PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #59

APRIL 1, 1989

Comprehensive Examination for Winter 1989

PART I

General Instructions

This Comprehensive Examination for Winter 1989 (#59) consists of eight problems of equal weight (20 points each). It has two parts. The first part (Problems 1-4) is handed out at 9:00 am (duration: 3 hours) and the second part (Problems 5-8) at 1:00 pm (duration 3 hours).

Work carefully, indicate your reasoning briefly, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. Please return it at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Physics Department Comprehensive Exam #59, April 1, 1989

PART I

Problem # 1

A pion of rest mass 140 MeV decays into a muon (rest mass = 105 MeV) and a massless neutrino:

A) Compute the kinetic energy of the muon in the π rest frame.

B) If the pion decays in flight, what are the limits of the muon energy in the laboratory frame? Express these limits in terms of the pion lab energy.

Problem #2

Consider an electrical current density given in cylindrical coordinates by $\vec{J} = \hat{k}\, J_o \sin(\kappa\rho)$, where J_o and κ are constants. This current density extends throughout space, and charge neutrality exists at every point in space.

A) Calculate the magnetic field everywhere.

B) Obtain an approximate expression for the B-field which is valid for all points near the z-axis, and then use this expression to determine the vector potential in that region. Hint: Make a reasonable assumption for the value of $\vec{A}$ along the z-axis.

C) The current density is now made time dependent by replacing J_o with $J_o't/\tau$, where J_o' and τ are constants. Calculate the electric field for points near the z-axis. Assume the B-field varies slowly, and use the results of part (b).

D) Determine the magnitude and direction of the Poynting vector for points near the z-axis.

Problem #3

Van der Waals proposed the following form for the Helmholtz free energy of a gas of N atoms in a volume V

$$F(\tau, V, N) = -N\tau \{\log [n_Q(V - N b)/N] + 1\} - \frac{N^2 a}{V}$$

where $\tau = k_\beta T$, $n_Q = \left(\frac{M\tau}{2\pi\hbar^2}\right)^{\frac{3}{2}}$, M is the mass of a molecule, and a and b are positive parameters.

A) Derive the Van der Waals equation of state.

B) What is the minimum value of V for which this equation is valid? What is the interpretation of this minimum value?

C) Calculate $\frac{\partial P}{\partial V}$. Calculate $\frac{\partial^2 P}{\partial V^2}$.

Evaluate the values of the critical temperature τ_c and pressure P_c for which both these derivatives are zero.

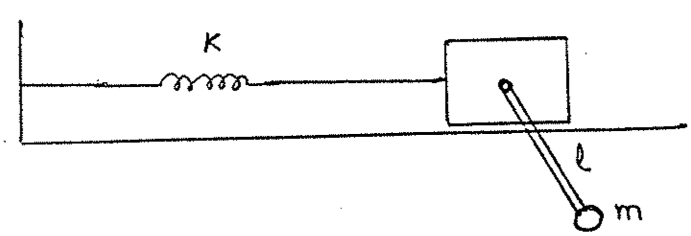
D) Sketch a pV diagram for $\tau>\tau_c$ and $\tau<\tau_c$.

E) If at a certain value of the pressure P more solutions exist for V, which quantity determines which solution is physical?

F) Sketch this physical solution for V as a function of pressure. Interpret this sketch.

Problem #4

A block of mass M slides without friction on a horizontal track, and is fastened to a spring with spring constant k. A pendulum with rigid weightless rod of length ℓ and bob of mass m hangs from it. Find the Lagrangian and the coupled equations of motion. For small displacements find the lowest order correction to the uncoupled oscillator frequencies.



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PART II

Problem #5

The "standard" form of the Born approximation is

$$f(\theta) = -\frac{2m}{\hbar^2 K}\int_0^\infty V(r)\,\sin Kr\; r\,dr$$

where $\vec{K} = \vec{k}_{inc} - \vec{k}_f$

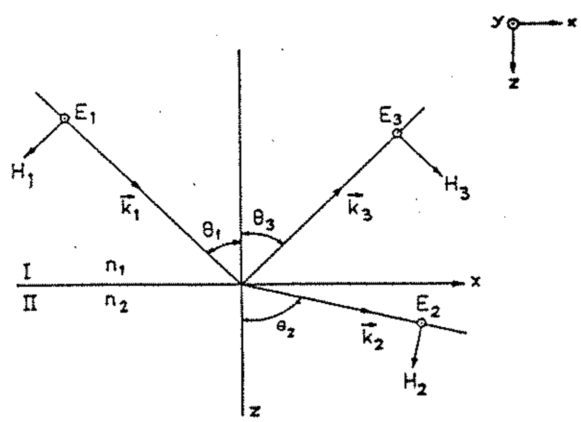
Calculate the differential cross section for scattering of particles of mass m and incident energy E by the repulsive spherical well potential

$$V = \begin{cases} V_o & 0 < r < a \\ 0 & r > a. \end{cases}$$

Show the explicit E and θ dependence.

Problem #6

The figure below depicts transmission and internal reflection of linearly polarized plane electromagnetic waves at a dielectric interface for the case that the index of refraction for medium I, n_1, is greater than n_2. For this problem, $n_1 = n$ and $n_2 = 1$. The E-field is polarized in the y-direction and the x-z plane is the plane of incidence. The amplitudes of the three E-fields shown in the figure are denoted by E_{1m}, E_{2m}, and E_{3m}. Also $\theta_1 = \theta_3$ and θ_2 are related by Snell's law. The dielectric is nonmagnetic.



