PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #56

JANUARY 9, 1988

Comprehensive Examination for Winter 1988

General Instructions

This Comprehensive Examination for Winter 1988 (#56) consists of eight problems of equal weight (20 points each). It has two parts. The first part (Problems 1-4) is handed out at 9:00 am (duration: 3 hours) and the second part (Problems 5-8) at 1:00 pm (duration 3 hours).

Work carefully, indicate your reasoning briefly, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you may find useful. Please return it at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solution. "Scratch" work will not be graded.

PART I

Problem #1

(a) Obtain the expression for the:
(1) energy levels, and
(2) the normalized wave functions
for a rigid rotator with moment of inertia, I, rotating in a plane.

(b) If the rotator has an electric dipole moment $\vec{d}$, calculate the first non-vanishing correction to the field free energy levels of the rotator when placed in a homogeneous electric field.

Problem #2

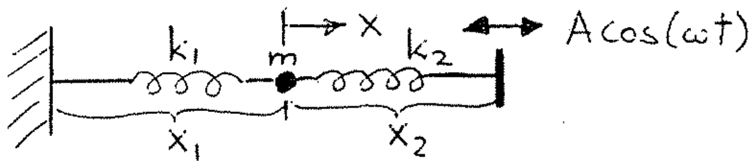
(a) (7) A one-dimensional simple harmonic oscillator with an energy level spacing of $\hbar\omega$ is in equilibrium with a heat reservoir at temperature T. Carry out the calculation of the partition function for this system.

(b) (3) Consider a system of five molecules in thermal equilibrium with a heat reservoir. The probability of any one molecule being in an excited state is 0.1. Determine the following probabilities:
i) Each molecule is in its ground state.
ii) Only one of the five molecules is in an excited state.
iii) At least two molecules are in an excited state.

(c) (10) A long slender rod of length L and heat capacity per unit length C_1 has its ends in contact with heat reservoirs at temperatures T_1 and T_2. The remainder of the rod is insulated. The reservoir at T_1 is removed suddenly and the entire rod is insulated except for its contact with the reservoir at T_2. Determine the entropy change of the rod as it comes to equilibrium. Assume $T_2 > T_1$.

Problem #3

A particle of mass m is connected to two springs as shown in the figure.



The right side where spring 2 is attached is vibrating back and forth with a displacement given by A cos (ωt). A friction force is proportional to the speed of the particle. Measure the displacement of the mass with respect to its equilibrium position in the absence of the external interaction.

(a) Find the equation of motion.

(b) Calculate the amplitude of the steady state solution by finding first the complete solution and then taking the real part of it. [Hint: replace cosωt by $e^{i\omega t}$]

(c) What is the phase relationship of x(t) with respect to the driving force?

(d) Find the resonance condition for ω

Problem #4

The grand partition function of a perfect Bose-Einstein gas is given by

$$(\text{G.P.F.}) = \prod_i \left[1 - \lambda \exp(-E_i/kT)\right]^{-1}$$

where i numbers the state of a single molecule.

(a) By appropriate series expansion show that

$$\ln(\text{G.P.F.}) = \sum_{j=1}^{\infty} j^{-1} \lambda^j \left(\sum_i e^{-jE_i/kT}\right) \text{for } \lambda < 1.$$

(b) Assuming the molecules to be monatomic and without internal states, and assuming the translational energy levels are essentially continuous, show that

$$\ln(\text{G.P.F.}) = v(2\pi kT/h^2)^{3/2} \sum_{j=1}^{\infty} j^{-5/2} \lambda^j$$

(c) Calculate the pressure and density of the gas.

PART II

Problem #5

(a) Calculate the energy spectrum of an energy level of hydrogen with quantum numbers n,l,s,j, and m_j when the atom is placed in a constant, homogeneous, magnetic field, $\vec{B}$, if the interaction energy of the electron and magnetic field is much larger than the spin-orbit interaction energy (Paschen-Back effect).

(b) Repeat if the electron-magnetic field interaction energy is much smaller than the spin-orbit term (Zeeman effect).

Hint: The Wigner-Eckart Theorem may be stated as

$$\langle lsjm_j|\vec{V}|lsjm_j\rangle = \text{constant} \times \langle lsjm_j|\vec{J}|lsjm_j\rangle,$$

where $\vec{J}$ is the total angular momentum operator and $\vec{V}$ is any vector operator (irreducible tensor operator of rank 1).

