PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #49

September 29, 1984

General Instructions

This Comprehensive Examination for Fall 1984 (#49) consists of six problems of equal weight (20 points each). Please check that you have all of them.

Work carefully, indicate your reasoning briefly, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you *may* find useful. Please return it at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solution. "Scratch" work will *not* be graded.

1. Two particles of mass m_1 and m_2 move about each other in circular orbits under the influence of gravitational forces only. The period of motion is τ. At a certain instant [t = 0] this motion is suddenly stopped and the particles are then released and allowed to fall toward each other. Find, in terms of the rotation period τ, the time between release and collision of the particles.

The following integrals might be useful.

$$\left[\begin{array}{ll} \int \sin^2 \theta d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4} & \int \cos^2 \theta d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} \\ \int \sin^3 \theta d\theta = -\cos\theta + \frac{1}{3}\cos^3\theta & \int \cos^3 \theta d\theta = \sin\theta - \frac{1}{3}\sin^3\theta \end{array}\right]$$

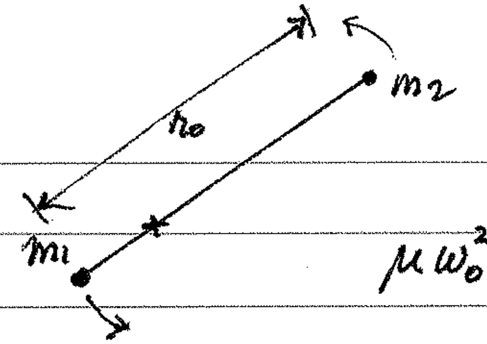
2. An infinite solenoid of radius R is oriented with its axis as the z-axis of a cylindrical coordinate system, (ρ, θ, z). The solenoid winding carries current to produce a uniform magnetic induction $\hat{k}$ B inside. Coaxial with the solenoid are two long cylinders which extended from $z = -h/2$ to $z = h/2$ and having radii a and b such that $a < R < b$. Both cylinders carry uniform surface charge, +Q on the inner cylinder and -Q on the outer. h is so large that end effects (fringing fields) may be neglected.

At first the two cylinders are at rest, but they are free to spin around their axis and are observed to do so when the current in the solenoid is reduced to zero.

(a) Calculate the torque on each cylinder, and the angular momentum of each cylinder when the solenoid current has been reduced to zero.

(b) Account for the angular momentum quantitatively.
Hint: In a problem Jackson gives the angular momentum density in Gaussian units as $\frac{\varepsilon\mu}{4\pi c}\vec{r} \times (\vec{E} \times \vec{H})$. For MKS this gives $\varepsilon_0 \vec{r} \times (\vec{E} \times \vec{B})$ in free space.

1.



$$\mu = \frac{m_1 m_2}{m_1+m_2}$$

$$\mu \omega_0^2 r_0 = \frac{G m_1 m_2}{r_0^2} \qquad \omega_0 = \frac{2\pi}{\tau}$$

$$r_0^3 = \frac{G m_1 m_2}{\mu \omega_0^2} = \frac{G \cancel{m_1 m_2}(m_1+m_2)}{\cancel{m_1 m_2}} \frac{\tau^2}{4\pi^2}$$

callapse time: from energy conservation

$$\frac{\mu}{2}\left(\frac{dr}{dt}\right)^2 - \frac{G m_1 m_2}{r} = -\frac{G m_1 m_2}{r_0}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{2}{\mu} G m_1 m_2 \left(\frac{1}{r} - \frac{1}{r_0}\right)$$ (use − sign in taking $\sqrt{\ }$)

$$\int_0^t dt = -\sqrt{\frac{\mu}{2 G m_1 m_2}} \int_{r_0}^{0} \frac{dr}{\left(\frac{1}{r} - \frac{1}{r_0}\right)^{1/2}}$$

to integrate - let $r = r_0 \sin^2\theta$

$dr = 2 r_0 \sin\theta \cos\theta \, d\theta$

$$\left(\frac{1}{r} - \frac{1}{r_0}\right)^{1/2} = \frac{1}{r_0^{1/2}}\left(\frac{1}{\sin^2\theta} - 1\right)^{1/2} = \frac{1}{r_0^{1/2}} \frac{\cos\theta}{\sin\theta}$$

