PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #52

January 11, 1986

General Instructions

This Comprehensive Examination for Winter 1986 (#52) consists of eight problems of equal weight (20 points each). It has two parts. The first part (Problems 1-4) is handed out at 9:00 a.m. and the second part (Problems 5-8) at 1:00 p.m.

Work carefully, indicate your reasoning briefly, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but *not* your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you *may* find useful. Please return it at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solution. "Scratch" work will *not* be graded.

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PART I.

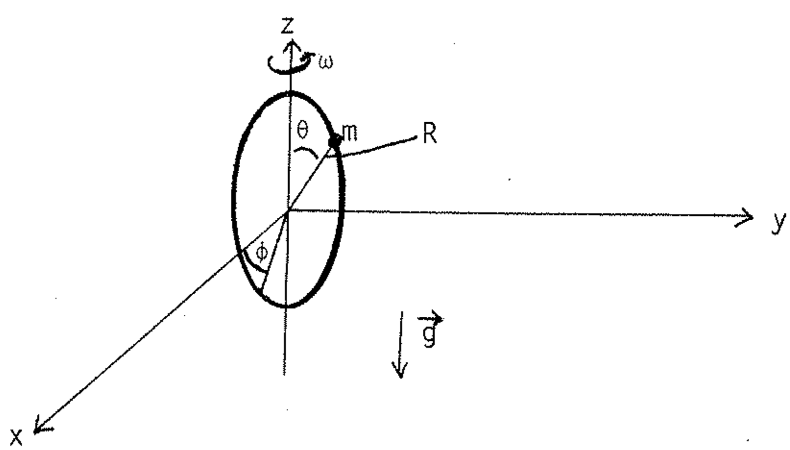
PROBLEM #1.

Consider an ideal gas contained in a volume V at temperature T. The gas is composed of N distinguishable particles of zero rest mass so that energy E and momentum p are related by $E = pc$. The number of single-particle energy states in the range p to p + dp is $4\pi V p^2 dp/h^3$.

Find the equation of state and the internal energy and compare your results with those for an ordinary gas.

PROBLEM #2.

A particle of mass m is constrained to move without friction on a circular wire which rotates with constant non-zero angular velocity ω about a vertical axis. (see Fig.)



a) Using the generalized coordinates θ and ϕ find the Lagrangian equation of motion.

b) Find the equilibrium position of the particle and calculate the frequency of small oscillation around this position.

c) Find and interpret physically a critical angular velocity $\omega = \omega_c$ which divides the motion of the particle into two distinct types.

PROBLEM #3.

The general form of the Heisenberg Uncertainty Principle for position and momentum is expressed as an inequality:

$$\Delta x \, \Delta p_x \geqslant \frac{\hbar}{2}$$

For a system with Gaussian state function,

$$\psi(x) = \frac{1}{\sqrt{\sigma \sqrt{\pi}}} e^{-\frac{x^2}{2\sigma^2}}$$

where σ is a constant,
find the exact form of the Principle (be careful of your definition of "uncertainty", Δx and Δp_x).

Grad E & M

PROBLEM #4.

A fuel pipe carrying ethyl alcohol (dielectric constant = 19) is built from a thin-walled insulating tube of square cross-section. The poles of a magnet which produces a field of 0.1 tesla are in contact with two opposite walls of the tube. The magnet gap is 3 cm. In the other two walls there are two conducting electrodes which are connected to a voltmeter of very high input impedance. If the alcohol is flowing through the tube with a velocity of 5.0 m/s, calculate the voltage developed across the electrodes.

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PART II.

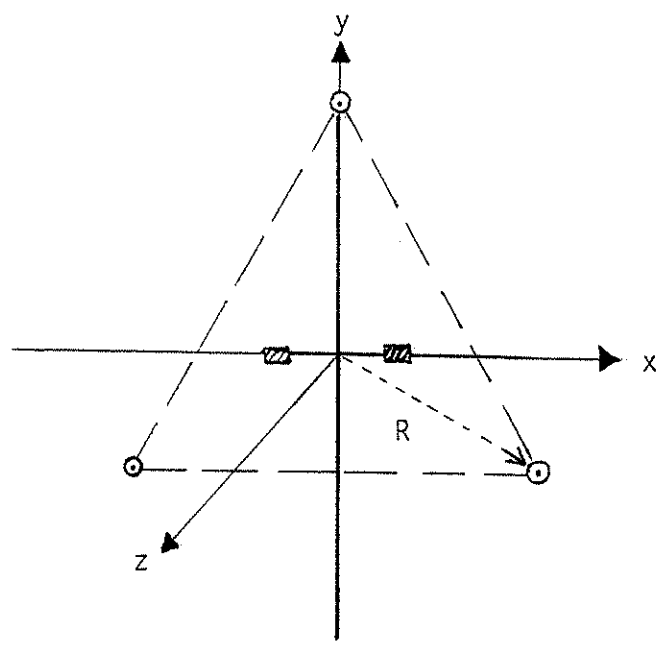
PROBLEM #5.