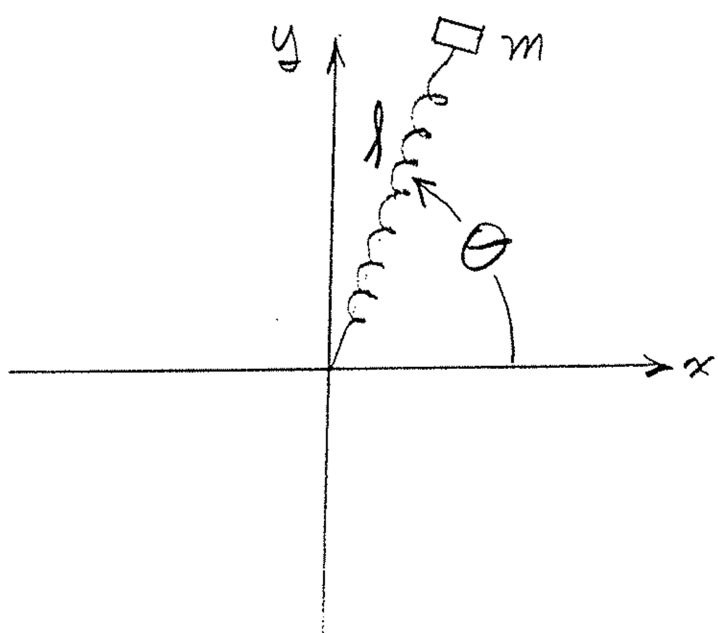
Problem #6

A mass, m, is connected to a center of force through a massless spring of unstretched length l_o, as indicated in the figure below. The mass moves on a frictionless horizontal platform. Both θ and l are variable; and the spring constant of the spring is K. The spring will not bend.

(a) Construct a Lagrangian for this system, and find the equations of motion.

(b) What are the constants of the motion?

(c) Identify a variable in the equations of motion which is likely to be a small quantity; and use an appropriate approximation to rewrite the equations of motion in a form which can be solved. Obtain an approximate solution of the equations of motion for the general case.



Problem #7

(a) Derive the nonrelativistic dispersion relation [$\omega=\omega(k)$] for the deBroglie wave of a particle of mass m_o moving in a region with constant potential energy U.

(b) Repeat the calculation of a) for a particle moving relativistically in free space (U=0).

(c) Assume that the deBroglie wave for part a) is given by

$$\psi(x,t) = C_1 e^{-i(\omega t+kx)} + C_2 e^{-i(\omega t-kx)}$$

Prove that ψ satisfies the Schrodinger equation.

$$i\hbar \frac{\partial \psi}{\partial t} = - \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + U\psi.$$

Problem #8

An isolated permanent electric point dipole $\vec{P} = P\hat{e}_z$ has a potential

$$U_0(r,\theta) = \frac{P \cos \theta}{r^2} ,$$

in spherical coordinates. If this dipole is placed at the center of a grounded conducting spherical shell of radius a, find

(a) the potential $U(r,\theta)$ inside the spherical shell.

(b) the surface charge density $\sigma(\theta)$ induced on the spherical shell.

1

$$\mathcal{H}\psi(\varphi) = E\psi(\varphi)$$

$$\mathcal{H} = T + V \qquad V = 0$$

$$T = \frac{1}{2} I\omega^2 = \frac{1}{2I} L_z^2 \quad \text{since } L_z = I\omega$$

$$L_z = -i\hbar \frac{\partial}{\partial\varphi}$$

$$\frac{-\hbar^2}{2I}\frac{\partial^2\psi(\varphi)}{\partial\varphi^2} = E\psi(\varphi)$$

$$\frac{\partial^2\psi}{\partial\varphi^2} + \frac{2IE}{\hbar^2}\psi = 0 \quad \text{— S.H. Osc. eqn}$$

$$\therefore \psi(\varphi) = A e^{ik\varphi} + B e^{-ik\varphi} \qquad k = \sqrt{\frac{2IE}{\hbar^2}}$$

