PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #50

January 12, 1985

## General Instructions

This Comprehensive Examination for Winter 1985 (#50) consists of six problems of equal weight (20 points each). Please check that you have all of them.

Work carefully, indicate your reasoning briefly, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you *may* find useful. Please return it at the end of the exam.

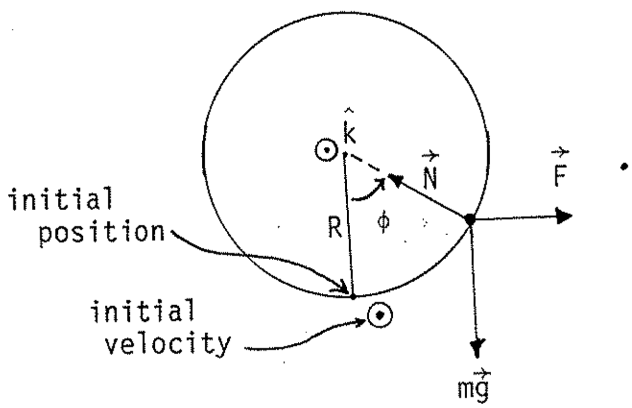
Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solution. "Scratch" work will *not* be graded.

PROBLEM #1.

A particle m moves on the frictionless inner surface of a horizontal cylinder of radius R. In addition to the weight and wall reaction force, there is a constant, horizontal force $\vec{F}$ acting on the particle as shown in the figure. The magnitude of $\vec{F}$ is (mg/3). If the particle is projected from rest at the position (R,0,0) (cylindrical coordinates) with initial velocity $\vec{v}_0$ in the z-direction.

a) Find the maximum angular rise of the particle.

b) Find the wall reaction force, $\vec{N}$, at that position.




PROBLEM #2.

A thin plastic circular disk of radius R has charge q uniformly distributed over its surface. The disk is rotating with angular velocity $\vec{\omega}$ around an axis through the center and perpendicular to the plane of the disk.

a) Compute the magnetic field along the axis.

b) Calculate the magnetic dipole moment of the disk.

c) A second identical disk rotating with the same angular velocity $\vec{\omega}$ is placed a distance L away from the first disk. The two disks have a common axis. Calculate the magnetic energy of interaction of the two disks. (Assume $L >> R$).

One or more of the following integrals may be useful:

$$\int \frac{dx}{x(x^2+a^2)} = \frac{1}{2a^2} \log \frac{x^2}{x^2+a^2}$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log (x+\sqrt{x^2+a^2})$$

$$\int \frac{x^3 dx}{(x^2+a^2)^{3/2}} = \sqrt{x^2+a^2} + \frac{a^2}{\sqrt{x^2+a^2}}$$

PROBLEM #3.

Recall that the fine structure splitting of energy levels of atoms is due to "spin-orbit coupling." The interaction Hamiltonian is:

$$H' = \frac{Ze^2}{(4\pi\varepsilon_0 c^2)2m^2} \frac{\vec{L}\cdot\vec{S}}{r^3}$$

where $\vec{L}$ and $\vec{S}$ are the electron orbital and spin angular momenta, respectively.

Considering this interaction as a perturbation, derive an expression for the energy shift of the $2^2P_{3/2}$ state of hydrogen.

$$\langle n\ell m| = \langle 210| = \frac{1}{4\sqrt{2\pi}} \left(\frac{Z}{a_0}\right)^{3/2} \frac{Zr}{a_0} e^{-\frac{Zr}{2a_0}} \cos\theta$$

$$\text{where } a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2} = 0.053 \text{ nm (Radius of first Bohr orbit)}$$

$$\Gamma(n) = (n-1)! = \int_0^\infty x^{n-1} e^{-x}\, dx$$

Using your result, make an estimate (in eV) of this energy shift.

PROBLEM #4.

In developing the theory to explain the spectral distribution of cavity (blackbody) radiation, Planck arrived at the average energy of the mode of frequency $\nu$:

$$\bar{\mathcal{E}} = \frac{h\nu}{e^{h\nu/kT} - 1}$$

a) Derive an expression for the number of modes per unit volume in the frequency range $d\nu$ at $\nu$. Find the energy density in the frequency range $d\nu$ at $\nu$.

b) The relationship between energy density $u_T(\nu)$ and the rate of emission (or absorption) by the cavity wall surface, $R_T(\nu)$ (watts/m$^2$) is:

$$R_T(\nu) = \frac{c}{4}\, u_T(\nu) \quad .$$

Assume the sun emits as a blackbody at 6000K; the diameter of the sun is roughly $10^6$ km.