A) Derive an expression for the ratios E_{3m}/E_{1m} and E_{2m}/E_{1m} in terms of n, θ_1, and θ_2. Hint: For plane waves, $\vec{B} = (\vec{k}\times\vec{E})/\omega$.

B) Use Snell's law to eliminate θ_2 from your equations of part (A).

C) For angles of incidence $\theta_1 > \theta_c$, where θ_c is the critical angle, $\sin\theta_2 > 1$ (requiring θ_2 to be complex), and, therefore, the ratios derived in part (A) can become complex. Derive an expression for $|E_{2m}/E_{1m}|$ for $\theta_1 > \theta_c$. Hint: Although θ_2 is complex, $\sin\theta_2$ is real.

D) Show that if $\theta_1 > \theta_c$, $\vec{k}_2$ is complex; and re-write the expression

$$E_2 = E_{2m}e^{i(\omega t - \vec{k}_2\cdot\vec{r}_2)}$$ so as to illustrate

the consequence of this fact. Hint: $\sin\theta_2 > 1$ and $\sin\theta_2$ is real.

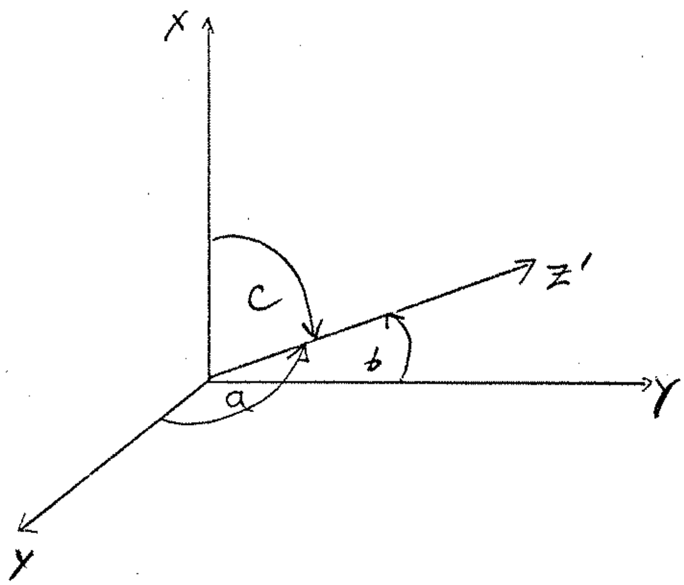
Problem #7

A) Write the expression for the component of the spin operator for an electron along the axis z', which makes respective angles (a,b,c) with the x,y,z) axis, in terms of the Pauli spin operators.

B) Write the matrix of the spin operator σ_z' in terms of the angles a,b,c and show that the eigenvalues of σ_z' are the same as those of σ_z.

C) Find the eigenvectors of σ_z' beloging to the eigenvalues found in (b) above.

D) An electron is known to be in the eigenstate $\alpha = \binom{1}{0}$ of σ_z. What are the probabilities that a measurement of S_z' finds the respective values $+\hbar/2$ and $-\hbar/2$ when the z' axis is specified by $a = \pi/2$, $b = \pi/2 - c$.



Problem #8

Consider a system of independent, identical particles with single particle energies $\epsilon_i(V)$.

A) Calculate the grand partition function $z(\tau, \mu, V)$ at temperature τ ($= k_\beta T$), chemical potential μ, and volume V. Do this calculation for bosons and fermions.

B) Calculate the average number of particles <N>, using the results of A.

C) Give the Fermi-Dirac and Bose-Einstein distribution function $f(\epsilon)$.

D) Show that the entropy is given by

$$- k_b \sum_i \{f(\epsilon_i) \log f(\epsilon_i) \mp (1 \pm f(\epsilon_i)) \log (1 \pm f(\epsilon_i))\}$$

With the top signs for Bose-Einstein and the bottom signs for Fermi-Dirac. (Hint: use the grand potential $\Omega = - \tau \log z$).

Problem #1

A pion of rest mass 140 MeV decays into a muon (rest mass = 105 MeV) and a massless neutrino:

$\boxed{\pi \to \mu + \nu}$ A. Compute the K.E. of the muon in the π rest frame.