But $\psi(\varphi) = \psi(\varphi + 2\pi)$

or $e^{\pm ik2\pi} = 1 \quad \therefore k = \text{integer} = m = 0, \pm 1, \pm 2$ etc

$$\psi_m(\varphi) = A e^{im\varphi}$$

$$E_m = \frac{\hbar^2 m^2}{2I}$$

$$\int_0^{2\pi} |\psi_m|^2 d\varphi = 1 = A^2 \int_0^{2\pi} d\varphi = 2\pi A^2 \quad \therefore A = \frac{1}{\sqrt{2\pi}}$$

$$\psi_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$$

(b) $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}' \qquad \mathcal{H}_0 = \frac{-\hbar^2}{2I}\frac{\partial^2\psi}{\partial\varphi^2}, \quad \mathcal{H}' = -\vec{d}\cdot\vec{\mathcal{E}}$

$$E_n = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} \ldots$$

$$E_n^{(0)} = E_n(\mathcal{E} = 0) = \frac{\hbar^2}{2I} m^2$$

$$E_n^{(1)} = \int_0^{2\pi} \psi_n^* \mathcal{H}' \psi_n \, d\varphi$$

Consider $\vec{\mathcal{E}} = \mathcal{E}\hat{x} \quad \therefore \vec{d}\cdot\vec{\mathcal{E}} = d\mathcal{E}\cos\varphi$

$$E_n^{(1)} = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\varphi}\cos\varphi \, e^{im\varphi} d\varphi = \int_0^{2\pi} \cos\varphi \, d\varphi = 0$$

\# 1. $E_m^{(2)} = \sum_{\substack{n=0 \\ n \neq m}} \frac{H'_{mn}}{E_m^{(0)} - E_n^{(0)}}$ $\qquad H'_{mn} = \int_0^{2\pi} \psi_m^* H' \psi_n \, d\varphi$

$$H'_{mn} = \frac{d\mathcal{E}}{2\pi} \int_0^{2\pi} e^{-i(m-n)\varphi} \cos\varphi \, d\varphi = \frac{d\mathcal{E}}{4\pi} \int_0^{2\pi} e^{-i(m-n-1)\varphi} + e^{-i(m-n+1)\varphi}$$

$$H'_{mn} = \frac{d\mathcal{E}}{2}\left[\delta_{n,m-1} + \delta_{n,m+1}\right]$$

$$E_m^{(2)} = \frac{d^2\mathcal{E}^2}{4}\left[\frac{1}{E_m^{(0)} - E_{m-1}^{(0)}} + \frac{1}{E_m^{(0)} - E_{m+1}^{(0)}}\right] = \frac{d^2\mathcal{E}^2 I}{2\hbar^2}\left[\frac{1}{m^2-(m-1)^2} + \frac{1}{m^2-(m+1)^2}\right]$$

$$= \frac{d^2\mathcal{E}^2 I}{2\hbar^2}\left[\frac{1}{2m-1} - \frac{1}{+2m+1}\right] = \frac{d^2\mathcal{E}^2 I}{2\hbar^2}\left[\frac{2m+1-(2m-1)}{4m^2-1}\right]$$

$$E_m^{(2)} = \frac{d^2\mathcal{E}^2 I}{\hbar^2} \frac{1}{4m^2-1} \qquad \text{or}$$

$$E_m = \frac{\hbar^2}{2I} m^2 + \frac{d^2\mathcal{E}^2 I}{\hbar^2} \frac{1}{4m^2-1}$$

#2 a) $Z = \sum_{n=0}^{\infty} e^{-\epsilon_n/kT}$, where $\epsilon_n = (n+\frac{1}{2})\hbar\omega$;

$\therefore Z = e^{-\frac{\hbar\omega}{2kT}} \sum_{n=0}^{\infty} x^n$, where $x = e^{-\hbar\omega/kT}$

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$; $\therefore Z(SHO) = \frac{e^{-\hbar\omega/2kT}}{1-e^{-\hbar\omega/kT}}$

b) i) $(.9)^5$, (ii) $.9^4/10$ (iii) $.01$

c) $\Delta S = \int \frac{đQ}{T}$. But all of the heat flow into the rod occurs at the high temperature, T_f;

$\therefore$, $\Delta S = \frac{\Delta Q}{T_f}$, where ΔQ is the total heat transferred to the rod.

