PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #46

October 1, 1983

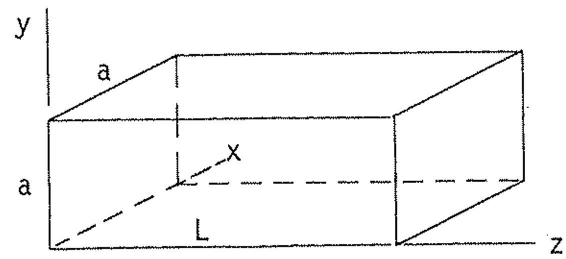
General Instructions

This Comprehensive Examination for Fall 1983 (#46) consists of six problems of equal weight (20 points each). Please check that you have all of them.

Work carefully, indicate your reasoning briefly, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you may find useful. Please return it at the end of the exam.

1. A metallic box of very large conductivity has width and height a in the x and y directions, and length L in the z direction. The box is filled with material of permittivity ε, permeability μ. A radiation field is excited in the box such that $E_z = E_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} e^{i\omega t}$, $H_z = 0$, where E_0 is a constant amplitude and ω is the angular frequency.



(6) (a) Find the value of ω in terms of the parameters given. (Hint: Satisfy wave equation.)

(6) (b) Find the other field components: E_x, E_y, H_x, H_y.

(8) (c) Find the direction and magnitude of the (time) average force on the faces at x = 0, y = 0, z = 0. (Hint: It turns out that the electromagnetic pressure is equal to the energy density.)

(20) 2. Calculated values of the energy levels of the hydrogen atom will depend on whether the proton is considered to be a point charge or an extended charge distribution. Consider a model in which the proton is represented as a uniformly charged sphere of radius $R = 1.0 \times 10^{-15}$ m. Use first-order perturbation theory to calculate the shift (in eV) of the ground state due to the finite size of the proton. Note, for the 1S state with a point proton,

$$\psi_{100} = (\pi a_0^3)^{-1/2} e^{-r/a_0}$$

where $a_0 = 0.53 \times 10^{-10}$ m.

3. This problem concerns a fictitious two-dimensional crystal which is a square lattice of n atoms and having side L (rigid boundaries). The lattice can carry displacement waves which are longitudinal, speed c_ℓ, and in-plane transverse waves, speed c_t, but the atoms cannot move perpendicular to the plane. Zero point energy is to be neglected.

(10) (a) Use the Debye model in derivation of the spectral density, that is, the number of modes, $f(\nu)\, d\nu$, with frequencies in the range $\nu, d\nu$.

(10) (b) Derive an expression for the so-called "Debye temperature", T_0, corresponding to the cut-off frequency of the spectral density function.

4. Particles in a uniform beam of cross-sectional area A and intensity I_0 experience scattering by a fixed repulsive center of force. The relationship between scattering angle Θ and impact parameter b is found to be

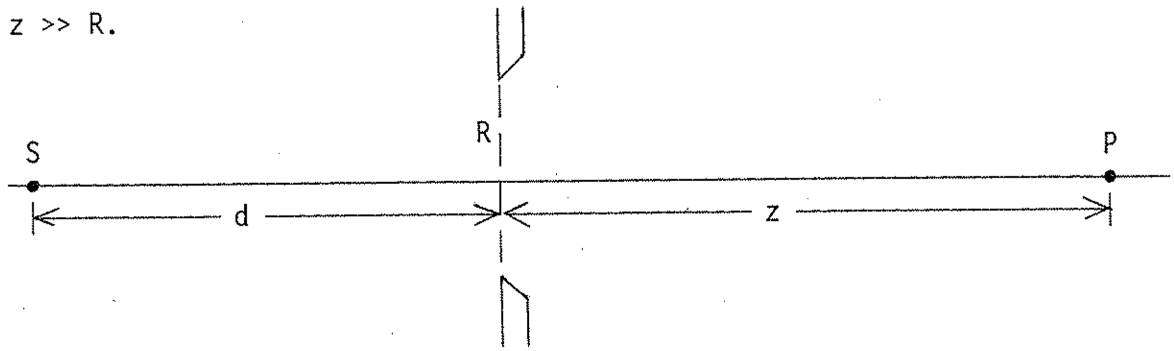
$$\Theta/\pi = 1 - \sqrt{b^2/(\frac{k}{E} + b^2)}$$

where k is a constant and E is the particle energy.

