PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #48

March 31, 1984

General Instructions

This Comprehensive Examination for Spring 1984 (#48) consists of six problems of equal weight (20 points each). Please check that you have all of them.

Work carefully, indicate your reasoning briefly, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you may find useful. Please return it at the end of the exam.

1. A gyrocompass is mounted in such a way that the only torque is that which constrains the symmetry axis to remain in a horizontal plane. That is, there is no vertical component of torque. The principal moments of inertia are C about the symmetry axis and A about any axis perpendicular to the symmetry axis.

   Let the gyrocompass be at the earth's equator where we establish a coordinate system with z-axis vertical, y-axis north, and x-axis east. The earth's spin angular velocity is then $\Omega\hat{j}$ and let the gyrocompass be spinning with angular speed $\omega >> \Omega$. Show that the axis will oscillate about the north-south direction, and find the period of oscillation.

2. A simple magnetron has a cathode which is a cylinder of radius a (a thermionically emitting wire), and an anode which is a coaxial cylinder of radius b > a. The cathode is at ground potential and the anode is +V volts. A uniform magnetic field $\vec{B}$ is directed parallel to the axis of symmetry, say, the z-axis.

   Consider that electrons are released from the cathode with zero total energy and travel on curved orbits toward the anode.

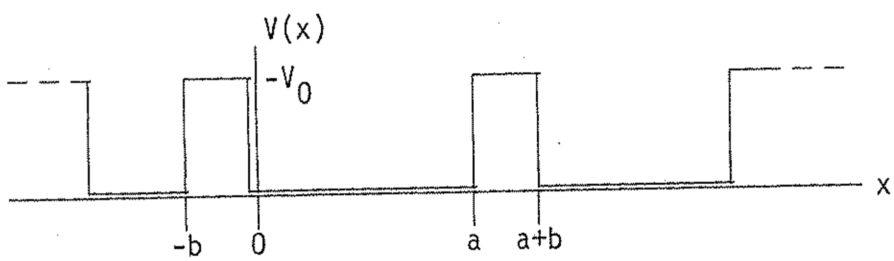
   (a) Show that the angular velocity ~~momentum~~ about the z-axis is $\dot{\theta} = (eB/2m)(1-a^2/r^2)$.

   (b) Compute the minimum value of V required for electrons to reach the anode.

3. Calculate the dipole-dipole magnetic interaction energy of a proton and antiproton at fixed separation a, in terms of the proton magnetic moment μ_0, for each eigenstate of total spin. The interaction Hamiltonian is

$$V = \frac{1}{r^2} \{\vec{\mu}_1 \cdot \vec{\mu}_2 - 2 \frac{(\vec{\mu}_1 \cdot \vec{r})(\vec{\mu}_2 \cdot \vec{r})}{r^2}\}.$$

4. The Kronig-Penney model of electrons in a crystal is a one-dimensional periodic potential:



Solutions of the one-dimensional Schroedinger equation for electrons of energy E are sought in the form of the so-called "Bloch wavefunctions", $\psi(x) = u_k(x)e^{ikx}$, where u is a periodic function, period (a+b) for this model. The allowed values of wavenumber k are, as usual, fixed by the overall size, or in this case length, of the crystal.

(a) Find the differential equation to the satisfied by $u_k(x)$ and show that $u = Ae^{i(\alpha-k)x} + Be^{-i(\alpha+k)x}$ is a solution for $0 < x < a$ if α has a certain value, and find this value.

(b) For $a < x < a+b$, show that a solution is $u = Ce^{(\beta-ik)x} + De^{-(\beta+ik)x}$ provided β has a certain value, and find this value.

5. A bead of mass m is constrained to slide without friction on a wire whose shape is $z = Ax^2$, with the plane of the wire normal to the earth's surface.

   (a) Use a Lagrangian method to determine the equation of motion of the bead.

   (b) Find the frequency of small oscillations of the bead about the origin.