Make a rough calculation of the total microwave power emitted by the sun in a one MHz bandwidth at 3 cm. wavelength.

PROBLEM #5.

A square flake of mica of permittivity $\epsilon$, area $w^2$, and thickness $t \ll w$ is in a vacuum in which there is a uniform electric field $\vec{E}_0$. With the flake oriented so the unit normal vector on one face is at angle $\alpha$ with respect to $\vec{E}_0$, find the torque on the flake.

PROBLEM #6.

Positronium is a bound state of an electron and a positron. The Hamiltonian for the system in a magnetic field $H$ can be written in the form

$$H = H_o + A(\vec{\sigma}_1 \cdot \vec{\sigma}_2) + \mu_B H(\sigma_{1z} - \sigma_{2z})$$

where $H_o$ contains kinetic energies and central force potentials, A is a constant and

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z}$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli spin matrices. The index 1 refers to the electron and the index 2 to the positron.

a) Find the value of A if in zero magnetic field the $1^1S$ and $1^3S$ states are separated by $2 \times 10^5$ MHz with the singlet state lower.

b) Discuss the principles which restrict the $1^1S$ and $1^3S$ states to decay primarily by two- and three-photon emission, respectively.

c) Find the energies of the states for non-vanishing magnetic field.

## #1

The motion in the $z$-direction is uniform: $z = v_0 t$,

a) The angular motion is given by

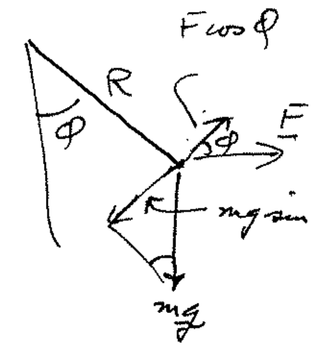
$$m R \ddot{\varphi} = F \cos\varphi - mg \sin\varphi$$

$$\ddot{\varphi}\,\dot{\varphi}\, dt = \frac{g}{R}\left[\frac{1}{3}\cos\varphi - \sin\varphi\right] d\varphi$$

$$\frac{\dot{\varphi}^2}{2} = \frac{g}{R}\left[\frac{1}{3}\sin\varphi + \cos\varphi\right] + \text{const},$$

$$0 = \frac{g}{R}\left[\frac{1}{3}\times 0 + 1\right] + \text{const}$$

$$\dot{\varphi}^2 = \frac{2g}{R}\left\{\frac{1}{3}\sin\varphi - (1 - \cos\varphi)\right\}$$




at $\dot{\varphi} = 0$, $\varphi = \varphi_{max}$

$$\frac{1}{9}\sin^2\varphi_{max} = \frac{1}{9}(1-\cos^2\varphi_{max}) = \frac{1}{9}(1-\cos\varphi_{max})(1+\cos\varphi_{max}) = (1-\cos$$

$$\frac{1}{9}(1+\cos\varphi_{max}) = 1 - \cos\varphi_{max}$$

$$\frac{10}{9}\cos\varphi_{max} = \frac{8}{9} \quad , \quad \underline{\cos\varphi_{max} = .8 \ , \ \varphi_{max} \doteq 37^\circ}$$

b) at $\varphi = \varphi_{max}$,

$$-mR\dot{\varphi}^2 = 0 = -N + F\sin\varphi_{max} + mg\cos\varphi_{max}$$

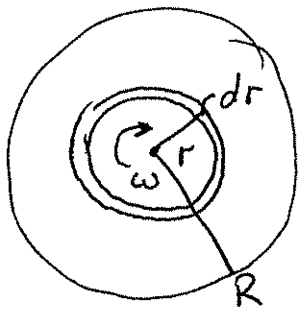
$$N = \frac{mg}{3}\left(\frac{3}{5}\right) + mg\left(\frac{4}{5}\right) = mg$$