For any small section of rod the heat transfer is $đQ = C_l\, dl\,(T_f - T)$, where T is the initial temperature at the position of interest, i.e. $đQ(x) = C_l\, dx\,[T_f - T(x)]$. Assume $T(x) = \frac{\Delta T}{L}x + T_i$ where $\Delta T = T_f - T_i$ and x is measured from the low temperature end of the rod. Thus,

$đQ = C_l\, dx\left[T_f - T_i - \frac{T_f - T_i}{L}x\right]$; and, $\Delta Q = C_l L\left[\Delta T - \frac{\Delta T}{2}\right]$

Finally, $\Delta S = \left(1 - \frac{T_i}{T_f}\right)\frac{C_l L}{2}$.

3 Comp Exam

Solution

equilibrium cond $k_1 X_1 = k_2 X_2$.

make displacement x

$$F = -k_1(x_1 + x) + k_2(x_2 - x) + k_2 A\cos\omega t - b\dot{x}$$

$$= -(k_1+k_2)x - b\dot{x} + k_2 A\cos\omega t.$$ (3)

a) $\underline{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = B\cos\omega t.}$ eq of motion

$$\beta = \frac{b}{2m} \quad \omega_0^2 = \frac{k_1+k_2}{m} \quad B = \frac{k_2 A}{m.}$$

b) let $\cos\omega t = \mathrm{Re}\,(e^{i\omega t})$

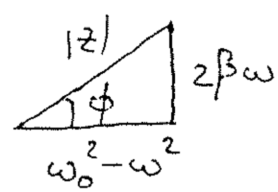
$$x(t) = X_0 e^{i\omega t}$$

$$\dot{x}(t) = i\omega X_0 e^{i\omega t} \qquad \ddot{x} = -\omega^2 X_0 e^{i\omega t}$$

substitute into eq of motion

(5)

$$\underbrace{\underbrace{(-\omega^2 + i2\beta\omega + \omega_0^2)}_{(\omega_0^2-\omega^2) + i\,2\beta\omega}}_{|z|e^{i\phi}} X_0 = B$$



$$|z| = \left[(\omega_0^2-\omega^2)^2 + (2\beta\omega)^2\right]^{\frac{1}{2}}$$

$$\phi = \tan^{-1}\left[\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right]$$

#3

$$X_0 = \frac{B}{|z|} e^{-i\phi} \qquad x(t) = \frac{\beta}{|z|} e^{i(\omega t - \phi)}$$

take real part

$$x(t) = \frac{B}{|z|} \cos(\omega t - \phi)$$

amplitude is $$\frac{B}{[(\omega_0^2-\omega^2)^2 + (2\beta\omega)^2]^{\frac{1}{2}}} \qquad (4)$$

c) phase is $$\tan^{-1} \frac{2\beta\omega}{\omega_0^2 - \omega^2} \qquad (3)$$

d) resonance cond.

$$\frac{d}{d\omega}\left[(\omega_0^2-\omega^2)^2+(2\beta\omega)^2\right]^{-\frac{1}{2}} = 0$$

or $$\omega\left[\omega_0^2-\omega^2-2\beta^2\right] = 0$$

$$\omega_1 = 0$$

$$\underline{\omega_2 = \sqrt{\omega_0^2 - 2\beta^2}} \qquad (5)$$

Problem 4

a) $(G.P.F) = \prod_i (1-\lambda e^{-\varepsilon_i/kT})^{-1}$

$\ln (G.P.F) = -\sum_i \ln(1-\lambda e^{-\varepsilon_i/kT})$

Now for $x<1$, $-\ln(1-x) = \sum_{j=1}^{\infty} \frac{x^j}{j}$

Hence, if $\lambda < 1$, $\lambda e^{-\varepsilon_i/kT} < 1$ for all i ($\varepsilon_i \geq 0$)

and thus

$$-\ln(1-\lambda e^{-\varepsilon_i/kT}) = \sum_{j=1}^{\infty} j^{-1}\lambda^j e^{-j\varepsilon_i/kT}$$

and

$$\ln(G.P.F.) = \sum_i \sum_{j=1}^{\infty} j^{-1}\lambda^j e^{-j\varepsilon_i/kT} = \sum_{j=1}^{\infty} j^{-1}\lambda^j \left(\sum_i e^{-j\varepsilon_i/kT}\right)$$