$\nu \longleftarrow \overset{\pi}{\circ} \longrightarrow \mu$

$E_\nu = pc \qquad E_\pi = m_\pi c^2 \qquad E_\mu^2 = p^2c^2 + (m_\mu c^2)^2$

conserv. of mom: $P_\nu = P_\mu \equiv P$

conserv. of Energy $E_\pi = E_\nu + E_\mu$

$$m_\pi c^2 = pc + \sqrt{p^2c^2 + (m_\mu c^2)^2}$$

$$m_\pi c = p + \sqrt{p^2 + (m_\mu c)^2}$$

squaring

① $$P = \frac{(m_\pi c)^2 - (m_\mu c)^2}{2 m_\pi c}$$

also

② $E_\mu = E_\pi - E_\nu = m_\pi c^2 - pc$

③ $$E_\mu = m_\pi c^2 - \frac{(m_\pi c)^2 - (m_\mu c)^2}{2 m_\pi}$$

$$\to E_\mu = \frac{(m_\pi c)^2 + (m_\mu c)^2}{2 m_\pi}$$

the μ Kinetic Energy is

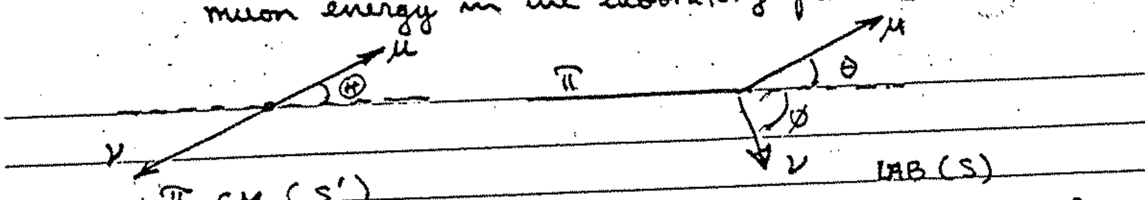
$$K_\mu = E_\mu - m_\mu c^2 = \frac{(m_\pi c)^2 + (m_\mu c)^2}{2 m_\pi} - m_\mu c^2$$

$$= (m_\pi - m_\mu)^2 c^2 / 2 m_\pi$$

using $m_\pi - m_\mu \approx 35\ \text{MeV}/c^2$

$$K_\mu \approx \frac{(35)^2}{280} = \underline{4.4\ \text{MeV}}$$

B. If the pion decays in flight, what are the limits of the muon energy in the laboratory frame?



the energy transformation is

$E = \gamma\,(E' + v\,P_x')$: $P_x' = p'\cos\theta$

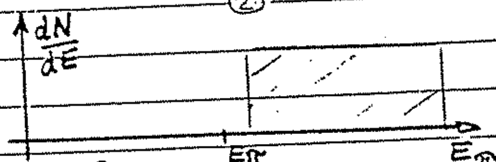
$E_{LAB} = \gamma\,(E_{CM} + \beta\, pc \cos\Theta)$

for the muon (set $\beta \approx 1$)

$E_{max/min} = \gamma\,(E_\mu \pm \beta pc)$

$$E_{min} = \gamma\,\underbrace{(E_\mu - pc)}_{\text{from ③ - ①}} = \gamma\,\frac{(m_\mu c)^2}{m_\pi} = \gamma\left(\frac{m_\mu}{m_\pi}\right)^2 m_\pi c^2$$

$$E^\mu_{max} = \gamma\,\underbrace{(E_\mu + pc)}_{②} \simeq \gamma\, m_\pi c^2 = E_\pi$$



and the LAB pion Energy, $E_\pi = \gamma\,(E' + v\,\underset{=0}{P_x'}) = \gamma E' = \gamma\,(m_\pi c^2)$

$$E_{min} = E_\pi \left(\frac{m_\mu}{m_\pi}\right)^2 = \left(\frac{105}{140}\right)^2 E_\pi = .56\ E_\pi$$

Problem # 2   UG E & M

a) $\oint \vec{B}\cdot d\vec{l} = \mu_0 \iint \vec{J}\cdot d\vec{A}$, $B_\phi \cdot 2\pi\rho = 2\pi\mu_0 J_0 \int_0^\rho \rho\, d\rho \sin(k\rho)$.

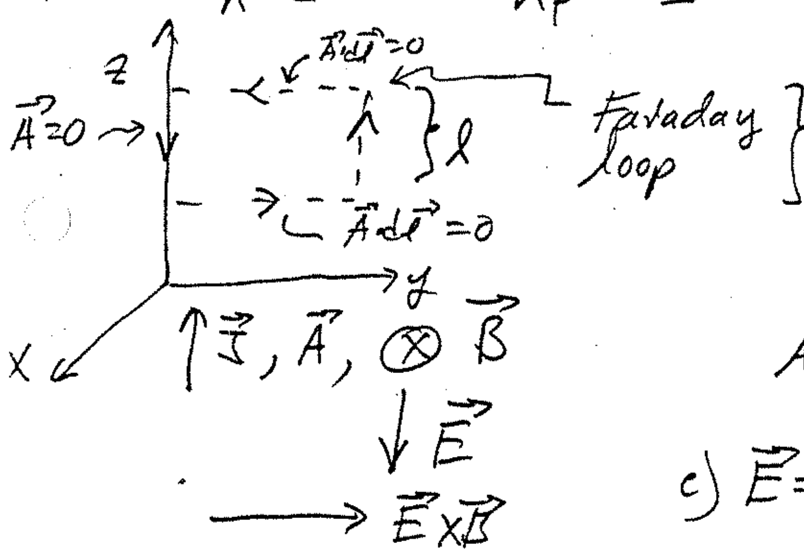
Let $u=\rho$, $dv = d\rho \sin(k\rho)$, $v = -\frac{\cos k\rho}{k}$. Then

$$B_\phi = \frac{\mu_0 J_0}{\rho}\left\{ - \frac{\rho\cos(k\rho)}{k}\Big|_0^\rho + \frac{1}{k}\int_0^\rho d\rho \cos(k\rho) \right\}$$

$$= \frac{\mu_0 J_0}{\rho}\left[- \frac{\rho\cos(k\rho)}{k} + \frac{1}{k^2}\sin(k\rho)\Big|_0^\rho \right] = \frac{\mu_0 J_0}{k\rho}\left[\frac{\sin(k\rho) - k\rho\cos(k\rho)}{k}\right]$$