An excited atom of mass m_0 initially at rest emits a photon and recoils. The internal energy of the atom decreases by ΔE. Find a relativistic expression for the energy of the photon in terms of ΔE and the rest energy of the excited atom.

PROBLEM #6.

a) Three long, thin, cylindrical wires are arranged (in cross-section) at the corners of an equilateral triangle which lies in the x-y plane. One wire carries a slowly varying sinusoidal current given by $I(t) = I_0 \sin(\omega_0 t)$. The other two wires carry currents of the same magnitude and frequency, but each current is 120 degrees out of phase with the other two. Take the positive current direction as the z-axis in the figure below. Find the magnetic field $\underline{B}$ at the center of the triangle. Describe in words the time dependence of $\underline{B}$. The medium is air.

b) Suppose that a small coil of n turns and of area A is placed at the center of the triangle, with the plane of the coil in the x-z plane (see fig. below). Calculate the frequency and magnitude of the induced e.m.f. in the coil when (a) the coil is held stationary and (b) the coil is rotated with an angular frequency of ω_1 rad/sec with the z axis as the axis of rotation.



PROBLEM #7.

a) Show that $\Psi_0(x,t) = \Phi_0(x) \exp[-i\beta t]$

where $\Phi_0(x) = (\alpha/\pi)^{\frac{1}{2}} \exp[-\frac{1}{2}\alpha^2 x^2]$
is a solution of the Schrödinger equation for a linear harmonic oscillator of mass m and characteristic frequency ω.

Find the values of α and β in terms of physical constants.

b) The first excited state of the same oscillator is described by the wave function $\Psi_1(x,t) = \Phi_1(x) \exp[-3i\omega t/2]$

where $\Phi_1 = (2\alpha^3/\sqrt{\pi})^{\frac{1}{2}} x \exp[-\frac{1}{2}\alpha^2 x^2]$

At t = 0, the particle is in a mixed state given by

$$\Psi(x,0) = \Phi_0 \cos\theta + \Phi_1 \sin\theta$$

where θ is a real mixing coefficient.

Find the mean position, $\overline{x}(t)$, of the particle at a subsequent time t.

PROBLEM #8.

A sample of germanium shows no Hall Effect. The mobility of electrons in germanium is $\mu_e = 3500\ cm^2$/volt sec and for holes is $\mu_h = 1400\ cm^2$/volt sec. What fraction of the current in the sample is carried by electrons?

#1 Solution

The eq of state may be obtained from the partition function.

$$Z = \int e^{-\frac{E}{KT}} dV_1 \cdots dV_n \; \frac{d^3p_1 \cdots d^3p_N}{h^{3N}}$$

$$= \left[\frac{4\pi V}{h^3} \int_0^{\infty} e^{-pc/KT} p^2 dp \right]^N$$

$$Z = \left[\left(\frac{kT}{c}\right)^{-} \left(\frac{8\pi V}{h^3}\right) \right]$$

Connect with thermodynamics through

$$F = -kT \ln Z$$

$$dF = -P dV - S dT \qquad \text{from which}$$

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = kT \frac{\partial \ln Z}{\partial V} = \frac{NKT}{V}$$

Internal energy is

$$U = -\frac{\partial \ln Z}{\partial \frac{1}{KT}} = 3NKT$$

The pressure is the same as that of an ordinary gas; the energy U is <u>twice</u> the one of an ordinary gas.

#2 $\quad \omega = \dot{\phi} \qquad L = \frac{1}{2} I_1 \dot{\phi}^2 + \frac{1}{2} I_2 \dot{\theta}^2 - mg\cos\theta$

$I_1 = mR^2 \sin^2\theta \qquad I_2 = mR^2$

Lagrange's eq.

