{2m(E_n-E_1)^2} \int_0^\infty dr\, u_{n1}(r) \frac{1}{r^2} u_{11}(r)\right]^{-1}$$

dominated by $n=2$, estimate radial integral $\sim \frac{1}{a_0}$

$$\Rightarrow G_0 << \left[\frac{32\hbar^3 a_0}{m E_1}\right]^{-1}$$