(10) (a) Derive an expression for the differential scattering cross-section $\sigma(\Theta)$.

(10) (b) Evaluate the fraction of beam particles which are back-scattered (i.e., with $\pi/2 \leq \Theta \leq \pi$).

5. A point source S of monochromatic light of wavelength λ is on the axis of a circular aperture of radius $R \gg \lambda$ in an otherwise opaque screen. S is at distance $d \gg R$ from the center of the aperture, but the waves falling on the aperture cannot be considered to be plane waves. The irradiance is observed at points P also on the axis, beyond the screen at distances $z \gg R$.



(15) (a) Find the values of z for which maxima and minima of irradiance will be observed.

(5) (b) What will be the approximate values of irradiance at these points relative to the irradiance when the screen is absent?

6. A particle of mass m moves in the field of a three-dimensional square-well potential

$$V(r) = \begin{cases} -V_0, & r < a, \\ 0, & r > a. \end{cases}$$

(10) (a) Find the transcendental equation to be solved for the bound state energy levels of zero orbital angular momentum. (Hint: The Schroedinger equation will be simplified by the substitution $\psi = \phi/r$.)

(5) (b) Find the minimum magnitude of the well depth, V_0, to bind just one level with $E \to 0$.

(5) (c) Apply the result of part (b) to compute V_0 (in MeV) for a model of the deuteron in which the (spinless) nucleons interact through $V(r)$, where r is now the relative coordinate in the equivalent one-body problem, and $a = 4.3 \times 10^{-15}$ m.

1. The components of the wave satisfy the wave eq. viz.

a) $\nabla^2 E_z - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = 0$

$$-\left(\frac{\pi^2}{a^2}+\frac{\pi^2}{a^2}+0\right)E_z - \frac{1}{c^2}\omega^2 E_z = 0 \implies \boxed{\omega = \sqrt{2}\,\frac{\pi c}{a}}$$

b) TM wave - continuity conditions on $B_\perp$ and $E_\parallel$ require $\boxed{H_z = 0}$ and $\boxed{E_x = E_y = 0}$

for other components look at Maxwell eqs.

$$\vec\nabla \times \vec E = -\mu\frac{\partial H}{\partial t} \qquad \vec\nabla \times \vec H = \varepsilon\frac{\partial E}{\partial t}$$

$$\frac{\partial E_z}{\partial y} - \cancel{\frac{\partial E_y}{\partial z}}^{\,0} = -\mu\frac{\partial H_x}{\partial t} \qquad \cancel{\frac{\partial H_z}{\partial y}}^{\,0} - \frac{\partial H_y}{\partial z} = \varepsilon\frac{\partial E_x}{\partial t} = 0$$

$$\frac{\pi}{a}E_0 \sin\frac{\pi x}{a}\cos\frac{\pi y}{a}\, e^{i\omega t} = -\mu\frac{\partial H_x}{\partial t}$$

$$\boxed{\begin{aligned} H_x &= \frac{\pi i}{a\omega\mu}E_0 \sin\frac{\pi x}{a}\cos\frac{\pi y}{a}\, e^{i\omega t} \\ \text{and } H_y &= \frac{\pi i}{a\omega\mu}E_0 \cos\frac{\pi x}{a}\sin\frac{\pi y}{a}\, e^{i\omega t}\end{aligned}}$$

c) For top & bottom $(y=0,a)$ or sides $(x=0,a)$ $E_x = E_y = E_z = 0$

viz. For side $(y=0,a)$, $H_y = 0$, $H_z = 0$; $F = \frac{\text{Pressure}}{\text{Area}} = \frac{\int(\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2)}{\text{Area}}$