b) $$\frac{\mu_0 J_0}{k}\left[\frac{\sin(k\rho) - k\rho\cos(k\rho)}{k\rho}\right] \simeq \frac{\mu_0 J_0}{k}\left[k\rho - \frac{(k\rho)^3}{3} - k\rho + \frac{(k\rho)^3}{2}\right] / k\rho$$

$$\simeq \frac{\mu_0 J_0}{k}\frac{(k\rho)^2}{6} \simeq B_\phi$$



$$\oint \vec{A}\cdot d\vec{l} \simeq \iint \vec{B}\cdot d\vec{A}$$

$$A_z l \simeq \frac{\mu_0 J_0}{6k}\int_0^\rho l\, d\rho\, k^2\rho^2$$

$$A_z \simeq \frac{\mu_0 J_0 k}{18}\rho^3,\quad \vec{A} \simeq \frac{\mu_0 J_0 k}{18}\rho^3 \hat{k}.$$

c) $$\vec{E} = -\frac{\partial \vec{A}}{\partial t} \simeq -\frac{\mu_0 k J_0'}{18\tau}\rho^3 \hat{k}$$

d) $$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} \simeq \hat{\rho}\ \frac{\mu_0 k J_0'}{18\tau}\rho^3\ \frac{\mu_0 J_0' t}{6k\tau} k^2\rho^2$$

$$\simeq \hat{\rho}\left(\frac{\mu_0 k J_0' t}{\tau}\right)^2 \frac{\rho^5}{108 t}$$

Problem #3

(A) $$P = -\left(\frac{\partial F}{\partial V}\right)_{\tau,N} = \frac{N\tau}{V-Nb} - \frac{N^2a^*}{V^2}$$

or $$\left(p + \frac{N^2a^*}{V^2}\right)(V - Nb) = N\tau$$

(B) $V > Nb$ $\qquad b =$ molecule volume.

(C) $$\frac{\partial p}{\partial V} = -\frac{N\tau}{(V-Nb)^2} + \frac{2N^2a^*}{V^3}$$

$$\frac{\partial^2 p}{\partial V^2} = \frac{2N\tau}{(V-Nb)^3} - \frac{6N^2a^*}{V^4}$$

$$\frac{\partial p}{\partial V} = 0 \Rightarrow N\tau V^3 = 2N^2a^*(V-Nb)^2 \qquad (1)$$

$$\frac{\partial^2 p}{\partial V^2} = 0 \Rightarrow 2N\tau V^4 = 6N^2a^*(V-Nb)^3 \qquad (2)$$

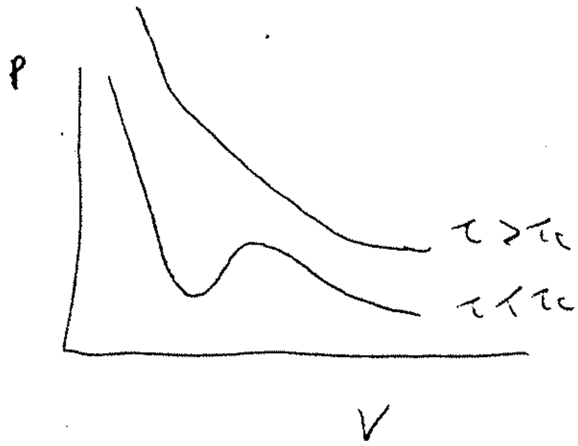
ratio $\Rightarrow 2V = 3(V - Nb)$

$$\Rightarrow V_c = 3Nb$$

in (1) $$N\tau_c\, 27N^3b^3 = 2N^2a^*\, 4N^2b^2$$

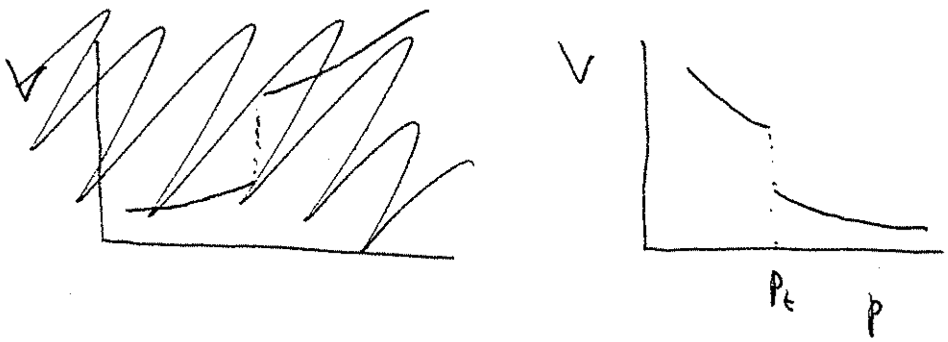
$$\tau_c = \frac{8a}{27b}$$

(D)



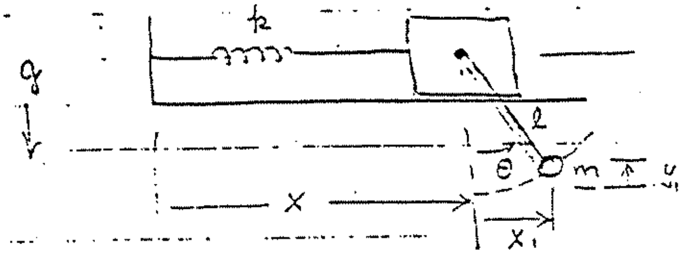
(E) Gibbs free energy $U - TS + pV$.
must be minimal.