$$t = \sqrt{\ } \int_0^{\pi/2} 2 r_0^{3/2} \frac{\sin^2\theta \cancel{\cos\theta} \, d\theta}{\cancel{\cos\theta}} = \sqrt{\ } \cdot 2 r_0^{3/2} \cdot \pi/4$$

eliminate r_0:

$$t = \sqrt{\frac{1}{2G(m_1+m_2)}} \cdot \cancel{2} \sqrt{G(m_1+m_2)} \frac{1}{\cancel{2\pi}} \tau \cdot \pi/4$$

$$t = \frac{\tau}{\sqrt{2}\, 4}$$

2.

a) Torque $\vec{N} = \vec{\rho} \times q\vec{E} = \vec{\rho} \times q\hat{\theta}\ \frac{d\Phi/dt}{2\pi\rho} = \frac{\hat{k}q}{2\pi}\frac{d\Phi}{dt}$

$$\vec{N_a} = \frac{\hat{k}Q}{2\pi}\pi a^2 \frac{dB}{dt} = \frac{d\vec{L_a}}{dt}, \quad \vec{L_a} = \tfrac{1}{2}\hat{k}Qa^2B$$

$$\vec{N_b} = \frac{-\hat{k}Q}{2\pi}\pi R^2 \frac{dB}{dt} = \frac{d\vec{L_b}}{dt}, \quad \vec{L_b} = -\tfrac{1}{2}\hat{k}QR^2B$$

Net ang. mom $\vec{L} = \underline{-\tfrac{1}{2}\hat{k}QB(R^2 - a^2)}$

b) Before current is switched off, the electric field is radial: $\vec{E} = \hat{\rho}\ \frac{Q/h}{2\pi\epsilon_0\rho}$, between cyl.

$$\vec{\mathcal{L}} = \epsilon_0 \vec{r} \times \left(\hat{\rho}\frac{Q/h}{2\pi\epsilon_0\rho} \times \hat{k}B\right)$$

$$= \epsilon_0(\hat{\rho}\rho + \hat{k}z) \times \left(-\hat{\theta}\ \frac{QB}{2\pi\epsilon_0\rho h}\right)$$

$$= \frac{QB}{2\pi\rho L}(-\hat{k}\rho + \hat{\rho}z), \quad \text{for } a \le \rho \le R, -\tfrac{h}{2} \le z \le \tfrac{h}{2}$$

$$\vec{L}_{EM} = \iiint \vec{\mathcal{L}}\, d\theta\, dz\, \rho\, d\rho$$

In the integration over θ the contributions due to the second term of $\vec{\mathcal{L}}$ will cancel leaving

$$\vec{L}_{EM} = \int_a^R -\hat{k}\frac{QB}{2\pi\cancel{\rho}h}\cancel{\rho}\,\rho\, d\rho \int_{-h/2}^{h/2} dz \int_0^{2\pi} d\theta$$

$$= -\hat{k}QB\frac{\rho^2}{2}\Big|_a^R = \underline{-\hat{k}QB\tfrac{1}{2}(R^2 - a^2)}$$ ✓

3. The water molecule is an assymetric top with molecular weight 18 (approx.). At 200°C a gas of these molecules at about 1 atm. will have full rotational excitation, but for the vibrational mode with smallest energy spacing, this spacing is about 1600 cm^{-1}, corresponding to 2200K.

(a) Use the equipartition theorem to compute the specific heat at constant volume, in cal/g °C for water at 200°C.

(b) The measured value of γ for water at 200°C is 1.31. Use this to compute the specific heat at constant volume, assuming ideal gas behavior, and compare with part (a). Comment on the difference, if any.

4. The equation

$$\vec{J} = \frac{h}{2mi}\left[\psi^* \vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right]$$

gives the probability that one particle per second will pass through a unit area having its plane normal to the direction of $\vec{J}$. A beam of particles with uniform speed v enters a region (containing N atoms per unit volume) where some of the particles can be absorbed by the atoms. This absorption may be represented by the introduction of a complex potential $(V_r - iV_i)$ to the time-dependent Schroedinger equation. Recall that the continuity equation with <u>no</u> absorption is

$$\frac{\partial}{\partial t}(\psi\psi^*) + \vec{\nabla}\cdot\vec{J} = 0.$$

Derive an equation for the absorption cross section per atom. This equation should contain N, v, term(s) of the complex potential and known constants.

