PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #47

January 7, 1984

General Instructions

This Comprehensive Examination for Winter 1984 (#47) consists of six problems of equal weight (20 points each). Please check that you have all of them.

Work carefully, indicate your reasoning briefly, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you may find useful. Please return it at the end of the exam.

1. The space between conducting concentric spheres of radii R_1 and $R_2 > R_1$ is filled with a linear dielectric which has dielectric constant $K = K_0 + K_1 \cos^2\theta$, where K_0 and K_1 are constants and θ is the polar angle in spherical coordinates. The outer sphere is grounded, and the inner sphere is charged positively to a potential V_0.

   (a) Prove that the electric field is radial, and find it.

   (b) Derive an expression for the capacitance.

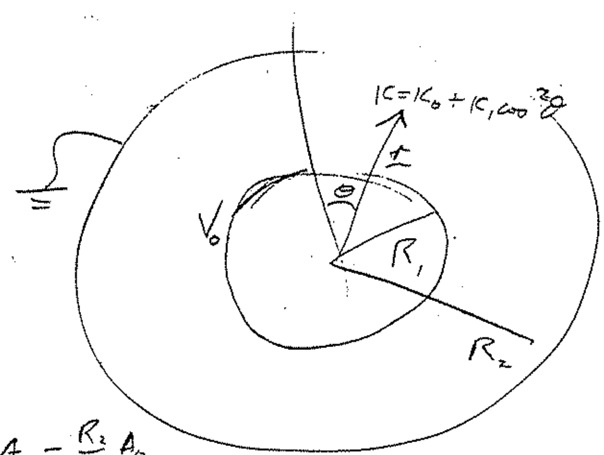
2. A vacuum ion beam carries a large current I_0 uniformly distributed within a tube of radius R. One ion leaves the central axis with velocity $\vec{v}$ making a <u>small</u> angle θ with the axis. Neglect electric fields and collisions.

   (a) Show that if the ion stays in the beam it will return to the axis in a distance $L \doteq 2\pi\sqrt{R\rho}$, where ρ is the radius of curvature of an ionic orbit at the edge of the beam (at radial distance R).

   (b) Prove that the condition on θ to keep the ion in the beam is $1 - \cos\theta \lessdot R/2\rho$.

1. $V = \sum (A_\ell r^\ell + B_\ell r^{-\ell-1}) P_\ell$

$V_0 = \left(A_0 + \frac{B_0}{R_1}\right) + \left(A_1 + \frac{B_1}{R_1^2}\right)\cos\theta + \cdots$



$A_1 = B_1 = A_2 = \cdots = 0$

$V_0 = A_0 + \frac{B_0}{R_1}$

$0 = A_0 + \frac{B_0}{R_2}$, $B_0 = -R_2 A_0$, $V_0 = A_0 - \frac{R_2}{R_1} A_0$

$A_0 = \frac{V_0}{1 - \frac{R_2}{R_1}}$

$V = A_0 - A_0 \frac{R_2}{r} = V_0 \frac{1 - \frac{R_2}{r}}{1 - \frac{R_2}{R_1}}$

$\underset{\sim}{E} = -\hat{r} \frac{\partial}{\partial r}\left(V_0 \frac{1 - \frac{R_2}{r}}{1 - \frac{R_2}{R_1}} \right) = -\hat{r} V_0 \frac{R_2/r^2}{1 - \frac{R_2}{R_1}} = \hat{r} V_0 \frac{R_1}{(1 - \frac{R_1}{R_2}) r^2}$

$\underset{\sim}{D} = \epsilon_0 K \underset{\sim}{E} = \frac{\hat{r} V_0 R_1 \epsilon_0}{1 - \frac{R_1}{R_2}} \frac{(K_0 + K_1 \cos^2\theta)}{r^2}$

$Q = \oint D \cdot da = \frac{V_0 R_1 \epsilon_0}{1 - \frac{R_1}{R_2}} \int_0^\pi \frac{K_0 + K_1 \cos^2\theta'}{r'^2} 2\pi r' \sin\theta' \, r' d\theta'$