where we exchange order of summation since the limits are independent

b) If we are dealing with a monatomic gas, we can either use the quantum value. $\varepsilon_i = \frac{h^2}{8mV^{2/3}}(p^2+q^2+r^2)$ or use the classical phase integral. If we choose the latter, $\sum_i$ is replaced by integration over phase space of a particle

$$\sum_i e^{-j\varepsilon_i/kT} = \frac{V}{h^3}\iiint e^{-j(p_x^2+p_y^2+p_z^2)/2mkT}\, dp_x\, dp_y\, dp_z$$

$$= \frac{V}{h^3}(2\pi m kT/j)^{3/2} = j^{-3/2}(2\pi m kT/h^2)^{3/2}\, V$$

thus

$$\ln(G.P.F.) = V(2\pi mkT/h^2)^{3/2}\sum_{j=1}^{\infty} j^{-5/2}\lambda^j$$

when $\lambda \ll 1$, $\sum_j j^{-5/2}\lambda^j \simeq \lambda$, and then

$$\ln(G.P.F.) = (2\pi mkT/h^2)^{3/2}V\lambda$$ the classical result.

#4

(2)

c) $P = -\left(\frac{\partial A}{\partial V}\right)_T$; $A = -kT \ln(G.P.F.)$

$$\frac{P}{kT} = \left[\partial \ln(G.P.F.)/\partial V\right]_{T,\lambda} = \left(2\pi m kT/h^2\right)^{3/2} \sum_j j^{-5/2} \lambda^j$$

(or we could use $PV = kT \ln(G.P.F.)$ directly)

$$N = \lambda\left[\partial \ln(G.P.F.)/\partial \lambda\right]_{T,V} = V\left(2\pi m kT/h^2\right)^{3/2} \sum_j j^{-3/2} \lambda^j$$

#5. $\mathcal{H}\psi = E\psi$

$\mathcal{H} = \mathcal{H}_0 + \xi(r)\,\vec{L}\cdot\vec{S} + g_e\mu_0\vec{B}\cdot\frac{\vec{L}}{\hbar} + g_s\mu_0\vec{B}\cdot\frac{\vec{S}}{\hbar}$

$\mathcal{H}_0 = T+V = -\frac{\hbar^2}{2m}\nabla^2 + V(r)$; $\mu_0 \equiv$ Bohr magneton

$\mathcal{H}_0\varphi_{n\ell s m_\ell m_s} = E_{n\ell s m_\ell m_s}\varphi_{n\ell s m_\ell m_s}$

&

$\left[\mathcal{H}_0 + \xi(r)\vec{L}\cdot\vec{S}\right]\psi_{n\ell s j m_j} = E_{n\ell s j m_j}\psi_{n\ell s j m_j}$

for $g_e\mu_0\vec{B}\cdot\frac{\vec{L}}{\hbar} + g_s\mu_0\vec{B}\cdot\frac{\vec{S}}{\hbar} \gg \xi(r)\vec{L}\cdot\vec{S}$

(a) neglect $\vec{L}\cdot\vec{S}$ term & use magnetic term as a perturbation with $\varphi_{n\ell s m_\ell m_s}$ as a basis set.

Consider $\vec{B} = \hat{k}B$ $\quad g_e \equiv -1$

$\therefore\ \mathcal{H}' = -\frac{\mu_0}{\hbar}B\left[L_z + g_s S_z\right]$

$E^{(1)}_{n\ell s m_\ell m_s} = -\mu_0 B\langle n\ell s m_\ell m_s | L_z + g_s S_z | n\ell s m_\ell m_s\rangle$

but $L_z|n\ell s m_\ell m_s\rangle = m_\ell\hbar|n\ell s m_\ell m_s\rangle$

& $S_z|n\ell s m_\ell m_s\rangle = m_s\hbar|n\ell s m_\ell m_s\rangle$

$E^{(1)}_{n\ell s m_\ell m_s} = \mu_0 B\left[m_\ell + g_s m_s\right]$

$E_{n\ell s m_\ell m_s} = E^{(0)}_{n,\ell} + \mu_0 B\left[m_\ell + g_s m_s\right]$