(F)



At p_t: phase transition gas $\Leftrightarrow$ liquid.

Problem # 4



Choose x and θ as generalized coords.

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{y}_1^2$$

$$x_1 = x + l\sin\theta \quad ; \quad y_1 = l(1-\cos\theta)$$

$$\dot{x}_1^2 = \dot{x}^2 + 2\dot{x}\dot{\theta}\, l\cos\theta + l^2\cos^2\theta\, \dot{\theta}^2$$

$$\dot{y}_1^2 = l^2\dot{\theta}^2\sin^2\theta$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta + \frac{ml^2}{2}\dot{\theta}^2\cos^2\theta + \frac{m}{2}l^2\dot{\theta}^2\sin^2\theta$$

$$T = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta + \frac{m}{2}l^2\dot{\theta}^2$$

$$U = \frac{1}{2}kx^2 + mgl(1-\cos\theta)$$

$$L = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta + \frac{ml^2}{2}\dot{\theta}^2 - \frac{1}{2}kx^2 - mgl(1-\cos\theta)$$

$$\frac{\partial L}{\partial x} = -kx \quad ; \quad \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + ml\dot{\theta}\cos\theta$$

$$\frac{\partial L}{\partial \theta} = -ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta = -ml(\dot{x}\dot{\theta}+g)\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml\dot{x}\cos\theta + ml^2\dot{\theta}$$

$$\boxed{\begin{aligned} &\text{for } x: \quad (M+m)\ddot{x} + ml(\cos\theta\,\ddot{\theta} - \dot{\theta}^2\sin\theta) + kx = 0 \\ &\text{for } \theta: \quad l\ddot{\theta} + \cos\theta\,\ddot{x} + g\sin\theta = 0 \end{aligned}}$$

The equations to 1st order are (uncoupled)

$$(M+m)\ddot{x} + kx = 0 \quad \Rightarrow \quad \omega_x^{2(0)} = k/(M+m)$$ uncoupled oscillator

$$l\ddot{\theta} + g\theta = 0 \quad \Rightarrow \quad \omega_\theta^{2(0)} = g/l$$ frequencies

assuming initial conditions $x(0) = x_0$; $\theta(0) = \theta_0$: $\dot{x}(0) = 0$ and $\dot{\theta}(0) = 0$

the 1st order solutions are

$$\theta = \theta_0 \cos\omega_\theta t \ ; \ \dot{\theta} = -\theta_0\omega_\theta \sin\omega_\theta t \ : \ \ddot{\theta} = -\theta_0\omega^2 \cos\omega_\theta t = -\omega_\theta^2\theta$$

$$x = x_0 \cos\omega_x t \ : \ \dot{x} = -x_0\omega_x \sin\omega_x t \ : \ \ddot{x} = -x_0\omega_x^2\cos\omega_x t = -\omega_x^2 x$$

For small oscillations — neglect since it's 3rd order

$$(m+M)\ddot{x} + kx + ml(\ddot{\theta} - \cancel{\dot{\theta}^2\theta}) = 0$$

$$(m+M)\ddot{x} + kx + ml\ddot{\theta} = 0 \qquad (1)$$

replace $\ddot{\theta}$ from (2) then

$$(m+M)\ddot{x} + kx - m(\ddot{x} + g\theta) = 0$$

$$m\ddot{x} + kx - mg\theta = 0 \qquad (3)$$

also $l\ddot{\theta} + \cos\theta\,\ddot{x} + g\sin\theta = 0$ or for small θ

$$l\ddot{\theta} + g\theta + \ddot{x} = 0 \qquad (2)$$

Now replace $\ddot{\theta}$ with the zero order $\ddot{\theta} = -\omega_\theta^2\theta$ in eq. (2)

then $-l\omega_\theta^2\theta + g\theta + \ddot{x} = 0$

or $$\theta = \ddot{x}/l\omega_\theta^2 - g \qquad (4)$$

sub (4) into (3) $$M\ddot{x} + kx - \frac{mg}{l\omega_\theta^2 - g}\ddot{x} = 0$$

$$\left(M - \frac{mg}{l\omega_\theta^2 - g}\right)\ddot{x} + kx = 0$$ so to 1st order $$\omega_x^{(1)^2} = \frac{k}{M + \frac{m}{1 - \frac{l\omega_\theta^2}{g}}}$$

If we eliminate $\ddot{x}$ between (1) and (2) we get

$$Ml\ddot{\theta} + g(M+m)\theta - kx = 0 \qquad (5)$$

Replacing $\ddot{x}$ with the zero order, $\ddot{x} = -\omega_x^2 x$ in (1), to get

$$-(m+M)\omega_x^2 x + kx + ml\ddot{\theta} = 0$$

or $$x = \frac{-ml\ddot{\theta}}{k - (m+M)\omega_x^2}$$

then substitute this result into (5)

$$Ml\ddot{\theta} + \frac{kml\ddot{\theta}}{k - (m+M)\omega_x^{2(0)}} + g(M+m)\theta = 0$$

$$\omega_\theta^{(1)^2} = \frac{g}{l}\,\frac{(M+m)}{M + m\left(\frac{k}{k - (M+m)\omega_x^{2(0)}}\right)}$$