$= \frac{V_0 R_1 \epsilon_0}{1 - \frac{R_1}{R_2}} 2\pi \left\{ -K_0 \cos\theta' \Big|_0^\pi - K_1 \frac{\cos^3\theta'}{3} \Big|_0^\pi \right\}$

$= \frac{V_0 R_1 \epsilon_0}{1 - \frac{R_1}{R_2}} 2\pi \left\{ K_0 (\cos(0) - \cos\pi) + \frac{1}{3}\left[\cos^3(0) - \cos^3\pi\right] \right\}$

$= \frac{V_0 R_1 \epsilon_0}{1 - \frac{R_1}{R_2}} 2\pi \left\{ 2K_0 + \frac{2}{3} K_1 \right\}$

$C = \frac{Q}{V_0} = \frac{4\pi \epsilon_0 R_1 (K_0 + \frac{1}{3} K_1)}{1 - \frac{R_1}{R_2}}$ ✓

2. (DB)

radial distance

$$\underset{\sim}{B} = \hat{\varphi}\, \frac{\mu_0 I_0}{2\pi R}\, \frac{b}{R} \quad , \quad \underset{\sim}{F} = +e\, \underset{\sim}{v} \times \underset{\sim}{B}$$

$$= \frac{\mu_0 I_0}{2\pi R^2}\, b\,(-\hat{i}\sin\varphi + \hat{j}\cos\varphi) = \frac{\mu_0 I_0}{2\pi R^2}(-\hat{i}y + \hat{j}x)$$

$$m(\hat{i}\ddot{x} + \hat{j}\ddot{y} + \hat{k}\ddot{z}) = +e(\hat{i}\dot{x} + \hat{j}\dot{y} + \hat{k}\dot{z}) \times \frac{\mu_0 I_0}{2\pi R^2}(-\hat{i}y + \hat{j}x)$$

$$= \frac{+e\mu_0 I_0}{2\pi R^2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x} & \dot{y} & \dot{z} \\ -y & x & 0 \end{vmatrix}$$

$$\hat{i}\ddot{x} + \hat{j}\ddot{y} + \hat{k}\ddot{z} = (+\alpha)\left[-\hat{i}x\dot{z} - \hat{j}y\dot{z} + \hat{k}(\dot{x}x + \dot{y}y)\right]$$

$$\ddot{x} = -\alpha x \dot{z}$$
$$\ddot{y} = -\alpha y \dot{z}$$
$$\ddot{z} = +\alpha \frac{d}{dt}\left(\frac{x^2 + y^2}{2}\right)$$

$$\dot{z} = +\frac{\alpha}{2}(x^2 + y^2) + C \quad , \quad C = \dot{z}_0$$

$$b = (x^2 + y^2)^{1/2}$$

$$\begin{cases} \ddot{b} = -\alpha b \dot{z} \\ \dot{z} = +\dfrac{\alpha b^2}{2} + \dot{z}_0 \end{cases}$$

$$\ddot{b} = -\alpha b \left(\dot{z}_0 - \frac{\alpha b^2}{2}\right)$$
$$\doteq -\alpha b \dot{z}_0$$

$$b = b_{max} \sin\sqrt{\alpha \dot{z}_0}\, t$$

$$\text{at } \sqrt{\alpha \dot{z}_0}\, t_1 = \pi \quad , \quad t_1 = \frac{\pi}{\sqrt{\alpha \dot{z}_0}} \quad , \quad L = \dot{z}_0 t_1 = \pi\sqrt{\frac{\dot{z}_0}{\alpha}}$$

$$L = \pi\sqrt{\frac{\dot{z}_0\, 2\pi R^2 m}{e\mu_0 I_0}} = \pi\sqrt{\frac{\dot{z}_0 R m}{e B(R)}} = \pi\sqrt{\frac{\dot{z}_0 m R}{m v/\rho}} \doteq \pi\sqrt{\rho R}$$