(b) for $g_e\mu_0\vec{B}\cdot\frac{\vec{L}}{\hbar} + g_s\mu_0\vec{B}\cdot\frac{\vec{S}}{\hbar} \ll \xi(r)\vec{L}\cdot\vec{S}$

use magnetic terms as a perturbation with

#5.

the ψ_{nlsjm_j} ... the "unperturbed" Hamiltonian is $H_0 + \xi(r)\vec{L}\cdot\vec{S} \rightarrow E^{0}_{nlj}$

$$E^{(1)}_{nlsjm_j} = \frac{\mu_0 B}{\hbar}\langle nlsjm_j \,|\, L_z + g_s S_z \,|\, nlsjm_j\rangle$$

but $|nlsjm_j\rangle$'s are not eigen states of L_z & S_z

use W-E th. as follows

$$\langle jm_j|\vec{L}|jm_j\rangle = C_L\langle jm_j|\vec{J}|jm_j\rangle = C_L m_J\hbar$$

Consider $\langle jm_j|\vec{L}\cdot\vec{J}|jm_j\rangle = \sum\langle jm_j|\vec{L}|jm_j'\rangle\langle jm_j'|J|jm_j\rangle$

by closure i.e. $\sum_{m_j'}|jm_j'\rangle\langle jm_j'| = 1$

$$= C_L\sum_{m_j'}\langle jm_j|\vec{J}|jm_j'\rangle\langle jm_j'|\vec{J}|jm_j\rangle$$

$$= C_L\langle jm_j|J^2|jm_j\rangle = C_L\, j(j+1)\hbar^2$$

$$\therefore\ C_L = \frac{\langle jm|\vec{L}\cdot J|jm\rangle}{j(j+1)}$$

& $\vec{L}\cdot\vec{J} = L^2 + L\cdot\vec{S} = L^2 + \frac{1}{2}(J^2 - L^2 - S^2) = \frac{1}{2}(J^2 + L^2 - S^2)$

since $J^2 = L^2 + S^2 + 2\vec{L}\cdot\vec{S}$

$$\therefore\ C_L = \frac{1}{2j(j+1)}\langle lsjm_j|J^2 + L^2 - S^2|lsjm_j\rangle = \frac{1}{2j(j+1)}\left[j(j+1) + l(l+1) - s(s+1)\right]$$

similarly for S_z

$$C_s = \frac{1}{2j(j+1)}\left[j(j+1) + s(s+1) - l(l+1)\right]$$

$$\langle lsjm_j|L_z + g_s S_z|lsjm_j\rangle = (C_L + g_s C_s)\langle lsjm_j|J_z|lsjm_j\rangle$$

#5. (3)

$$E_{n\ell s j m_j} = (C_\ell + g_s C_s)\, \mu_0 B\, m_J$$

$$(C_\ell + g_s C_s) \equiv g_J = \frac{1}{2J(J+1)} \left\{ J(J+1) + \ell(\ell+1) - s(s+1) + g_s \left[J(J+1) + s(s+1) - \ell(\ell+1) \right] \right\}$$

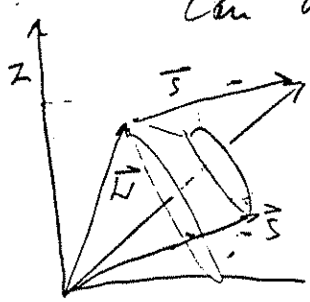
$$g_J = 1 + \frac{(g_s - 1)\left[J(J+1) + s(s+1) - \ell(\ell+1) \right]}{2J(J+1)} \quad \text{[Landé g factor]}$$

for $g_s \cong 2$ $\qquad g_J = 1 + \dfrac{J(J+1) + s(s+1) - \ell(\ell+1)}{2J(J+1)}$

$$\therefore\ E^{(1)}_{n\ell s j m_j} = g_J\, \mu_0 B\, m_J$$

$$E_{n\ell s j m_j} = E^{(0)}_{n\ell_j} + g_J\, \mu_0 B\, m_J$$

Can also use Vector model



where $J \equiv \sqrt{J(J+1)}$
$L = \sqrt{\ell(\ell+1)}$
$S = \sqrt{s(s+1)}$

$\mu_J = g_J \mu_0 m_J$; $\mu_s = g_s \mu_0 m_s$; $\mu_\ell = g_\ell \mu_0 m_\ell$

$$\mu_J = g_J \mu_0 J \cos(J, z)$$

Let $\quad g_J \mu_0 J = g_\ell \mu_0 L \cos(L, \vec{J}) + g_s \mu_0 S \cos(S, J)$