Problem # 5 UG.QM 5/89

$$f(\theta) = \frac{2mV_0}{\hbar^2 K}\int_0^a \sin Kr \, r\,dr$$

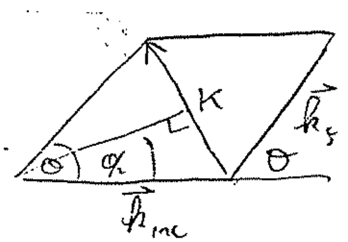
$$\boxed{\int \sin ax \, x\,dx = -\frac{\partial}{\partial a}\int \cos ax \,dx = -\frac{\partial}{\partial a}\left(\frac{1}{a}\sin ax\right) = \frac{1}{a^2}\sin ax - \frac{1}{a}x\cos ax}$$

$$= \frac{2mV_0}{\hbar^2 K}\left[\frac{1}{K^2}\sin Kr - \frac{1}{K}r\cos Kr\right]_0^a$$

$$= \frac{2mV_0}{\hbar^2 K^3}\left(\sin Ka - Ka\cos Ka\right)$$

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left(\frac{2mV_0}{\hbar^2 K^3}\right)^2\left(\sin Ka - Ka\cos Ka\right)^2}$$

elastic $|\vec{k}_{inc}| = |\vec{k}_f|$



$$K/2 = k\sin\theta/2$$

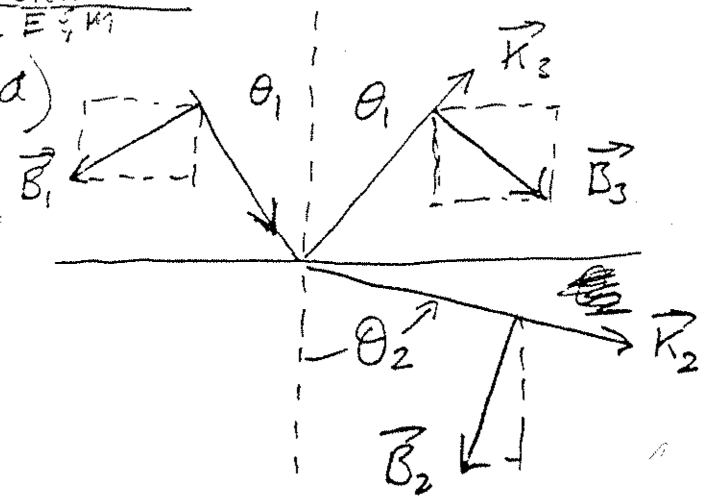
$$\frac{p_{inc}^2}{2m} = E \qquad p_{inc} = \hbar k_{inc}$$

$$k = \frac{1}{\hbar}\sqrt{2mE}$$

$$\boxed{\hbar K = 2\hbar k\sin\theta/2 = 2\sqrt{2mE}\sin\theta/2}$$

Problem # 6
Grad E & M

a)



Boundary conditions:

$$E_{1m} + E_{3m} = E_{2m}$$

$$B_{1m}\cos\theta_1 - B_3\cos\theta_1 = B_2\cos\theta_2,$$

or, $$E_{1m} - E_{3m} = \frac{E_{2m}\cos\theta_2}{n\cos\theta_1}$$

Add: $$2E_{1m} = E_{2m}\,\frac{n\cos\theta_1 + \cos\theta_2}{n\cos\theta_1},$$

$$\left(E_{2m}/E_{1m}\right)_\perp = \frac{2n\cos\theta_1}{n\cos\theta_1 + \cos\theta_2}.$$

Subtract: $$2E_{3m} = \frac{n\cos\theta_1 - \cos\theta_2}{n\cos\theta_1}E_{2m}$$

$$2E_{3m} = \frac{n\cos\theta_1 - \cos\theta_2}{n\cos\theta_1}\,\frac{2n\cos\theta_1}{n\cos\theta_1 + \cos\theta_2}\,E_{1m},$$

$$\left(E_{3m}/E_{1m}\right)_\perp = \frac{n\cos\theta_1 - \cos\theta_2}{n\cos\theta_1 + \cos\theta_2}$$

b) $$\cos\theta_2 = \sqrt{1-\sin^2\theta_2}$$

$$= \sqrt{1-n^2\sin^2\theta_1};$$

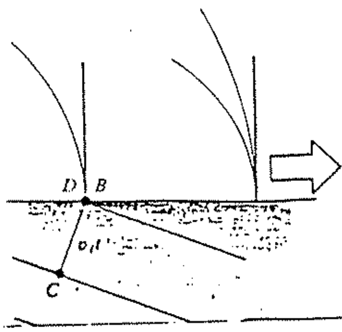
$$\therefore,\ E_{2m}/E_{1m} = \frac{2n\cos\theta_1}{n\cos\theta_1 + \sqrt{1-n^2\sin^2\theta_1}},\quad \frac{E_{3m}}{E_{1m}} = \frac{n\cos\theta_1 - \sqrt{1-n^2\sin^2\theta_1}}{n\cos\theta_1 + \sqrt{1-n^2\sin^2\theta_1}}$$

c) $$\frac{E_{3m}}{E_{1m}} = \frac{n\cos\theta_1 - i\sqrt{n^2\sin^2\theta_1 - 1}}{n\cos\theta_1 + i\sqrt{n^2\sin^2\theta_1 - 1}} = \frac{x-iy}{x+iy} = \frac{(x-iy)^2}{x^2+y^2}$$

Let $z = \rho e^{i\phi}$. Then $\rho^2 = x^2+y^2$ and $\tan\phi = -\frac{y}{x}$.