$$\frac{mv^2}{\rho} = B(R)\,ev = \frac{\mu_0 I_0 e v}{2\pi R} \qquad \frac{e\mu_0 I_0}{2\pi R} = \frac{mv}{\rho}$$

$$b = b_{max} \sin\left(\sqrt{\alpha \dot{z}_0}\, t\right)$$

$$\dot{b} = \sqrt{\alpha \dot{z}_0}\, b_m \cos\left(\sqrt{\alpha \dot{z}_0}\, t\right)$$

$$\dot{b}(0) = \sqrt{\alpha \dot{z}_0}\, b_m = v_0 \sin\theta$$

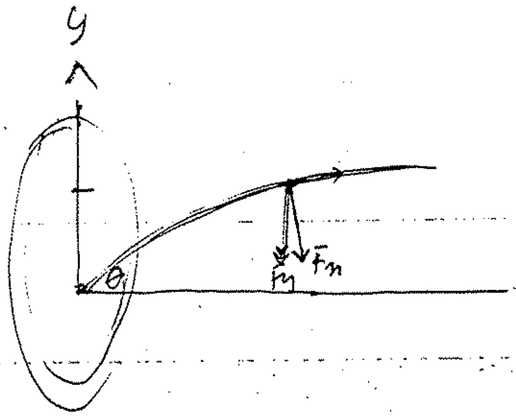
$$b_m = \frac{v_0 \sin\theta}{\sqrt{\alpha \dot{z}_0}} < R$$

$$\sin\theta < \frac{R}{v}\sqrt{\frac{e\mu_0 I_0 \dot{z}_0}{2\pi R^2 m}} = \frac{R}{v}\sqrt{\frac{mv}{\rho}\frac{\dot{z}_0}{Rm}} \doteq \sqrt{\frac{R}{\rho}}$$

$$\sin^2\theta = 1-\cos^2\theta = (1-\cos\theta)(1+\cos\theta) \doteq 2(1-\cos\theta) \lesssim \frac{R}{\rho}$$

$$1-\cos\theta < \frac{R}{2\rho}$$

2(HE)



θ small $F_n \approx F_y$

$\oint \vec{B}\cdot d\vec{l} = \mu_0 I$

$B \cdot 2\pi y = \mu_0 I \frac{y^2}{a^2}$

$B = \frac{\mu_0 I}{2\pi a^2} y$

F_n $y = a$

$Bev = \frac{mv^2}{\rho}$

$\downarrow$

$\frac{\mu_0 I}{2\pi a} e\cancel{v} = \frac{mv^{\cancel{2}}}{\rho}$

~~and eliminate~~

$\frac{\mu_0 I e}{2\pi m} = \frac{av}{\rho}$

$m\ddot{y} = -evB = -\frac{ev\mu_0 I}{2\pi a^2} y$

$\ddot{y} + \frac{ev\mu_0 I}{2\pi m a^2} y = 0$

$\ddot{y} + \frac{\cancel{a}v^2}{\rho a^{\cancel{2}}} y = 0$

shm motion vertically

$\omega = \frac{v}{\sqrt{\rho a}} = \frac{2\pi}{T}$

time to get back to axis is $\frac{T}{2}$ so distance to get back is

$L = v\frac{T}{2} = \cancel{2}\pi\sqrt{\rho a}$ QED analogy with a spring:

want $\frac{1}{2}mv^2 < \frac{1}{2}ka^2$

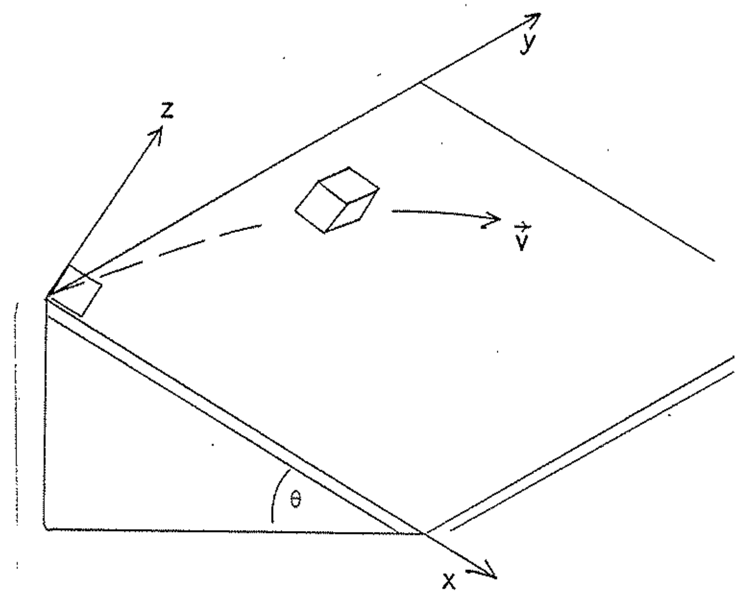
where v is initial y-velocity and $k = m\omega^2$

$$\cancel{\tfrac{1}{2}}\cancel{mv^2}\underbrace{\sin^2\theta} < \cancel{\tfrac{1}{2}}\cancel{m}\left(\frac{\cancel{v^2}}{\rho\cancel{a}}\right)a^{\cancel{2}}$$

$$1-\cos^2\theta = (1-\cos\theta)(1+\cos\theta) \approx 2(1-\cos\theta) < \frac{a}{\rho}$$

$$\text{or } (1-\cos\theta) < \frac{a}{2\rho} \qquad \text{QED}$$

3. A particle of mass m moves on the surface of an inclined plane for which the coefficient of friction is $\mu = \tan\theta_0$, where θ_0 is the "angle of repose" or "critical angle" below which a particle at rest will not slip. With $\theta < \theta_0$, the particle is projected from the origin at $t = 0$ with $\vec{v}_0 = \hat{j}v_0$, using the coordinate system shown.



(a) Write the equations of motion (differential equations for v_x, v_y).

(b) Transform to the dimensionless variables $V_{x,y} = v_{x,y}/v_0$, $T = \frac{gt}{v_0}\mu\cos\theta$.

(c) For the first part of the motion $V_x << V_y$. Use this to decouple the equations of motion, and solve for $V_y(T)$, $V_x(T)$. (Hint: The transformation $V_x = (1 - \alpha T)\, f(T)$ will facilitate integration, with suitable choice of α.)

4. A typical magnetic resonance experiment can be described as follows: A system of spin 1/2 is placed in a strong uniform field of strength B_0 along the z-axis. A weak, oscillating uniform field $B_1 \sin\omega t$ is applied along the x-direction. Transitions are readily induced if ω corresponds to the so-called "Larmor frequency". Derive this result by the use of first-order, time-dependent perturbation theory.

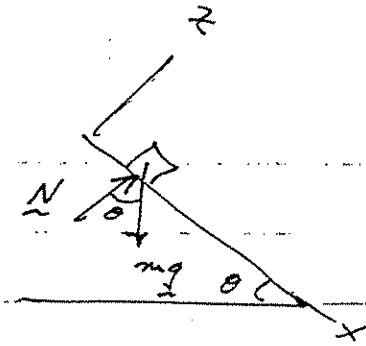
Note: The Pauli spin operators are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3.

$$\underset{\sim}{W} = mg(\hat{i}\sin\theta - \hat{k}\cos\theta)$$

$$\underset{\sim}{f} = -\mu N \underset{\sim}{v}/v \quad , v = \sqrt{v_x^2+v_y^2}$$



$$\begin{cases} m\dot{v}_z = 0 = mg\cos\theta - N \\ m\dot{v}_x = mg\sin\theta - \mu N \dfrac{v_x}{v} = mg\sin\theta - \dfrac{v_x}{v}\mu mg\cos\theta \\ m\dot{v}_y = -\mu N\dfrac{v_y}{v} = -\dfrac{v_y}{v}\mu mg\cos\theta \end{cases}$$

$$\dot{v}_i = \frac{dv_i}{dV_i}\frac{dV_i}{dT}\frac{dT}{dt} = v_0 \frac{dV_i}{dT}\frac{\mu g\cos\theta}{v_0}$$