$\cos(L \cdot J) = \dfrac{\vec{L} \cdot \vec{J}}{LJ} \qquad \because\ (\vec{J} - \vec{L})^2 = J^2 + L^2 - 2\vec{L}\cdot\vec{J} = S^2$

$\therefore\ L \cos(\vec{L}, \vec{J}) = \dfrac{J^2 + L^2 - S^2}{2J} \quad$ & $\quad S\cos(\vec{S}, \vec{J}) = \dfrac{J^2 + S^2 - L^2}{2J}$

$$\therefore\ g_J = g_\ell \left(\frac{J^2 + L^2 - S^2}{2J^2} \right) + g_s \left(\frac{J^2 + S^2 - L^2}{2J^2} \right)$$

Since $g_\ell = 1 \qquad g_s = 2$

$$\therefore\ g_J = 3/2 + \frac{S^2 - L^2}{2J^2} = 1 + \frac{J(J+1) + s(s+1) - \ell(\ell+1)}{2J(J+1)}$$

#6.

$$T = \tfrac{1}{2} m (\dot{l}^2 + l^2 \dot{\theta}^2)$$

$$V = \tfrac{1}{2} k (l - l_0)^2 = \tfrac{1}{2} k \eta^2, \eta = l - l_0$$

$$T = \tfrac{1}{2} m (\dot{\eta}^2 + l_0^2 \dot{\theta}^2 + 2 l_0 \eta \dot{\theta}^2 + \eta^2 \dot{\theta}^2)$$

$$\mathcal{L} = \frac{m}{2}[\dot{\eta}^2 + (l_0 + \eta)^2 \dot{\theta}^2] - \frac{\eta^2}{2} k \,.\ \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m l^2 \dot{\theta},\ \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{d}{dt}(m l^2 \dot{\theta}) = 0,\ m l^2 \dot{\theta} = \text{const} = L_z$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\eta}} = m \dot{\eta} \ ,\ \frac{\partial \mathcal{L}}{\partial \eta} = 2m (l_0 + \eta) \dot{\theta}^2 - k\eta$$

$$m \ddot{\eta} - 2 m l \dot{\theta}^2 + k\eta = 0 = m\ddot{\eta} - 2ml \frac{L_z^2}{m^2 l^4} + k\eta$$

$$\ddot{\eta} - 2\left(\frac{L_z^2}{m^2 l^3}\right) + \frac{k}{m}\eta = 0,\ l^{-3} = \frac{1}{l_0^3}\left(1 + \frac{\eta}{l_0}\right)^{-3} \simeq \frac{1}{l_0^3}\left(1 - 3\frac{\eta}{l_0}\right);$$

$$\ddot{\eta} - \frac{2L_z^2}{m^2 l_0^3}\left(1 - 3\frac{\eta}{l_0}\right) + \omega_0^2 \eta = 0$$

$$\ddot{\eta} + \left(\omega_0^2 + \frac{6L_z^2}{m^2 l_0^4}\right)\eta = \frac{2L_z^2}{m^2 l_0^3} \ .\ \text{Assume } \eta = \eta_0 \cos(\omega t + \phi) + \cancel{A}$$

$$-\omega^2 \eta_0 \cos(\omega t + \phi) + \left(\omega_0^2 + \frac{6L_z^2}{m^2 l_0^4}\right)\eta_0 \cos(\omega t + \phi) + \left(\omega_0^2 + \frac{6L_z^2}{m^2 l_0^4}\right) A = \frac{2L_z^2}{m^2 l_0^3}$$

$$\omega^2 = \omega_0^2 + 6\left(\frac{L_z}{m l_0^2}\right)^2,\quad A = \frac{2L_z^2 / m^2 l_0^3}{\omega^2} \ .\ \text{Choose } \phi = 0$$

$$\dot{\theta} = \frac{L_z}{m(l_0 + \eta)^2} = \frac{L_z}{m l_0^2}\left(1 + \frac{\eta}{l_0}\right)^{-2} \simeq \frac{L_z}{m l_0^2}\left(1 - 2\frac{\eta}{l_0}\right)$$

$$\dot{\theta} \simeq \frac{L_z}{m l_0^2}\left[1 - \frac{2\eta_0}{l_0}\cos \omega t\right]$$

$$\theta \simeq \frac{L_z}{m l_0^2}\left[t - \frac{2\eta_0}{\omega l_0}\sin \omega t\right]$$