3. a) There are $3N = 9$ degrees of freedom, of which 3 are unexcitable at 200°C. That leaves $C_V = (9-3)\frac{R}{2} = 3R \doteq 6 \frac{\text{cal}}{\text{mol °C}}$

Then $c_V = \frac{6}{18} = \underline{0.33}$ cal/gm °C.

b) $\gamma = \frac{C_p}{C_V} = \frac{C_V + R}{C_V} = 1 + \frac{R}{C_V} = 1.31$

$C_V = \frac{R}{.31} = 3.2\underline{3}R \doteq 6.3\underline{5}$ cal/mol °C

$c_V = \frac{6.35}{18} = \underline{0.35 \text{ cal/gm °C}}$

The result suggests that there may already be some excitation of vibration.

4.

$$\vec{J} = \frac{\hbar}{2mi}\left[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right]$$

look at:

$$\frac{\partial}{\partial t}(\psi\psi^*) = -\vec{\nabla}\cdot\vec{J} + (\quad)$$

absorption

$$\vec{\nabla}\cdot\vec{J} = \frac{\hbar}{2mi}\left[\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right]$$

S. Eq. including complex potential:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + (V_r - iV_i)\psi = -\frac{\hbar}{i}\frac{\partial\psi}{\partial t} \qquad \times\psi^*$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi^* + (V_r + iV_i)\psi^* = \frac{\hbar}{i}\frac{\partial\psi^*}{\partial t} \qquad \times\psi \qquad \text{and add eq's}$$

$$-\frac{\hbar^2}{2m}\left[\psi^*\nabla^2\psi\right] + (V_r - iV_i)\psi\psi^* = -\frac{\hbar}{i}\psi^*\frac{\partial\psi}{\partial t}$$

$$-\frac{\hbar^2}{2m}\left[\psi\nabla^2\psi^*\right] + (V_r + iV_i)\psi\psi^* = \frac{\hbar}{i}\psi\frac{\partial\psi^*}{\partial t} \qquad \text{subtract}$$

$$\underbrace{-\frac{\hbar^2}{2mi}\left[\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right]} - \frac{2\cancel{i}V_i}{\hbar}\psi\psi^* = \mp\frac{\cancel{\hbar}}{\cancel{i}}\frac{\partial}{\partial t}(\psi\psi^*) \qquad \div i\hbar$$

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}(\psi\psi^*) = \underbrace{-\frac{2V_i}{\hbar}(\psi\psi^*)}_{\text{rate of absorption per unit volume.}}$$

$$\frac{\text{rate of absorption}}{\text{vol}} = -N\sigma(\psi^*\psi)$$

$$\therefore \quad \sigma = \frac{2V_i}{N\hbar v}$$

5. In order to observe the sun in monochromatic light, the French astronomer B. Lyot invented the birefringent filter consisting of a series of birefringent crystals (C). Each element, after the first, is twice as thick as the previous one. Polarizing films (P) are mounted between the crystals and at each end [see fig. 1]. All the crystals are mounted with their optic axes parallel and at right angles to the direction of propagation of light. The polarization axes of the polarizing films are also all parallel and at 45° to the direction of the optic axes [see fig. 2]. Only certain bands of light are able to penetrate the filter. For a filter containing s elements, calculate the transmission as a function of wavelength λ. (Hint: The expression for the transmission can be simplified by using $\cos\theta = \frac{\sin 2\theta}{2\sin\theta}$.) Also find the width $\Delta\lambda$ of the bands that can pass the filter, and the wavelength separation between such bands.