$$\frac{E_{3m}}{E_{1m}} = \frac{\left(\sqrt{n^2\cos^2\theta_1 + n^2\sin^2\theta_1 - 1}\ e^{i\phi}\right)^2}{n^2\cos^2\theta_1 + n^2\sin^2\theta_1 - 1} = e^{2i\phi}$$

$$\left|\frac{E_{2m}}{E_{1m}}\right| = 1$$



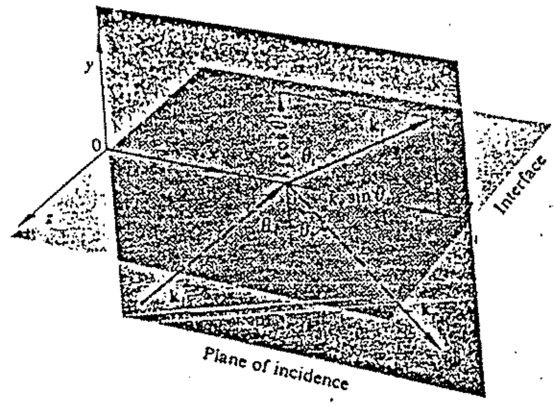


Figure 4.36 Propagation vectors for internal reflection.

$$\mathbf{E}_t = \mathbf{E}_{0t} \exp i(\mathbf{k}_t \cdot \mathbf{r} - \omega t),$$

where

$$\mathbf{k}_t \cdot \mathbf{r} = k_{tx}x + k_{ty}y,$$

there being no z-component of $\mathbf{k}$. But

$$k_{tx} = k_t \sin \theta_t$$

and

$$k_{ty} = k_t \cos \theta_t,$$

as seen in Fig. 4.36. Once again using Snell's law, we find that

$$k_t \cos \theta_t = \pm k_t \left(1 - \frac{\sin^2 \theta_i}{n_{ti}^2}\right)^{1/2} \tag{4.72}$$

or, since we are concerned with the case where $\sin \theta_i > n_{ti}$,

$$k_{ty} = \pm i k_t \left(\frac{\sin^2 \theta_i}{n_{ti}^2} - 1\right)^{1/2} \equiv \pm i\beta$$

and

$$k_{tx} = \frac{k_t}{n_{ti}} \sin \theta_i.$$

Hence

$$\mathbf{E}_t = \mathbf{E}_{0t} e^{\mp \beta y} e^{i(k_i x \sin \theta_i / n_{ti} - \omega t)}. \tag{4.73}$$

#7 Problem 7 Grad Q.M.

(a) $\sigma_{z'} = \vec{\sigma}\cdot\hat{z}' = \sigma_x \cos a + \sigma_y \cos b + \sigma_z \cos c$

$\sigma_{z'} = \cos a\begin{pmatrix}0&1\\1&0\end{pmatrix} + \cos b\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \cos c\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$\sigma_{z'} = \begin{pmatrix}\cos c & \cos a - i\cos b\\ \cos a + i\cos b & -\cos c\end{pmatrix}$

(b) Check E-values are $+1$ & -1 – Diagonalize $\sigma_{z'}$

$\begin{vmatrix}\cos c - \lambda & \cos a - i\cos b\\ \cos a + i\cos b & -\cos c - \lambda\end{vmatrix} = 0$

$-\cos^2 c + \lambda^2 - \cos^2 a - \cos^2 b = 0$

but $\cos^2 a + \cos^2 b + \cos^2 c = 1$ i.e. direction cosines

$\therefore \lambda^2 = 1 \quad ; \lambda = \pm 1$ QED

(c) Find $\alpha' = \begin{pmatrix}d\\e\end{pmatrix}$ & $\beta' = \begin{pmatrix}f\\g\end{pmatrix}$ of $+1$ & -1 resp.

when $\sigma_{z'}\alpha' = \alpha'$ & $\sigma_{z'}\beta' = -\beta'$

$\begin{pmatrix}\cos c - 1 & \cos a - i\cos b\\ \cos a + i\cos b & -\cos c - 1\end{pmatrix}\begin{pmatrix}d\\e\end{pmatrix} = 0$

$(\cos c - 1)d + (\cos a - i\cos b)e = 0$

$(\cos a + i\cos b)d - (\cos c + 1)e = 0$

$d = \dfrac{\cos a - i\cos b}{1-\cos c}\, e$

& $|d|^2 + |e|^2 = 1$

$\left[\dfrac{\cos^2 a + \cos^2 b}{(1-\cos c)^2} + 1\right]|e|^2 = 1 = \dfrac{\cos^2 a + \cos^2 b + \cos^2 c - 2\cos c + 1}{(1-\cos^2 c)^2}$

$e = \sqrt{\dfrac{1-\cos c}{2}} \quad ; \quad d = \dfrac{\cos a - i\cos b}{1-\cos c}\sqrt{\dfrac{1-\cos c}{2}} = \dfrac{\cos a - i\cos b}{\sqrt{2(1-\cos c)}}$

#7 $\alpha' = \frac{1}{\sqrt{2(1-\cos c)}} \begin{pmatrix} \cos a - i\cos b \\ 1-\cos c \end{pmatrix}$

Similarly

$$\beta' = \frac{1}{\sqrt{2(1+\cos c)}} \begin{pmatrix} -\cos a + i\cos b \\ 1+\cos c \end{pmatrix}$$