6. Let f_ν be the distribution function for a system, the probability that a random member of an ensemble of such systems will be in a quantum state characterized by ν. For a canonical ensemble we require that there be an ensemble average energy U characterized by a temperature T when the entropy

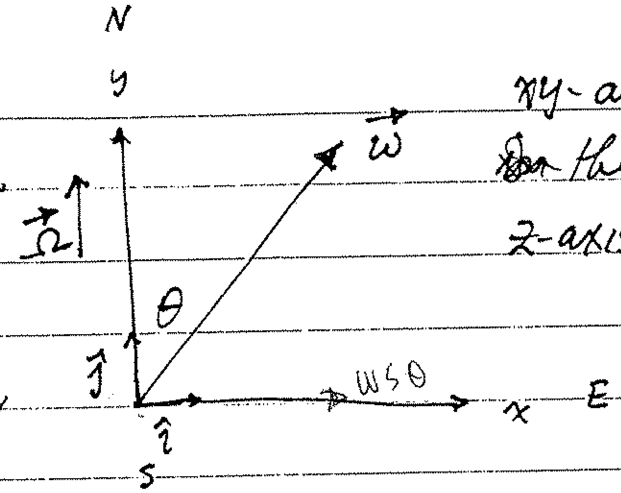
$$S = \frac{U-A}{T} = - k \sum_\nu f_\nu \ln f_\nu$$

is maximal, A being the Helmholz free energy and k is Boltzmann's constant.

   (a) Find an expression for f_ν by the method of Lagrange multipliers α_0 and α_e by maximizing the expression $S + \alpha_0 \sum f_\nu + \alpha_e \sum f_\nu E_\nu$, where E_ν are the eigenvalues of the states ν, subject to the constraints $\sum f_\nu = 1$, $\sum f_\nu E_\nu = U$.

   (b) Complete the determination of f_ν by finding interpretations of α_0 and α_e.

1.



xy-axes are in the plane of the earth's surface;
z-axis is out of the paper (i.e. vertical).
θ = angle of symmetry axis with North, $\theta \to 0$
angular velocity of earth is $\Omega\hat{j}$

angular momentum:

$$L_x = C\omega\sin\theta \approx C\omega\theta$$

$$L_y = C\omega\cos\theta \approx C\omega$$

$$L_z = A\dot{\theta}$$

NOTE: The only reason for the oscillation of the gyro is ~~because~~ the earth frame is non-inertial.

$$\frac{d\vec{L}}{dt} = \vec{N} - \vec{\Omega}\times\vec{L}$$

know $N_z = 0$, look at $\frac{dL_z}{dt}$

$$\vec{\Omega}\times\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \Omega & 0 \\ C\omega\theta & C\omega & A\dot{\theta} \end{vmatrix}$$

$$\frac{dL_z}{dt} = A\ddot{\theta} = -(\vec{\Omega}\times\vec{L})_z = +C\omega\Omega\theta$$

$$\ddot{\theta} + \frac{C\omega\Omega}{A}\theta = 0$$

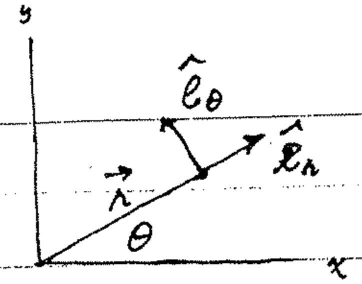
SHMotion (QED)

$$\text{"}\omega\text{"} = \frac{2\pi}{\tau} = \sqrt{\frac{C\omega\Omega}{A}}$$

period $\boxed{\tau = 2\pi\sqrt{\frac{A}{C\omega\Omega}}}$

2.

a)



cylindrical coordinates:

$\hat{e}_r \times \hat{e}_z = -\hat{e}_\theta \qquad \hat{e}_r \times \hat{e}_\theta = \hat{e}_z$

$\hat{e}_\theta \times \hat{e}_z = \hat{e}_r$

$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

$\vec{B} = B\hat{e}_z$

$\vec{F} = -e\vec{v}\times\vec{B} = -e(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta)\times B\hat{e}_z - eE\hat{e}_r$
$\quad -eE\hat{e}_r$

$= e\dot{r}B\hat{e}_\theta - er\dot{\theta}B\hat{e}_r - eE\hat{e}_r$

$\vec{N} = \vec{r}\times\vec{F} = e\dot{r}rB\hat{e}_z = \dfrac{d\vec{L}}{dt}$

By looking at torque & r angular momentum instead of force (and $m\vec{a}$), the E-field can be eliminated from the equation.
$L_z = mr^2\dot{\theta}$

$$\frac{d}{dt}(mr^2\dot{\theta}) = eBr\dot{r}$$

$$mr^2\dot{\theta}\Big|_0^{(\ )} = eB\frac{r^2}{2}\Big|_a^r = \frac{eB}{2}(r^2 - a^2)$$

$$mr^2\dot{\theta} = \frac{eB}{2}(r^2 - a^2)$$

so angular <u>velocity</u> $\quad \dot{\theta} = \dfrac{eB}{2m}\left(1 - \dfrac{a^2}{r^2}\right)$ QED

b) from conservation of energy:

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - eV(r) = 0 \qquad r\dot{\theta} = \frac{eB}{2mr}(r^2 - a^2)$$

$$\frac{1}{2}m\left\{\dot{r}^2 + \left(\frac{eB}{2mr}\right)^2(r^2 - a^2)^2\right\} = eV(r)$$

at $r = b$ want $\dot{r} = 0$ for $V(r{=}b) = V_m$

so $\frac{1}{2}m\left\{\left(\frac{eB}{2mb}\right)^2(b^2-a^2)^2\right\} = eV_m$

$$\boxed{V_m = \frac{eB^2}{8mb^2}(b^2-a^2)^2}$$

Problem 3

Calculate the dipole-dipole magnetic interaction energy of a proton and an antiproton at a fixed distance a, in eigenstates of total spin, in terms of the proton magnetic moment μ_0. The two magnetic dipoles have the interaction energy

$$V = \frac{1}{r^3}\left\{ \vec{\mu}_1 \cdot \vec{\mu}_2 - \frac{3(\vec{\mu}_1 \cdot \vec{r})(\vec{\mu}_2 \cdot \vec{r})}{r^2} \right\}$$

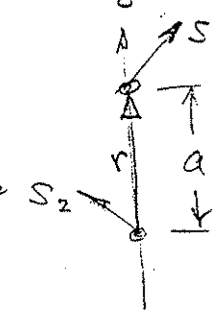
solution:

The proton and anti-proton magnetic moments are given by

$$\vec{\mu}_1 = 2\mu_0 \vec{S}_1 \quad \text{and} \quad \vec{\mu}_2 = 2\mu_0 \vec{S}_2$$

choosing the axis joining the two particles for the z-axis,

$$\frac{(\vec{\mu}_1 \cdot \vec{r}_1)(\vec{\mu}_2 \cdot \vec{r})}{r^2} = 4\mu_0^2 \, S_{1z} S_{2z}$$



and

$$2 S_{1z} S_{2z} = \vec{S}_z^2 - (\vec{S}_{1z})^2 - (\vec{S}_{2z})^2 \qquad \vec{S}_z = \vec{S}_{z1} + \vec{S}_{z2}$$

also

$$\vec{\mu}_1 \cdot \vec{\mu}_2 = 4\mu_0^2 \, \vec{S}_1 \cdot \vec{S}_2$$

$$\text{and } 2\vec{S}_1 \cdot \vec{S}_2 = \vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2 \qquad \vec{S} = \vec{S}_1 + \vec{S}_2$$

the eigen values are

$$\left.\begin{aligned} 2 S_{1z} S_{2z} &= S_z^2 - \tfrac{1}{2} \\ 2\vec{S}_1 \cdot \vec{S}_2 &= S(S+1) - \tfrac{3}{2} \end{aligned}\right\} \quad \text{for spin } \tfrac{1}{2} \text{ particles}$$

the eigen states are $S = 0,\ S_z = 0$

and $S = 1,\ S_z = -1, 0, +1$

the corresponding energies are

$$V_{S,S_z} = \frac{2\mu_0^2}{a^3}\left\{ S(S+1) - \frac{3}{2} - 3\left(S_z^2 - \frac{1}{2}\right)\right\}$$

$$V_{0,0} = 0 \quad , \quad V_{1,0} = \frac{4\mu_0^2}{a^3} \quad , \quad V_{1,\pm 1} = -\frac{2\mu_0^2}{a^3}$$

For the case $$V = \frac{1}{r^2}\left\{ \vec{\mu}_1 \cdot \vec{\mu}_2 - \frac{2(\vec{\mu}_1 \cdot \vec{r})(\vec{\mu}_2 \cdot \vec{r})}{r^2} \right\}$$