$$F_y = \mu\int_0^a dx\int_0^L dz\, H_x = \mu\int_0^a dx\int_0^L dz\,\frac{\pi^2}{a\omega^2\mu^2}E_0^2\sin^2\frac{\pi x}{a}\,e^{2i\omega t}$$

$$\underline{F_y = \frac{\pi^2 L E_0^2}{8a^2\omega^2\mu}} \text{ (outward)} \qquad \text{same on side } (x=0,a)$$

on Ends $z=0,L$: $F_E = \frac{\varepsilon}{2}\int_0^a dx\int_0^a dy\, E_0^2\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a}\,e^{2i\omega t} = \frac{\varepsilon E_0^2 a^2}{16}$ (IN)

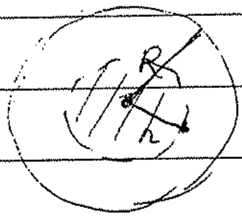
$E_x = E_y = 0$

$H_z = 0$

$$F_H = \frac{\mu}{2}\int_0^a dx\int_0^a dy\left(\frac{H_x^2+H_y^2}{2}\right) = \frac{\pi^2 E_0^2}{8\omega^2\mu} \qquad \text{(OUT)}$$

$$F_z = -F_E + F_H = \left(\frac{\pi^2-\pi^2}{8\omega^2\mu}\right)E_0^2 = 0$$

2. Need potential inside a uniformly charged (e) sphere of radius R $\qquad r \geq R \quad E = \frac{e}{4\pi\varepsilon_0}\frac{1}{r^2}$



$r \leq R$

$$E \cdot 4\pi r^2 = \frac{1}{\varepsilon_0}\frac{e}{\frac{4}{3}\pi R^3}\cdot\frac{4}{3}\pi r^3$$

$$E = \frac{e}{4\pi\varepsilon_0}\frac{r}{R^3}$$

$$V(r) = -\int_\infty^r E\,dr = +\frac{e}{4\pi\varepsilon_0}\int_\infty^R \frac{dr}{r^2} - \frac{e}{4\pi\varepsilon_0}\int_R^r \frac{r\,dr}{R^3}$$

$$= +\frac{e}{4\pi\varepsilon_0}\frac{1}{R} - \frac{e}{4\pi\varepsilon_0}\frac{r^2}{2R^3}\Big|_R^r$$

$$= \frac{e}{4\pi\varepsilon_0}\left\{\frac{1}{R} - \frac{r^2}{2R^3} + \frac{1}{2R}\right\}$$

$$= \frac{e}{4\pi\varepsilon_0 R}\left\{\frac{3}{2} - \frac{r^2}{2R^3}\right\} = \frac{e}{4\pi\varepsilon_0}\cdot\frac{3}{2R}\left\{1 - \frac{r^2}{3R^2}\right\}$$

Use perturbation potential of form: $\qquad U(r) = -eV(r)$

$$\delta U = \begin{cases} 0 & r > R \\ \frac{e^2}{4\pi\varepsilon_0}\left\{\frac{1}{r} - \frac{3}{2R}\left(1 - \frac{r^2}{3R^2}\right)\right\} & r \leq R \end{cases}$$

$$\Delta E = \langle \psi | \delta U | \psi \rangle$$

$$\Delta E = \frac{4\cancel{\pi}}{\cancel{\pi}(a_0)^3} \cdot \frac{e^2}{4\pi\varepsilon_0}\left\{\int_0^R \frac{1}{r} e^{-2r/a_0} r^2\,dr - \frac{3}{2R}\int_0^R e^{-2r/a_0} r^2\,dr + \frac{1}{2R^3}\int_0^R r^2 e^{-2r/a_0}\,dr\right\}$$