$$(\mu g\cos\theta)\frac{dV_x}{dT} = \cancel{\mu} g\sin\theta - \frac{V_x}{V}\mu g\cos\theta$$

$$\mu g\cos\theta\frac{dV_y}{dT} = -\frac{V_y}{V}\mu g\cos\theta$$

$$\begin{cases} \dfrac{dV_x}{dT} = \dfrac{\tan\theta}{\tan\theta_0} - \dfrac{V_x}{V} = \beta - \dfrac{V_x}{V} \\ \dfrac{dV_y}{dT} = -\dfrac{V_y}{V} \end{cases}$$

For first part of the motion, $V = \sqrt{V_x^2+V_y^2} \doteq V_y$

$$\frac{dV_y}{dT} \doteq -1 \quad , \quad V_y = 1-T$$

$$\frac{dV_x}{dT} \doteq \beta - \frac{V_x}{1-T}$$

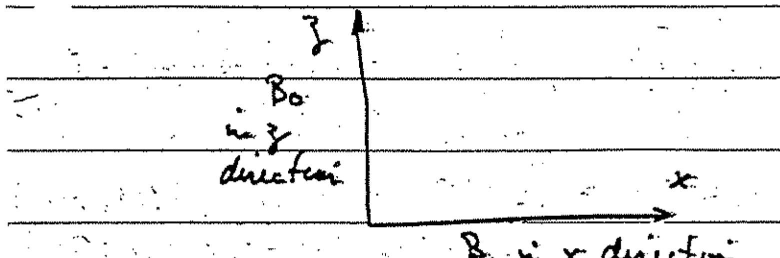
$$V_x = (1-\alpha T)f(T) \quad , \quad \frac{dV_x}{dT} = -\alpha f + (1-\alpha T)\frac{df}{dT}$$

$$-\alpha f + (1-\alpha T)\frac{df}{dT} = \beta - \frac{(1-\alpha T)f}{1-T}$$

For $\alpha = 1$, $\quad \dfrac{df}{dT} = \dfrac{\beta}{1-\alpha T} \quad , f = -\beta\ln(1-T)$

$$V_x = -\beta(1-T)\ln(1-T) = \beta(1-T)\left\{T + \tfrac{1}{2}T^2 + \cdots\right\}$$

4. Question 2



Let the Hamiltonian be given by $H = H_0 + H'$, where H' is the perturbation. then

$H_0 = -\mu_0 \vec{\sigma}\cdot\vec{B}_0 = -\mu_0 \sigma_z B_0$

$H' = -\mu_0 \sigma_x B_1 \sin\omega t$

where μ_0 is the mag. mom. of the electron.

The eigenfunctions of H_0 are $\psi^\uparrow$ (spin up) and $\psi^\downarrow$ (spin down)
$\psi^\uparrow = \binom{1}{0}$ and $\psi^\downarrow = \binom{0}{1}$, $\sigma_z\psi^\uparrow = \psi^\uparrow$, $\sigma_z\psi^\downarrow = -\psi^\downarrow$, $\sigma_x\psi^\uparrow = \psi^\downarrow$, $\sigma_x\psi^\downarrow = \psi^\uparrow$. Thus if at time $t=0$ the system is placed in the state $\psi^\uparrow$, then H' will induce transitions to the state $\psi^\downarrow$. The probability amplitude for this trans. in 1st of order perturbation theory is,

$$a(t) = \frac{1}{i\hbar}\int_0^t dt \langle \psi^\downarrow | -\mu_0 \sigma_x B_1 \sin\omega t | \psi^\uparrow \rangle e^{i\omega_0 t}$$

where $\hbar\omega_0 = 2\mu_0 B_0$

$$a(t) = -\frac{\mu_0 B_1}{i\hbar}\int_0^t dt\, e^{i\omega_0 t}\left[\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right]$$

$$a(t) = -\frac{\mu_0 B_1}{i\hbar}\frac{1}{2i}\left[\frac{(e^{i(\omega+\omega_0)t} - 1)}{i(\omega+\omega_0)} - \frac{(e^{i(\omega_0-\omega)t} - 1)}{i(\omega_0-\omega)}\right]$$