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

a) $\quad \ddot{\theta} - \omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta = 0$

equilibrium position $\quad (\ddot{\theta} = 0)$

b) $\quad \cos\theta_0 = -\frac{g}{R\omega^2} \qquad \omega \neq 0$

(in lower half of the ring)

(Mechanics (G) solution)

frequency of small oscillation:

let $\theta = \theta_0 + \varepsilon$ $\qquad \varepsilon$ small.

$$\ddot{\varepsilon} - \omega^2 (\sin\theta_0 + \varepsilon\cos\theta_0)(\cos\theta_0 - \varepsilon\sin\theta_0)$$

$$- \frac{g}{R}(\sin\theta_0 + \varepsilon\cos\theta_0) = 0$$

neglecting terms of order ε^2 we have

$$\ddot{\varepsilon} - \varepsilon\left[\omega^2(\cos^2\theta_0 - \sin^2\theta_0) + \frac{g}{R}\cos\theta_0\right]$$

$$\underbrace{-\omega^2\sin\theta_0\cos\theta_0 - \frac{g}{R}\sin\theta_0}_{0} = 0$$

since $\cos\theta_0 = -\dfrac{g}{R\omega^2}$

$$\ddot{\varepsilon} + \varepsilon\underbrace{\left[\omega^2(\sin^2\theta_0 - \cos^2\theta_0) - \frac{g}{R}\cos\theta_0\right]}_{\omega_0^2} = 0$$

b) $$\omega_0^2 = \omega^2\left[1 - \frac{g^2}{R^2\omega^4}\right]$$

c) from $\cos\theta_0 = -\frac{g}{R\omega^2}$

we must have $\omega \geqslant \left(\frac{g}{R}\right)^{\frac{1}{2}}$

thus $\underline{\omega_c = \left(\frac{g}{R}\right)^{\frac{1}{2}}}$

for $\omega > \omega_c$ ω_0 is real and the motion is oscillatory with frequency ω_0

for $\omega < \omega_c$ the motion is damped in time

#3 The "uncertainty" for a general variable α is defined as:

$$(\Delta\alpha)^2 = \langle\alpha^2\rangle - \langle\alpha\rangle^2$$

$$\int_0^\infty x^{2n} e^{-a^2x^2} dx = \frac{1.3.5\cdots(2n-1)\sqrt{\pi}}{2^{n+1} \quad a^{2n+1}}$$

$$\langle x^2\rangle = \frac{2}{\sigma\sqrt{\pi}} \int_0^\infty x^2 e^{\frac{-x^2}{\sigma^2}} dx \qquad a = \frac{1}{\sigma}$$

$$= \frac{2}{\cancel{\sigma\sqrt{\pi}}} \frac{\sigma^{\cancel{3}\,2}}{4} \sqrt{\pi} = \frac{\sigma^2}{2}$$

$\langle x\rangle = 0 \leftarrow$ --- (integrand is odd)

$$\langle p_x\rangle = \frac{2}{\sigma\sqrt{\pi}} \frac{\hbar}{i} \int_0^\infty e^{-x^2/\sigma^2}\left(\frac{-2x}{\sigma^2}\right) dx \qquad \hat{p}_x = \frac{\hbar}{i}\frac{d}{dx}$$

$= 0 \leftarrow$

$$\frac{d}{dx}\left(e^{-x^2/\sigma^2}\right) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}$$

$$\frac{d^2}{dx^2}\left(\text{"} \right) = -\frac{1}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} + \frac{x^2}{\sigma^4} e^{-\frac{x^2}{2\sigma^2}}$$

$$\langle p_x^2 \rangle = \frac{-2\hbar^2}{\sigma\sqrt{\pi}} \left\{ -\frac{1}{\sigma^2}\int_0^\infty e^{-x^2/\sigma^2}dx + \frac{1}{\sigma^4}\int_0^\infty x^2 e^{-\frac{x^2}{\sigma^2}} dx \right\}$$

$$\text{"} \left\{ -\frac{\sigma\sqrt{\pi}}{2\sigma^2} + \frac{1}{\sigma^4}\frac{\sigma^3\sqrt{\pi}}{4} \right\}$$

$$= \hbar^2/2\sigma^2$$

$$(\Delta x)^2 (\Delta p_x)^2 = \frac{\sigma^2}{2} \cdot \frac{\hbar^2}{2\sigma^2} \qquad \therefore \Delta x \Delta p_x = \hbar/2$$