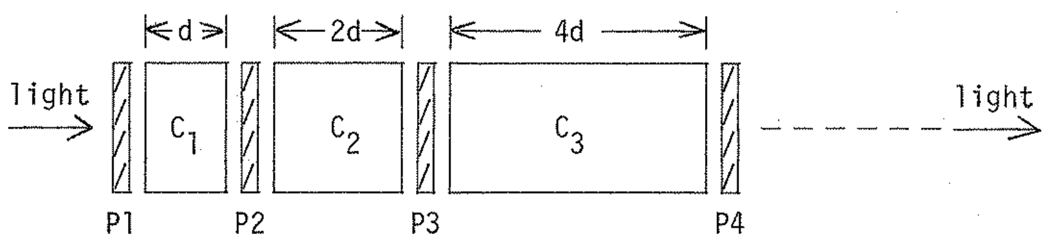


Figure 1.

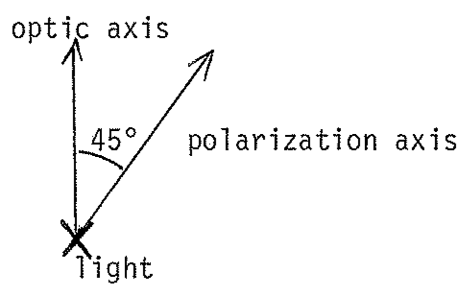
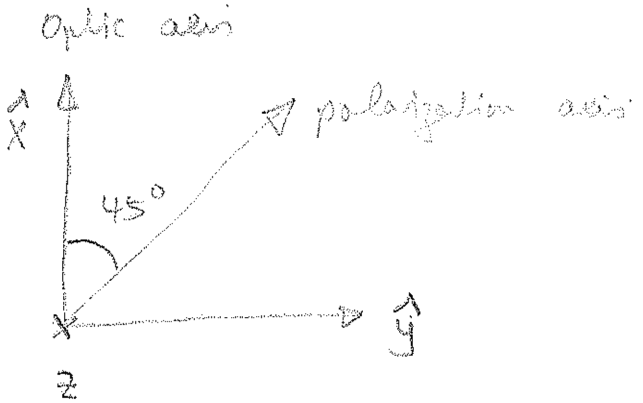


Figure 2.

5.

In a birefringent crystal, the e-m waves polarized $\parallel$ and $\perp$ to the optic axis travel with different phase velocities



After the first polarizer the field is

$$\vec{E} = E_0\,(\hat{x}+\hat{y})/\sqrt{2}$$

After traveling a distance z in the birefringent crystal we have

$$\vec{E} = E_0\left(\hat{x}\,e^{in_1kz} + \hat{y}\,e^{in_2kz}\right)e^{-i\omega t}/\sqrt{2}$$

n_1 = index of refraction along optic axis; phase vel: $\frac{c}{n_1}$

n_2 = index of refraction $\perp$ to optic axis; phase vel: $\frac{c}{n_2}$

$k = \frac{2\pi}{\lambda}$

After traveling the complete length $2^{m-1}d$ of crystal m the light transmitted through the next polarizer is

$$\vec{E}\cdot(\hat{x}+\hat{y})/\sqrt{2} = \frac{E_0}{2}\left[e^{in_1k2^{m-1}d} + e^{in_2k2^{m-1}d}\right]$$

$$= E_0\left[e^{ikd2^{m-2}(n_1+n_2)}\right]\cos(2^{m-1}\phi)$$

where $\phi = (n_1-n_2)kd/2$

After passing through 5 elements the <u>intensity</u> is reduced by the factor

Solution: Optics Problem.

$$T = \left[\cos\phi \cos 2\phi \cdots \cos(2^{S-1}\phi)\right]^2$$

use $\cos\theta = \dfrac{\sin 2\theta}{2\sin\theta}$ repeatedly. This yields

$$T = \left(\frac{\sin(2^S\phi)}{2^S \sin\phi}\right)^2.$$

Note that $T(\phi+\pi) = T(\phi)$ transmission occurs primarily for $\phi = p\pi$ p = integer.

Width $\delta\phi \simeq \dfrac{2\pi}{2^S}$

Thus transmission occurs in bands located at

$$\lambda = \frac{(n_1 - n_2)d}{p} \quad \text{and width} \quad \frac{\delta\lambda}{\lambda} \simeq \frac{2}{2^S p}$$

6. In a hydrogen-like atom the 2S and 2P levels are separated by a small energy difference Δ. The effect causing Δ has a negligible influence on the wave functions of the states. The atom is placed in a constant, uniform electric field E. (Neglect electronic and nuclear spin in your calculations.)