From this we see that $a(t)$ has a resonance for $\omega = \pm\omega_0$, and hence transitions are readily induced for these frequency (ω_0 is called the Larmor frequency)

5. Processes which can occur when antiprotons are captured in deuterium are indicated in these reactions:

$$\bar{p} + D \rightarrow n + \pi^0 \rightarrow n + 2\gamma.$$

(a) If the first reaction occurs with the $\bar{p}$ and D at rest, derive an expression for the (lab frame) kinetic energy of the pions. Your final equation should contain the masses of the particles involved, $m_{\bar{p}}$, m_D, m_n, m_π, and known constants.

(b) A pion at rest will decay into two gammas, each having energy $(E_\gamma)_0$ = (135/2) MeV, and the gammas will have an angle of 180° between their propagation vectors. On the other hand, if the pion has energy $E_\pi > m_\pi c^2$, there will be an angle $\theta < 180°$ between the gammas observed in the lab frame, and θ will be smallest if the gammas have equal energy. Derive an expression for θ as a function of $(m_\pi c^2/E_\pi)$ for this case, and evaluate it for $E_\pi = 2m_\pi c^2 = 270$ MeV.

5.

a) $\begin{cases} E = E_\pi + E_n = m_p c^2 + m_D c^2 \\ p_n^2 c^2 = E_n^2 - m_n^2 c^4 = p_\pi^2 c^2 = E_\pi^2 - m_\pi^2 c^4 \end{cases}$

$$\left[(m_p c^2 + m_D c^2) - E_\pi\right]^2 - m_n^2 c^4 = E_\pi^2 - m_\pi^2 c^4$$

$$(m_p c^2 + m_D c^2)^2 - 2E_\pi (m_p c^2 + m_D c^2) + \cancel{E_\pi^2} = \cancel{E_\pi^2} - m_\pi^2 c^4 + m_n^2 c^4$$

$$2E_\pi (m_p c^2 + m_D c^2) = (m_p c^2 + m_D c^2)^2 + m_\pi^2 c^4 - m_n^2 c^4$$

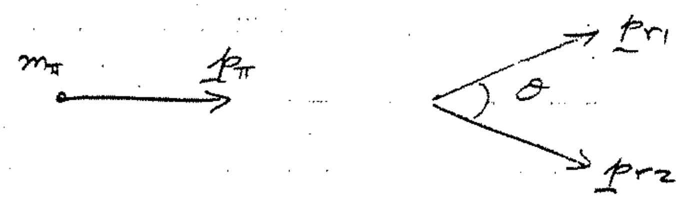
$$E_\pi = \frac{(m_p c^2 + m_D c^2)^2 + m_\pi^2 c^4 - m_n^2 c^4}{2(m_p c^2 + m_D c^2)} \Longleftarrow$$

$$= \frac{(1000 + 2000)^2 + 100^2 - 1000^2}{2(1000 + 2000)} = \frac{3000}{2} + \frac{10000}{6000} - \frac{1000}{6}$$

$$= \frac{8000}{6} - \frac{10}{6} = \frac{7990}{6} = 1332 \text{ MEV}$$

$$KE = E_\pi - m_\pi c^2 = 1332 - 100 = \underline{1.23 \text{ BeV}}$$

b)



$$p_{r1} = p_{r2} = p_r$$

$$2 p_r \cos\frac{\theta}{2} = p_\pi = \sqrt{\frac{E_\pi^2}{c^2} - m_\pi^2 c^2}$$

$$2E_r = 2\sqrt{p_r^2 c^2} = E_\pi \ , \quad p_r = \sqrt{E_\pi^2 / 4c^2}$$

$$\cos\frac{\theta}{2} = \frac{\sqrt{\frac{E_\pi^2}{c^2} - m_\pi^2 c^2}}{2\sqrt{E_\pi^2/4c^2}} = \sqrt{1 - \left(\frac{m_\pi c^2}{E_\pi}\right)^2}$$