4 Grad E & M.

Consider Coordinate Systems S magnets, electrode at rest & S' where liquid at rest

magnetic field B in S E' in S'

& $\vec{E}' = \vec{v} \times \vec{B}$; $\vec{B}' = \vec{B}$

terms also contain $\Gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \simeq 1 + \frac{1}{2}(v/c)^2 + O(v/c)^4$

$(v/c)^2 = \left(\frac{5\ m/s}{3\times10^8}\right)^2 \approx 10^{-16}$ so neglect

E' polarized dielectric in S' and from

$\vec{D}_1 \cdot \hat{n} = \vec{D}_2 \cdot \hat{n}$ across boundary

$\vec{E}_d' = \epsilon_0/\epsilon\ \vec{E}'$ where $\begin{cases} E' = \text{space value of } E' \\ E_d' = \text{electric field in dielectric} \end{cases}$

Consider $\vec{E}_d' = \vec{E}' - \vec{E}_{pol}'$ E_{pol}' = field from dielectric polarization but $\vec{E} = E'_{pol}$ as $\vec{E}'$ transforms out

when observing from S system

$\therefore V = \ell E = \ell E'_{pol} = \ell(E' - E_d') = \ell E'(1 \mp 1/K_\ell) = \ell v B(1 + 1/K_\ell)$

where ℓ = gap & K_ℓ = dielectric constant of liquid

$V = .03 \times 5 \times 0.1(1 - 1/14) = 1.42 \times 10^{-2} = 14.2$ milli volts

#5 Conservation of energy:

$$m_o c^2 = h\nu + \frac{m_o c^2 - \Delta E}{\sqrt{1 - \frac{v^2}{c^2}}}$$

conservation of momentum:

$$0 = \frac{h\nu}{c} - \frac{\left(m_o - \frac{\Delta E}{c^2}\right)v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$1 - \frac{v^2}{c^2} = \frac{\left(m_o - \frac{\Delta E}{c^2}\right)^2 c^4}{\left(m_o - \frac{\Delta E}{c^2}\right)^2 c^4 + (h\nu)^2}$$

substitute into eq for conservation of energy

$$(m_o c^2 - h\nu)^2 \left(1 - \frac{v^2}{c^2}\right) = (m_o c^2 - \Delta E)^2$$

$$= \left(m_o - \frac{\Delta E}{c^2}\right)^2 c^4$$

$$(m_0c^2 - h\nu)^2 = \left(m_0 - \frac{\Delta E}{c^2}\right)^2 c^4 + (h\nu)^2$$

$$h\nu = \Delta E\left(1 - \frac{\Delta E}{2m_0c^2}\right)$$

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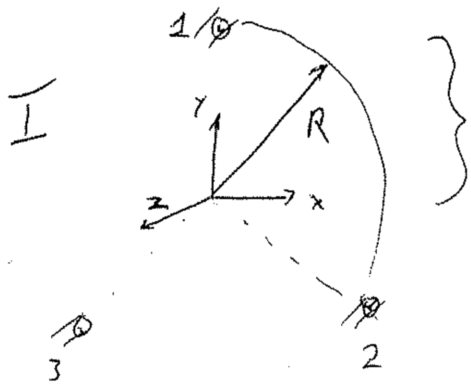
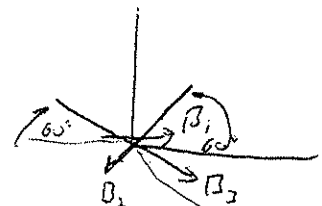




Positive Current in z direction

$$I_1 = I_0 \sin \omega_0 t$$
$$I_2 = I_0 \sin(\omega_0 t - 2\pi/3)$$
$$I_3 = I_0 \sin(\omega_0 t + 2\pi/3)$$

using $\oint \vec{H}\cdot d\vec{l} = \int_A \vec{J}\cdot d\vec{a} = I$

$$\therefore \quad B_i = \frac{\mu_0 I_i}{2\pi R}$$

$$B_x = \frac{\mu_0 I_0}{2\pi R}\left(\sin\omega_0 t - \cos 60^\circ \sin\left(\omega_0 t - \frac{2\pi}{3}\right) - \cos 60^\circ \sin\left(\omega_0 t + \frac{2\pi}{3}\right)\right)$$

$$B_y = \frac{\mu_0 I_0}{2\pi R}\left(-\sin 60^\circ\left(\sin\left(\omega_0 t - \frac{2\pi}{3}\right)\right) + \sin 60^\circ \sin\left(\omega_0 t + \frac{2\pi}{3}\right)\right)$$