7 Comp Exam

Solution

a) $E = h\nu \qquad E = \frac{p^2}{2m_0} + U \;; \quad p = \frac{h\nu}{c}$

$$h\nu = \frac{h^2}{\lambda^2 2m_0} + U \qquad h\nu = \hbar\omega$$

$$k = \frac{2\pi}{\lambda}.$$

(5)

$$\underline{\hbar\omega = \frac{\hbar^2 k^2}{2m_0} + U}$$

b) $E = h\nu + m_0 c^2$

$$E^2 = c^2 p^2 + (m_0 c^2)^2 \qquad p = \frac{h\nu}{c}$$

(10)

$$h\nu + m_0 c^2 = \left[\frac{c^2 h^2}{\lambda^2} + m_0^2 c^4\right]^{\frac{1}{2}}$$

$$\underline{\hbar\omega = m_0 c^2\left[\sqrt{1 + \frac{\hbar^2 k^2}{m_0^2 c^2}} - 1\right]}$$

for $\hbar k \ll m_0 c$

$$\hbar\omega \simeq m_0 c^2\left[1 + \frac{1}{2}\frac{\hbar^2 k^2}{m_0^2 c^2} - 1\right]$$

$$\simeq \frac{\hbar^2 k^2}{2m_0}$$ the nonrelativistic expression for $U = 0$

#7 c) $\frac{\partial \psi}{\partial t} = -i\omega\psi$

$$\frac{\partial \psi}{\partial x} = -ikC_1 e^{-i(\omega t + kx)} + ikC_2 e^{-i(\omega t - kx)}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2\psi.$$

from dispersion relationship (part a)

$$-i\omega\psi = -i\frac{\hbar k^2}{2m_0}\psi - i\frac{U}{\hbar}\psi$$

$$\underline{i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + U\psi}$$ Q.E.D. (5)

Problem 8

The general solution of Laplaces eq in spherical coordinates is

a) $$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l\, r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

the B.C. to be satisfied are

$$\Phi(r,\theta) \to \Phi_0(r,\theta) = \frac{P\cos\theta}{r^2} \quad \text{as } r \to 0 \qquad (1)$$

and $$\Phi(a,\theta) = 0 \qquad (2)$$

we may specialize the expansion to

$$\Phi(r,\theta) = \left(A_0 + \frac{B_0}{r}\right) + \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta + \dots$$

B.C. (1) requires $A_0 = B_0 = 0$ and $B_1 = P$, and all higher order terms vanish

B.C. (2) requires $\left(A_1 a + \frac{P}{a^2}\right)\cos\theta = 0 \quad \therefore \quad A_1 = -\frac{P}{a^3}$

$$\therefore \quad \Phi(r,\theta) = \left(-\frac{P}{a^3} r + \frac{P}{r^2}\right)\cos\theta = \underline{\underline{P\left(\frac{1}{r^2} - \frac{r}{a^3}\right)\cos\theta}}$$

$$= -\frac{P r}{a^3}\cos\theta + \Phi_0(r,\theta)$$

b) The surface charge is induced on the inside of the sphere. The outside is closed and shielded by the conductor. By Gauss' Law

$$\int \vec{E}\cdot\hat{n}\, da = -E_r\Big|_{r=a}(4\pi R^2) = \frac{Q}{\epsilon_0} \qquad : Q = \text{induced charge}$$

$$\sigma = Q/4\pi R^2$$

$$-E_r\Big|_{r=a} = \frac{\sigma}{\epsilon_0} = \frac{\partial\Phi}{\partial r}\Big|_{r=a} = -P\left(\frac{1}{a^3} + \frac{2}{r^3}\right)_{r=a}\cos\theta = -\frac{3P}{a^3}\cos\theta$$

$$\boxed{\sigma = \frac{-3\epsilon_0 P\cos\theta}{a^3}}$$

the total induced charge is zero.