(as stated on the Comp Exam)

$$V_{S,S_3} = \frac{2\mu_0^2}{a^2}\left\{ S(S+1) - \frac{3}{2} - 2\left(S_3^2 - \frac{1}{2}\right)\right\}$$

$$V_{0,0} = -\frac{\mu_0^2}{a^2} = -\left(\frac{2\mu_0}{a}\right)^2\left(\frac{1}{4}\right)$$

$$V_{1,0} = \frac{3\mu_0^2}{a^2} = -\left(\frac{2\mu_0}{a}\right)^2\left(-\frac{3}{4}\right)$$

$$V_{1,1} = -\frac{\mu_0^2}{a^2} = -\left(\frac{2\mu_0}{a}\right)^2\left(\frac{1}{4}\right)$$

$$V_{1,-1} = -\frac{\mu_0^2}{a^2} = -\left(\frac{2\mu_0}{a}\right)^2\left(\frac{1}{4}\right)$$

4, a) Schroedinger's eqn in one-dimension:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

or $\quad \psi'' + \frac{2m}{\hbar^2}(E - V)\psi = 0$

$\psi = u(x)\,e^{ikx}, \quad \psi' = u'e^{ikx} + iku\,e^{ikx}$

$$u''e^{ikx} + 2iku'e^{ikx} - k^2u\,e^{ikx} + \frac{2m}{\hbar^2}(E-V)u\,e^{ikx} = 0$$

$$\underline{u'' + 2iku' + \frac{2m}{\hbar^2}(E - V - E_k)u = 0}$$

where $E_k = \frac{\hbar^2 k^2}{2m}$.

For $0 < x < a$, $V=0$, assume $u = A e^{i(\alpha-k)x} + B e^{-i(\alpha+k)x}$

$u' = i(\alpha-k)A e^{i(\alpha-k)x} - i(\alpha+k)B e^{-i(\alpha+k)x}$

$u'' = -(\alpha-k)^2 A e^{i(\alpha-k)x} - (\alpha+k)^2 B e^{-i(\alpha+k)x}$

$$\begin{aligned}
&-(\alpha^2 - 2\alpha k + k^2)A e^{i(\alpha-k)x} - (\alpha^2 + 2\alpha k + k^2)B e^{-i(\alpha+k)x} \\
&-2k\,(\alpha-k)A e^{i(\alpha-k)x} + 2k\,(\alpha+k)\,B e^{-i(\alpha+k)x} \\
&+\frac{2m}{\hbar^2}(E - E_k)A e^{i(\alpha-k)x} + \frac{2m}{\hbar^2}(E - E_k)B e^{-i(\alpha+k)x} = 0
\end{aligned}$$

$$-\alpha^2 + 2\alpha k - k^2 - 2k\alpha + 2k^2 + \frac{2mE}{\hbar^2} - k^2 = 0$$

requires $\quad \underline{\alpha = \sqrt{\frac{2mE}{\hbar^2}}}$

4. b) For $a < x < a+b$, $V = V_0$

assume $u = Ce^{(\beta - ik)x} + De^{-(\beta + ik)x}$

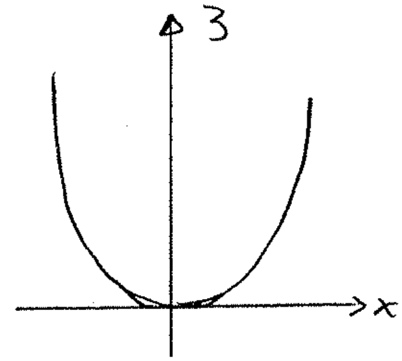
$$u' = (\beta - ik)Ce^{(\beta - ik)x} - (\beta + ik)De^{-(\beta + ik)x}$$

$$u'' = (\beta - ik)^2 Ce^{(\beta - ik)x} + (\beta + ik)^2 De^{-(\beta + ik)x}$$

$$(\beta^2 - 2ik\beta - k^2)Ce^{(\beta - ik)x} + (\beta^2 + 2ik\beta - k^2)De^{-(\beta + ik)x}$$
$$+ (2ik\beta + 2k^2)Ce^{(\beta - ik)x} - (2ik\beta - 2k^2)De^{-(\beta + ik)x}$$
$$+ \frac{2m}{\hbar^2}(E - V_0 - E_k)Ce^{(\beta - ik)x} + \frac{2m}{\hbar^2}(E - V - E_k)De^{-(\beta + ik)x}$$
$$= 0$$

Requires $\beta = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$

Problem 5



The Lagrangian may be written in cartesian coordinates

$$L = \frac{m}{2}(\dot{x}^2 + \dot{z}^2) - mgz$$

the constraint is $z = Ax^2$
If we choose to eliminate z using the constraint condition we get

$$L = \frac{m}{2}\left[\dot{x}^2 + (2Ax\dot{x})^2\right] - mgAx^2$$

$$\frac{\partial L}{\partial x} = x\left[4A^2m\dot{x}^2 - 2mAg\right] \qquad \frac{\partial L}{\partial \dot{x}} = \dot{x}\left[m + 4A^2mx^2\right]$$