Using $\sin\left(\omega t \pm \frac{2\pi}{3}\right) = \sin\omega_0 t \cos\frac{2\pi}{3} \pm \cos\omega_0 t \sin 2\pi/3$

$$= -\frac{1}{2}\sin\omega_0 t \pm \sqrt{3}/2 \cos\omega_0 t$$

$$B_x = \frac{\mu_0 I_0}{2\pi R}\left(\sin\omega_0 t + \frac{1}{2}\cdot\frac{1}{2}\sin\omega_0 t + \frac{1}{2}\cdot\frac{1}{2}\sin\omega_0 t\right) = \frac{3\mu_0 I}{4\pi R}\sin\omega_0 t$$

$$B_y = \frac{\mu_0 I_0}{2\pi R}\left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cos\omega_0 t + \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}\cos\omega_0 t\right) = \frac{3\mu_0 I}{4\pi R}\cos\omega_0 t$$

$$\vec{B} = \frac{3\mu_0 I_0}{4\pi R}\left(\hat{i}\sin\omega_0 t + \hat{j}\cos\omega_0 t\right)$$

{ Clock-wise rotation at angular frequency ω.

II $\oint \vec{E}\cdot d\vec{l} = \text{e.m.f.} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{a}$

but $\int \vec{B}\cdot d\vec{a} = \int \vec{B}\cdot\hat{j}\, da = \int B_y\, da = B_y A$

a) $\text{e.m.f.} = -A\frac{\partial}{\partial t}B_y = \frac{3I_0\mu_0}{4\pi R}\frac{\partial}{\partial t}(\cos\omega_0 t) = -\frac{3I_0\mu_0\omega_0}{4\pi R}\sin\omega_0 t$

If coil rotates CW at ω_1

$\sin\omega_0 t \rightarrow \sin(\omega_0-\omega_1)t$

when $\omega_1 = \omega_0$ emf $= 0$

if coil rotates CCW at ω_1

$\sin\omega_0 t \rightarrow \sin(\omega_0+\omega)t$

i.e. CW (CCW) $d\vec{a} = da(\hat{i}\sin\omega_1 t \pm \hat{j}\cos\omega_1 t)$

$\vec{B}\cdot d\vec{a} = da\, B_0(\sin\omega_1 t\sin\omega_0 t \pm \cos\omega_1 t\cos\omega_0 t)$

$= B_0\, da\,(\cos(\omega_0 \mp \omega_1)t)$

$B_0 \equiv \frac{3\mu_0 I_0}{4\pi R}$

7 Grad Q.M. Sol.

$$\mathcal{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t} \; ; \; \mathcal{H} = \frac{p^2}{2m} + \tfrac{1}{2} k x^2 \qquad k = m\omega^2$$

$$p = -i\hbar \, \partial/\partial x$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{2} m\omega^2 x^2 \Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

$$-\frac{\hbar^2}{2m}(\alpha^4 x^2 - \alpha^2)\Psi_0 + \frac{1}{2} m\omega^2 x^2 \Psi_0 = \hbar\beta \Psi_0$$

$$\left(\frac{1}{2} m\omega^2 - \frac{\hbar^2}{2m}\alpha^4\right)x^2 + \left(\frac{\hbar^2}{2m}\alpha^2 - \hbar\beta\right) = 0 \quad \text{for } \underline{\text{all}} \; x$$

$$\therefore \; \alpha^2 = \frac{m\omega}{\hbar} \; ; \; \beta = \frac{\hbar}{2m}\alpha^2 = \omega/2$$

$$\overline{x}(t) = \langle \Psi(t) | x | \Psi(t) \rangle = \langle \psi_0 | x | \psi_0 \rangle \cos^2\theta + \langle \psi_1 | x | \psi_1 \rangle \sin^2\theta$$

$$+ \left[\langle \psi_0 | x | \psi_1 \rangle + \langle \psi_1 | x | \psi_0 \rangle\right] \sin\theta \cos\theta$$

but $= \langle \psi_0 | x | \psi_1 \rangle = \langle \psi_1 | x | \psi_0 \rangle^*$

$$\overline{x}(t) = \langle \psi_0 | x | \psi_0 \rangle \cos^2\theta + \langle \psi_1 | x | \psi_1 \rangle \sin^2\theta + 2\,\mathrm{Re}\left[\langle \psi_0 | x | \psi_1 \rangle \sin\theta\cos\theta\right]$$