(2

(d) Expand $\alpha = \langle\alpha'|\alpha\rangle\alpha' + \langle\beta'|\alpha\rangle\beta' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\langle\alpha'|\alpha\rangle$ & $\langle\beta'|\alpha\rangle$ are amplitudes for measuring $\langle\sigma_{z'}\rangle = +1$ & -1

$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ $\therefore$ $P(\hbar/2) = |\langle\alpha'|\alpha\rangle|^2$ & $P(-\frac{\hbar}{2}) = |\langle\beta'|\alpha\rangle|^2$

$\cos a = \cos \pi/2 = 0$

$\cos b = \cos(\frac{\pi}{2} - c) = \sin c$ $\quad \alpha' = \frac{1}{\sqrt{2(1-\cos c)}} \begin{pmatrix} -i\sin c \\ 1-\cos c \end{pmatrix}$

$$\langle\alpha'|\alpha\rangle = \frac{1}{\sqrt{2(1-\cos c)}} \begin{pmatrix} i\sin c & 1-\cos c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{i\sin c}{\sqrt{2(1-\cos c)}}$$

$$\langle\beta'|\alpha\rangle = \frac{1}{\sqrt{2(1+\cos c)}} \begin{pmatrix} -i\sin c & 1+\cos c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{-i\sin c}{\sqrt{2(1+\cos c)}}$$

$$\therefore P(+\tfrac{\hbar}{2}) = \frac{\sin^2 c}{2(1-\cos c)} = \frac{1-\cos^2 c}{2(1-\cos c)} = \underline{\frac{1+\cos c}{2}}$$

$$P(-\tfrac{\hbar}{2}) = \frac{\sin^2 c}{2(1+\cos c)} = \frac{1-\cos^2 c}{2(1+\cos c)} = \underline{\frac{1-\cos c}{2}}$$

Problem # 8

(A)

$$\mathcal{Z}(\tau, \mu, V) = \sum_{\text{states}} e^{(n_s \mu - \varepsilon_s)/\tau}$$

$$= \sum_{\{n_1 n_2 n_3 \quad\}} \prod_i e^{(n_i \mu - n_i \varepsilon_i)/\tau}$$

$$= \prod_i \sum_n e^{n(\mu - \varepsilon_i)/\tau} = \prod_i \mathcal{Z}(\mu, \tau, \varepsilon_i)$$

FD $$\sum_{n=0}^{1} e^{n(\mu - \varepsilon_i)/\tau} = \underbrace{1}_{n=0} + \overbrace{e^{(\mu - \varepsilon_i)/\tau}}^{n=1}$$

BE $$\sum_{n=0}^{\infty} e^{n(\mu - \varepsilon_i)/\tau} = \left[1 - e^{(\mu - \varepsilon_i)/\tau} \right]^{-1} \qquad \mu < \varepsilon_i$$

(B)

$$\langle N \rangle = \tau \frac{\partial}{\partial \mu} \log \mathcal{Z} =$$

$$= \tau \sum_i \frac{\partial}{\partial \mu} \log \mathcal{Z}(\varepsilon_i) = \sum_i f_i$$

FD $$f_i = \frac{1}{e^{(\varepsilon_i - \mu)/\tau} + 1}$$ BE $$f_i = \frac{1}{e^{(\varepsilon_i - \mu)/\tau} - 1}$$

(C) $$f_{\substack{FD\\BE}}(\varepsilon) = \left[e^{(\varepsilon-\mu)/\tau} \pm 1 \right]^{-1}$$

(D) $$\Omega = -\tau \log \mathcal{Z}$$

$$\frac{1}{k_B} S = -\frac{\partial \Omega}{\partial \tau} =$$

$$\sum_i \left\{ \log Z(\varepsilon_i) + \tau \frac{\partial}{\partial \tau} \log Z(\varepsilon_i) \right\}$$

$$f_i = \tau \frac{\partial}{\partial \mu} \log Z(\varepsilon_i) = -\frac{\tau^2}{(\mu-\varepsilon_i)} \frac{\partial}{\partial \tau} \log Z(\varepsilon_i)$$

$$= \frac{-1}{(\mu-\varepsilon_i)/\tau} \tau \frac{\partial}{\partial \tau} \log Z(\varepsilon_i)$$

$\underline{\underline{FD}}$ $$\frac{\mu-\varepsilon_i}{\tau} = \log(Z_i - 1) \qquad f_i = \frac{Z_i - 1}{Z_i} \qquad Z_i = \frac{1}{1-f_i}$$

$$\Rightarrow \frac{1}{k_B} S = \sum_i \left\{ \log (1-f_i)^{-1} - f_i \log \left(\frac{f_i}{1-f_i} \right) \right\}$$

$$= -\sum_i \left\{ f_i \log f_i + (1-f_i) \log (1-f_i) \right\} \quad \text{QED}$$

$\underline{\underline{\text{BE}}}$. $\quad \dfrac{\mu - \varepsilon_i}{\tau} = \log\left(1 - \dfrac{1}{Z_i}\right) =$

$$f_i = Z_i - 1 \qquad Z_i = f_i\left(\frac{1}{f_i} + 1\right) = f_i + 1$$

$$\frac{1}{k_B} S = \sum_i \left\{ \log(f_i + 1) - f_i \log\left(\frac{f_i}{f_i + 1}\right) \right\}$$

$$= - \sum_i \left\{ f_i \log f_i - (1 + f_i) \log(1 + f_i) \right\} \quad \text{QED}$$