$$\vec{N} = -\hat{e}_r\, mg$$

#2

Solution

a) Divide the disk into small circular rings

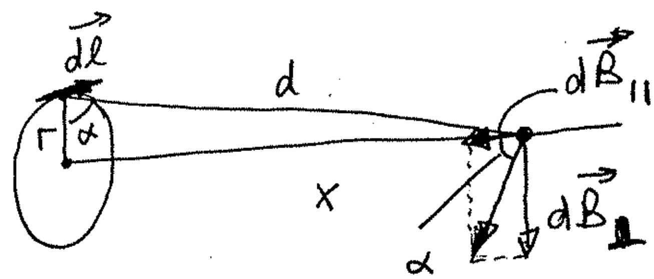



$$dq = \frac{2\pi r\,dr}{\pi R^2}\,q$$

$$di = \frac{dq}{T} = \frac{\omega r\,dr}{\pi R^2}\,q$$

$$\left(T = \frac{2\pi}{\omega}\right)$$

Apply Biot - Savart law $\quad d(d\vec{B}) = \frac{\mu_0 (di)}{4\pi}\,\frac{d\vec{l} \times \vec{d}}{d^3}$




from symmetry $\vec{B}_{\perp} = 0 \qquad d(dB_{\parallel}) = \frac{\mu_0 (di)}{4\pi}\,\frac{dl}{d^2}\cos$

$$d(dB_{\parallel}) = \frac{\mu_0 (di)}{4\pi}\,\frac{r}{(x^2+r^2)^{3/2}}\,dl$$

$$dB_{\parallel} = \frac{\mu_0\,di}{2}\,\frac{r^2}{(x^2+r^2)^{3/2}}$$

$$= \frac{\mu_0\,\omega q}{2\pi R^2}\,\frac{r^3\,dr}{(x^2+r^2)^{3/2}}$$

$$B = \frac{\mu_0}{2\pi}\,\frac{\omega q}{R^2}\left[(x^2+R^2)^{\frac{1}{2}} + \frac{x^2}{(x^2+R^2)^{1/2}} - 2x\right]$$

solution cont.

[ use integral $\displaystyle\int \frac{y^3\,dy}{z^{3/2}} = z^{1/2} + \frac{x^2}{z^{1/2}}$

$z = x^2 + y^2$ ]

b) dipole moment of disk.

$$d\mu = di\,A \qquad di = \frac{\omega q}{\pi R^2}\, r\,dr \qquad A = \pi r^2$$

$$\mu = \int_0^R \frac{\omega q}{R^2}\, r^3\,dr = \underline{\underline{\frac{\omega q R^2}{4}}}$$

c) interaction energy. (for $L \gg R$)

$$\mathcal{E} = -\vec{\mu}\cdot\vec{B} = -\frac{\mu_0 \omega^2 q^2}{8\pi}\left[(L^2+R^2)^{1/2} + \frac{L^2}{(L^2+R^2)^{1/2}} - 2L\right]$$

for $L \gg R$ expand $(L^2+R^2)^{\pm 1/2}$.

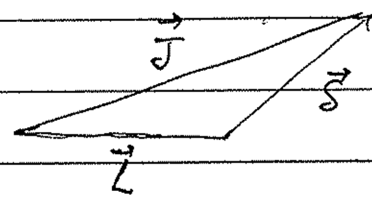
$$\mathcal{E} = -\frac{\mu_0 \omega^2 q^2 R^4}{32\pi L^3}$$

# #3

$$\Delta E = \langle (nlm) | H' (nlm) \rangle$$

$$H' = \frac{Ze^2}{(4\pi\varepsilon_0 c^2)\, 2m^2} \; \frac{\vec{L}\cdot\vec{S}}{r^3}$$

$\langle \vec{L}\cdot\vec{S} \rangle$ :




$$J^2 = L^2 + S^2 + 2\vec{L}\cdot\vec{S}$$

$$\vec{L}\cdot\vec{S} = \frac{J^2 - L^2 - S^2}{2}$$

$j = 3/2 \quad l = 1 \quad s = \frac{1}{2}$

$$\langle J^2 \rangle = \hbar^2 j(j+1) = \frac{15}{4}\hbar^2$$

$$\langle L^2 \rangle = \hbar^2 l(l+1) = 2\hbar^2$$

$$\langle S^2 \rangle = \hbar^2 s(s+1) = 3/4\,\hbar^2$$

$$\langle \vec{L}\cdot\vec{S} \rangle = \frac{\hbar^2}{2}\left\{ 15/4 - 2 - 3/4 \right\} = \frac{\hbar^2}{2}$$