For $\frac{m_\pi c^2}{E_\pi} = \frac{1}{2}$, $\cos\frac{\theta}{2} = \sqrt{1 - \frac{1}{4}} = \cos 30^\circ$

$$\underline{\theta = 60^\circ}$$

6. A simple, isothermal, mass-point gas is one in which the molecules do not interact with each other. In the absence of an external force field, the distribution function $f(\underset{\sim}{r},\underset{\sim}{v},t)$, $\iint f \, d^3r \, d^3v = N$ = total number of particles, is given by the Maxwell function:

$$f_M = n(m/2\pi kT)^{3/2} \exp\,(-mv^2/2kT),$$

where n is the concentration (number density) and the other symbols have their usual meaning.

(a) A differential equation for f is the Boltzmann equation:

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \nabla f + \underset{\sim}{a} \cdot \nabla_v f = \left.\frac{\partial f}{\partial t}\right)_{coll.}$$

in which $\underset{\sim}{a}$ is acceleration, ∇_v represents the gradient operator in v-space, and the right-hand side gives the rate of change of f due to collisions. Show that if there is no external field applied to the gas, f_M will be a solution of the Boltzmann equation provided n is independent of $\underset{\sim}{r}$.

(b) Now suppose that a field is applied, and that this field can be represented by a potential $\phi(\underset{\sim}{r})$ such that each molecule experiences force $\underset{\sim}{F} = -\nabla\phi$. Show that f_M is still in equilibrium solution but with $n = n(\underset{\sim}{r})$, and find this function.

6.

a) For no interactions, $(\partial f/\partial t)_{coll} = 0$, and $\partial f_M/\partial t = 0$, $\underset{\sim}{a} = \underset{\sim}{F}/m = 0$; so

$$\underset{\sim}{v} \cdot \nabla f = 0$$

But if $n \neq n(\underset{\sim}{r})$, then $f_M \neq f(\underset{\sim}{r})$, $\nabla f = 0$, so the equation is satisfied.

b) If $\underset{\sim}{a} = \underset{\sim}{F}/m = -\nabla\phi/m \neq 0$, the BE is

$$\underbrace{\left(\frac{\partial f}{\partial t}\right)}_{=0} + \underset{\sim}{v} \cdot \nabla\left[(n)\left(\frac{f}{n}\right)\right] - \frac{\nabla\phi}{m} \cdot \nabla_v f = 0$$

$\frac{f_M}{n}$ is independent of $\underset{\sim}{r}$, and n is ind. of $\underset{\sim}{v}$, so

$$\frac{f}{n}\, \underset{\sim}{v} \cdot \nabla n = \frac{n}{m} \nabla\phi \cdot \nabla_v\left(\frac{f}{n}\right)$$

$$\nabla_v\left(\frac{f}{n}\right) = \hat{v}\, \frac{\partial(f/n)}{\partial v} = \hat{v}\left(-\frac{mv}{kT}\, \frac{f}{n}\right) = -\underset{\sim}{v}\left(\frac{m}{kT}\, \frac{f}{n}\right)$$

$$\left(\cancel{\frac{f}{n}}\right) \underset{\sim}{v} \cdot \nabla n = \frac{n}{\cancel{m}}\left(\frac{\cancel{m}}{kT}\, \cancel{\frac{f}{n}}\right)\left(-\nabla\phi \cdot \underset{\sim}{v}\right)$$

$$\underset{\sim}{v} \cdot \left[kT\, \frac{\nabla n}{n} + \nabla\phi\right] = 0$$

Since this must be true at very point and for all $\underset{\sim}{v}$,

$$\frac{\nabla n}{n} = \nabla(\ln n) = -\nabla\left(\frac{\phi}{kT}\right)$$

$$\ln n = -\frac{\phi}{kT} + \text{const}$$

$$n(\underset{\sim}{r}) = n_0\, e^{-\frac{\phi(\underset{\sim}{r})}{kT}}$$

where n_0 is the concentration at the position where $\phi = 0$.