$$= x_{00}\cos^2\theta + x_{11}\sin^2\theta + 2x_{01}\cos\omega t \, \sin\theta\cos\theta$$

where $x_{ij} = \langle \Psi_i | x | \Psi_j \rangle$ & $\langle \psi_0 | x | \psi_1 \rangle = x_{01} \,(\text{real})\, e^{i\omega t/2} e^{-i3\omega t/2}$

$$= x_{01} e^{-i\omega t}$$

$x_{00} = x_{11} = 0$ from cons. of parity

or $x_{00} = \alpha/\sqrt{\pi} \int_{-\infty}^{\infty} x e^{-\alpha^2 x^2} dx = 0$

$x_{11} = 2\alpha^3/\sqrt{\pi} \int_{-\infty}^{\infty} x^3 e^{-\alpha^2 x^2} dx = 0$

$\Bigg\{ \dfrac{\text{odd}}{\text{functions}} \quad \int_{-a}^{a} f(x) = 0$ for $f(x)$ odd

$$X_{10} = X_{01} = \left(2\alpha^4/\pi^{3/2}\right)^{1/2} \int_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2} dx$$

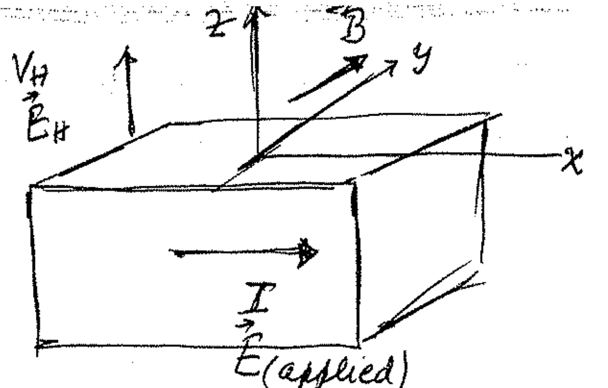
$$= \frac{2\sqrt{2}\alpha^2}{\pi^{3/4}} \frac{1}{4\alpha^2} \frac{\pi^{\frac{1}{2}}}{\alpha} = \frac{1}{\sqrt{2}\alpha\pi^{1/4}}$$

$$\overline{x}(t) = \frac{\sqrt{2}}{\alpha\pi^{1/4}} \cos\theta \sin\theta \cos\omega t$$

$$= \frac{1}{\sqrt{2}\alpha\pi^{1/4}} \sin 2\theta \cos\omega t$$

$$= \sqrt{\frac{\hbar}{2m\omega\sqrt{\pi}}} \sin 2\theta \cos\omega t$$

#8



$\mu_e = 3500\,\text{cm}^2/\text{V sec}$ electrons

$\mu_h = 1400$ " holes

$\vec{B} = B_0 \hat{j}$ (applied magnetic field)

$\vec{I} = I_0 \hat{i}$ (sample current)

Hall voltage (and field) will be along z-axis)

$\vec{v}_e = -\mu_e \vec{E}$ $\quad\quad \vec{v}_h = \mu_h \vec{E}$

$\vec{F}_e = -e\vec{v}_e \times \vec{B}$ $\quad\quad \vec{F}_h = +e\vec{v}_h \times \vec{B}$

induced transverse velocities

$$v_e' \hat{k} = -\mu_e \vec{v} \times \vec{B}$$

$$= \mu_e^2 BE\hat{k} \quad\quad \text{and } v_h'\hat{k} = \mu_h^2 BE\hat{k}$$

Hall current (in z-direction)

$$I' = en_h v_h' - en_e v_e' = eBE(n_h\mu_h^2 - n_e\mu_e^2)$$

Since this vanishes $n_h\mu_h^2 = n_e\mu_e^2$

Sample current is

$$\vec{I} = e(\mu_h n_h + \mu_e n_e)\vec{E}$$

fraction of current due to e⁻ is: $\dfrac{\mu_e n_e}{\mu_e n_e + \mu_h n_h}$

$$= \left(1 + \frac{\mu_h n_h}{\mu_e n_e}\right)^{-1} = \left(1 + \frac{\mu_e}{\mu_h}\right)^{-1} = \left(1 + \frac{7}{2}\right)^{-1}$$

$$= \frac{2}{9}$$