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{16(2\pi)} \frac{1}{a_0^3} \frac{1}{a_0^2} \int_0^\infty \int_0^\pi \frac{r^2}{r^3} e^{-r/a_0} \cos^2\theta \; 2\pi r^2 \sin\theta \, d\theta \, dr$$

$$\left. -\frac{\cos^3\theta}{3} \right|_0^\pi = \frac{2}{3} \qquad a_n = n^2 a_0$$

$$a_0^2 \int_0^\infty \frac{r}{a_0} e^{-r/a_0} \, d r/a_0 = a_0^2$$

so $\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{24} \frac{1}{a_0^3}$

$$|E_2| = \frac{e^2}{4\pi\varepsilon_0 (8a_0)} = \frac{13.6}{2^2} = 3.4\,\text{eV}$$

$$a_0^2 = \frac{(4\pi\varepsilon_0)^2 \hbar^4}{m^2 e^4}$$

$$\Delta E = \frac{e^2}{4\pi\varepsilon_0 c^2 m^2} \; \frac{\hbar^2}{96 a_0^3}$$

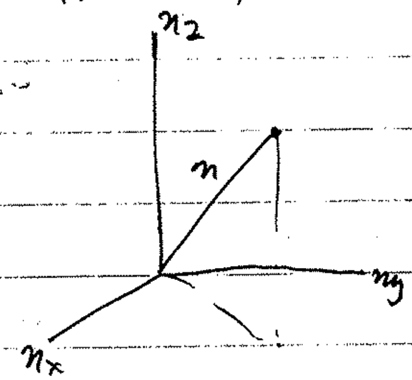
$$\Delta E = |E_2| \frac{\cancel{\hbar^2}}{12 \cancel{m^2}} \; \frac{\cancel{m^2} e^4}{(4\pi\varepsilon_0)^2 \hbar^{\cancel{4}2} c^2}$$

$$= \frac{\alpha^2 E_2}{12} = \frac{3.4}{(1.4\times10^2)^2 \times 12} \cong 1\times10^{-5}\,\text{e}$$

$2.6\times10^4$

(note: shift is +)

#4. a)




Assume cubical cavity $L$ on a side

# of modes $0 \to \nu$ :

$$N(\nu) = \frac{2}{8} \frac{4}{3}\pi n^3 \quad (2 \leftarrow \text{polarization})$$

$$= \frac{8\pi}{3} \frac{L^3}{c^3} \nu^3$$

0 —— L

$$n\lambda/2 = L$$

$$n = \frac{2L}{c}\nu$$

# of modes per unit vol $\nu \to \nu + d\nu$ : $dN_\nu = \frac{8\pi}{3}\nu^2 d\nu$

so energy density $u_\nu(\nu)\,d\nu = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/kT}-1} d\nu$

$$R_\nu(\nu)\,d\nu = u_\nu(\nu)\,d\nu \cdot c/4$$

$$= \frac{2\pi h\nu^3}{c^2} \frac{d\nu}{e^{h\nu/kT}-1}$$

for $\lambda = 3.$ cm

$T = 6000K$

$h\nu/kT \ll 1$

total radiation : $\lambda = 3$cm

$d\nu = 10^6 Hz$

$$\frac{1}{e^{h\nu/kT}-1} \approx \frac{kT}{h\nu}$$

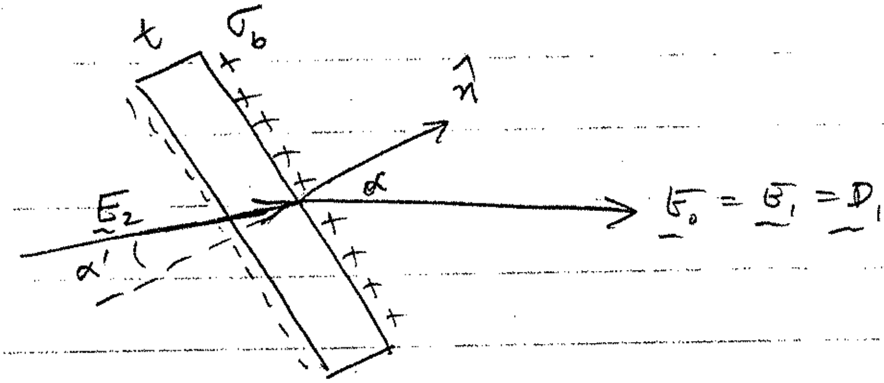
surface area $\pi d^2 = \pi(10^9)^2$

$$\lambda = \frac{c}{\nu} = 3\times10^{-}$$

$$E = \frac{2\pi}{(3\times10^{-2})^2} 1.4\times10^{-23} \times 6\times10^3 \times 10^6$$

$E = 18\times10^8$ watts

#5



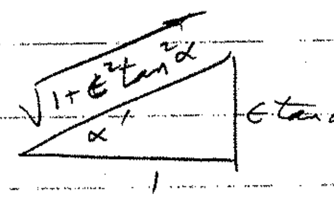

At the surfaces, $D_{norm}$ + $E_{tang}$ are continuous:

$D_n = D_2 \cos\alpha' = D_1 \cos\alpha$, $\quad \epsilon E_2 \cos\alpha' = E_0 \cos\alpha$

$E_{tang} =$ $\quad E_2 \sin\alpha' = E_0 \sin\alpha$

$\tan\alpha' = \epsilon \tan\alpha$

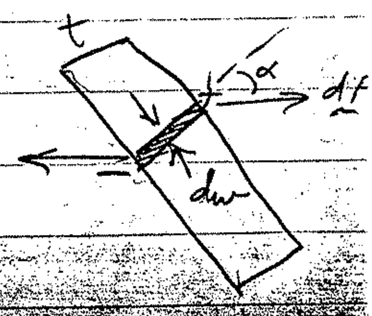
$\sin\alpha' = \dfrac{\epsilon \tan\alpha}{\sqrt{1+\epsilon^2\tan^2\alpha}}$ ; $\cos\alpha' = \dfrac{1}{\sqrt{\phantom{x}}}$




$$E_2 = E_0 \frac{\sin\alpha}{\sin\alpha'} = E_0 \frac{\sin\alpha}{\epsilon \frac{\sin\alpha}{\cos\alpha}\sqrt{\phantom{x}}} = \frac{E_0}{\epsilon}\cos\alpha\sqrt{1+\epsilon^2\tan^2\alpha}$$

$$D_2 = \epsilon E_2 = E_2 + 4\pi P_2, \quad P_2 = \frac{\epsilon-1}{4\pi} E_2$$

$$\sigma_b = \pm\, \hat n \cdot P_2 = \pm P_2 \cos\alpha' = \pm \frac{\epsilon-1}{4\pi} E_2 \frac{1}{\sqrt{\phantom{x}}} = \pm \frac{\epsilon-1}{4\pi\epsilon} E_0 \cos\alpha$$




A segment of width dw will have forces

$dF = \pm (\sigma_b\, w\, dw)\, E_0$ exerted,

and torque

$$\left.\begin{aligned} dN &= t \sin\alpha\, dF \\ N &= t \sin\alpha\, \sigma_b E_0 w^2 \end{aligned}\right\} = \frac{(\epsilon-1)E_0^2\, t\, w^2 \sin\alpha\cos\alpha}{4\pi\epsilon}$$

$$= \frac{(\epsilon-1)}{8\pi\epsilon}\, t w^2 E_0^2 \sin 2\alpha$$

#6

Solution:

a)

The eigenstates are

$$\psi_{1,1} = \alpha_1 \alpha_2$$

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \psi_{1,0} = \frac{1}{\sqrt{2}}(\alpha_1 \beta_2 + \beta_1 \alpha_2)$$

$$\psi_{1,-1} = \beta_1 \beta_2$$

$$\psi_{0,0} = \frac{1}{\sqrt{2}}(\alpha_1 \beta_2 - \beta_1 \alpha_2)$$

Eigen values of $A(\vec{\sigma}_1 \cdot \vec{\sigma}_2)$

write $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z}$.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

one gets

$$E_{0,0} = -3A \qquad E_{1,0} = E_{1,\pm 1} = A$$

thus $1S - 3S$ splitting is $\Delta E = 4A$

a) $\frac{4A}{\hbar} = 2 \cdot 10^5 \text{ MHz} \qquad \underline{A = 5 \times 10^{-17} \text{ erg}}$

b) Must conserve linear and angular momentum.

Single photon decay is impossible since linear momentum cannot be conserved.
The $^1S$ state has zero angular momentum.
It can decay via two photon emission.
The photons go off in opposite direction and carry no net angular momentum.
The $^3S$ state requires three photons to conserve both linear and angular momentum.

c) For $\mathcal{H} \neq 0$ one must solve

$$\begin{array}{c} 1,1 \\ 1,0 \\ 1,-1 \\ 0,0 \end{array} \begin{pmatrix} A-E & 0 & 0 & 0 \\ 0 & A-E & 0 & 2\mu_B\mathcal{H} \\ 0 & 0 & A-E & 0 \\ 0 & 2\mu_B\mathcal{H} & 0 & -3A-E \end{pmatrix} = 0$$

The solutions are

$$E_{1,\pm 1} = A \qquad E_{\substack{0,0 \\ 1,0}} = -A \pm 2\left[A^2 + \mu_B^2\mathcal{H}^2\right]^{1/2}$$