Lagrange's Eq. is $\quad m\ddot{x} + 4mA^2[\ddot{x}x^2 + x\dot{x}^2] + 2mAgx = 0$
or $\quad \boxed{m\ddot{x} + 2Amgx + 4A^2mx(\dot{x}^2 + x\ddot{x}) = 0}$

the last term is proportional to the third power of the amplitude so it will be small and in the limit of small oscillations we have

$$m\ddot{x} + 2Amgx = 0 \qquad \therefore \boxed{\omega^2 = 2Ag}$$

Alternatively using the method of Lagrange multipliers

$$\begin{cases} m\ddot{x} - 2A\lambda x = 0 & (1) \\ m\ddot{z} + mg + \lambda = 0 & (2) \\ A^2 - z = 0 & (3) \end{cases}$$

differentiate (3) twice to obtain

$$\ddot{z} - 2A(\dot{x}^2 + x\ddot{x}) = 0$$

then (2) becomes

$$2Am(\dot{x}^2 + x\ddot{x}) + mg + \lambda = 0$$

replace λ in (1)

$$m\ddot{x} + 2Ax\left[(2Am)(\dot{x}^2 + x\ddot{x}) + mg\right] = 0$$

which is the same as equation of motion above
and $\lambda = -2Am(\dot{x}^2 + x\ddot{x}) - mg = \frac{m\ddot{x} + 2mAgx}{2Ax} - mg$

$$\lambda \xrightarrow[\text{small } x]{\lim} = -mg \qquad \text{so} \quad \ddot{x} + 2Agx = 0$$

and again $\omega^2 = 2Ag$

Other methods of finding ω include defining and effective force $f' = -4mA^2[x^2\ddot{x} + x\dot{x}^2] - 2mAgx = m\ddot{x}$ and expanding about the equilibrium position x_0 (where $\dot{z} = 0$)

6. a)

We need to find f_ν such that $(S + \alpha_0 \sum f_\nu + \alpha_e \sum f_\nu E_\nu)$ is a maximum, subject to $\sum f_\nu = 1$, $\sum f_\nu E_\nu = U$.

Therefore, we will set

$$\frac{\partial}{\partial f_\nu}\left(S + \alpha_0 \sum f_\nu + \alpha_e \sum f_\nu E_\nu\right) = 0$$

or

$$\frac{\partial}{\partial f_\nu}\left(-k\sum f_\nu \ln f_\nu + \alpha_0 \sum f_\nu + \alpha_e \sum f_\nu E_\nu\right)$$

$$= -k\left(\ln f_\nu + f_\nu \frac{1}{f_\nu}\right) + \alpha_0 + \alpha_e E_\nu = 0$$

Then

$$\boxed{f_\nu = e^{(\alpha_0 + \alpha_e E_\nu - k)/k}}$$

$$S = -k\sum f_\nu \ln f_\nu = -k \sum f_\nu \left(\frac{\alpha_0 + \alpha_e E_\nu - k}{k}\right)$$

$$= \sum\left(k f_\nu - \alpha_0 f_\nu - \alpha_e E_\nu f_\nu\right)$$

$$= (k - \alpha_0)\left(\sum f_\nu\right) - \alpha_e \sum f_\nu E_\nu$$

$$= (k - \alpha_0) - \alpha_e U$$

Since

$$TS = -A + U = (kT - \alpha_0 T) - \alpha_e T U$$

we can interpret $\alpha_e T = -1$, $\boxed{\alpha_e = -\frac{1}{T}}$,

and $\alpha_0 T = kT + A$, or $\boxed{\alpha_0 = k + A/T}$.

Then $f_\nu = e^{\left(\frac{\alpha_0}{k} - 1\right)} e^{+\alpha_e E_\nu / k} = e^{\left(\frac{\alpha_0}{k} - 1\right)} e^{-E_\nu / kT}$

and because $\sum f_\nu = e^{\left(\frac{\alpha_0}{k} - 1\right)} \sum e^{-E_\nu / kT} = 1$,

$$\boxed{f_\nu = \frac{e^{-E_\nu / kT}}{\sum e^{-E_\nu / kT}}}$$