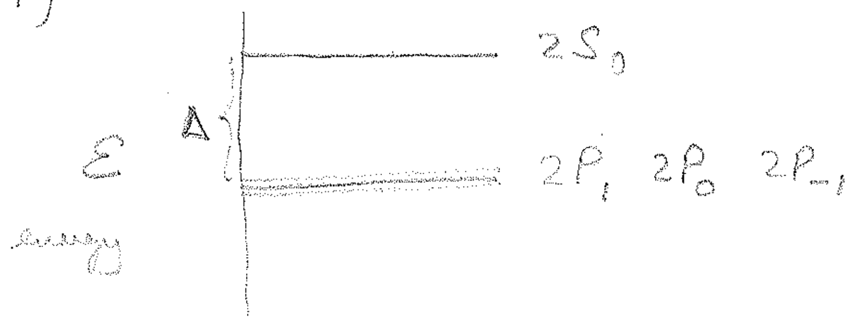
   (a) Sketch an energy level diagram of the $n = 2$ states for $E = 0$. Label all the states with the appropriate quantum numbers. Indicate any degeneracies of the levels.

   (b) Neglecting the influence of more distant levels, obtain general expressions for the energy shifts of <u>all</u> $n = 2$ levels as a function of field strength E. [You do not have to evaluate explicitly non-zero matrix elements.]

   (c) Plot the energy of the levels as a function of E. Label all the curves.

6.

1)



2). only non-zero matrix element is

$$\langle 2S_0 | F | 2P_0 \rangle = \langle 2S_0 | eEz | 2P_0 \rangle = aE.$$

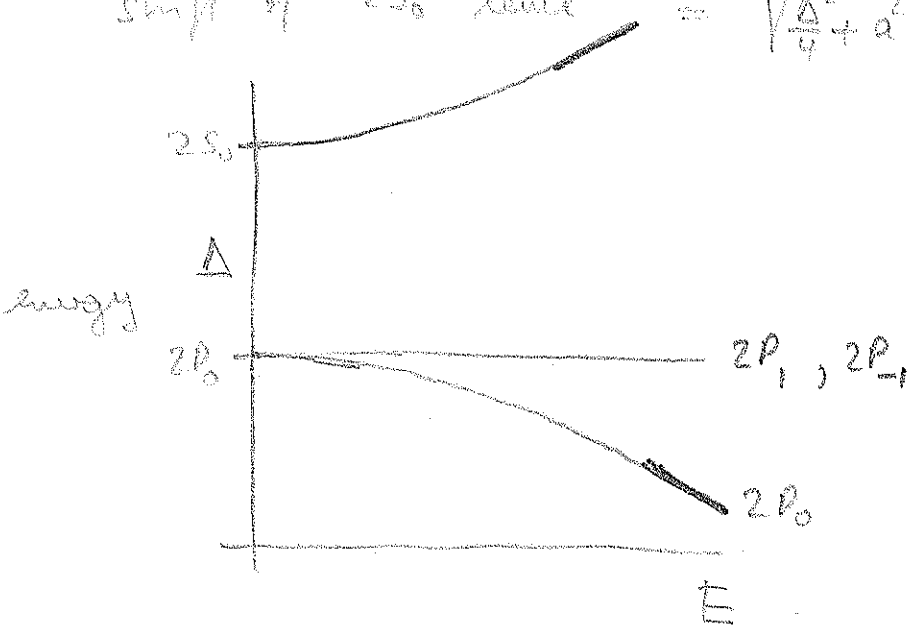
reduced matrix is

$$\begin{array}{c} 2P_0 \\ 2S_0 \end{array} \begin{bmatrix} 0 & aE \\ aE & \Delta \end{bmatrix}$$

diagonalization yields

shift of $2P_0$ level $= \frac{\Delta}{2} - \sqrt{\frac{\Delta^2}{4} + a^2 E^2}$

shift of $2S_0$ level $= \sqrt{\frac{\Delta^2}{4} + a^2 E^2} - \frac{\Delta}{2}$



for large E energy shifts of $2S_0$ and $2P_0$ are linear in E.