Since $R \ll a_0$, over the range of integration $r \ll a_0$ and $e^{-2r/a_0} \approx 1$ — so $\int$'s simplify:

$$\Delta E = \frac{4e^2}{4\pi\varepsilon_0 a_0^3}\left\{\frac{R^2}{2} - \frac{\cancel{3}}{2\cancel{R}}\frac{\cancel{R^3}^{\,2}}{\cancel{3}} + \frac{1}{2\cancel{R^3}}\frac{\cancel{R^5}^{\,2}}{5}\right\}$$

$$\Delta E = \underbrace{+\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{a_0}}_{13.6\,\text{eV}}\ \underbrace{\left(\frac{4}{5}\frac{R^2}{a_0^2}\right)}_{\frac{4}{5}\left(\frac{1.0\times10^{-15}}{0.53\times10^{-10}}\right)^2}$$

$$\Delta E = 3.8\times10^{-9}\ \text{eV}$$

(shift is +, i.e., less binding)

3

a) For either longitudinal or transverse waves the solutions of the wave equations must be of the form $e^{-i\omega t} \sin\left(j_x \frac{\pi x}{L}\right) \sin\left(j_y \frac{\pi y}{L}\right)$,

where j_x, j_y are positive integers. A mode with (j_x, j_y) has propagation constant $\kappa = \pi j / L$, where $j = \sqrt{j_x^2 + j_y^2}$. The number of modes in a quarter circle of radius j is $\pi j^2/4$, and between j and $j + dj$ there are $\frac{1}{2}\pi j\, dj$ modes.

The wavelength of a mode is $\lambda = \frac{2\pi}{\kappa} = \frac{2L}{j}$, frequency $\nu = c/\lambda = cj/2L$. Thus

$$j = \frac{2L\nu}{c}, \quad dj = \frac{2L}{c} d\nu,$$

$$f(\nu)\, d\nu = \tfrac{1}{2}\pi j\, dj = \tfrac{1}{2}\pi \left(\frac{2L}{c}\right)^2 \nu\, d\nu$$

for either type of wave. We can have two types, so

$$\underline{f(\nu)\, d\nu = 2\pi L^2 \left(\frac{1}{c_t^2} + \frac{1}{c_\ell^2}\right) \nu\, d\nu}$$

b) With n atoms each having two degrees of freedom, the total number of modes possible is $2n$.

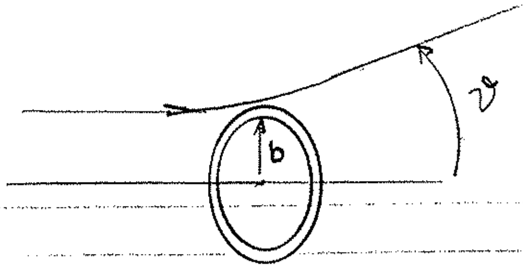
$$2n = \int_0^{\nu_{max}} f(\nu)\, d\nu = 2\pi L^2 \left(\frac{1}{c_t^2} + \frac{1}{c_\ell^2}\right) \frac{\nu_{max}^2}{2}$$

This finds

$$\nu_{max} = \sqrt{\frac{2n}{\pi L^2 \left(\frac{1}{c_t^2} + \frac{1}{c_\ell^2}\right)}} = \frac{k T_D}{h}$$

or

$$\underline{T_D = \frac{h c_t c_\ell}{k L} \sqrt{\frac{2n}{\pi (c_t^2 + c_\ell^2)}}}$$



4.

a) $b^2 = \left(\frac{k}{E} + b^2\right)\left(1 - \frac{\theta}{\pi}\right)^2$

let $\alpha = \left(1 - \frac{\theta}{\pi}\right)$ then $b^2(1-\alpha^2) = \frac{k}{E}\alpha^2$

$$b\frac{db}{d\theta} = -\frac{1}{2\pi}\frac{k}{E}\frac{(1-\alpha^2)2\alpha - \alpha^2(-2\alpha)}{(1-\alpha^2)^2} = -\frac{1}{2\pi}\frac{k}{E}\frac{2\alpha}{(1-\alpha^2)^2}$$

$$\sigma(\theta) = \frac{2\pi b\, db}{d\Omega} = \frac{2\pi b\, db}{2\pi \sin\theta\, d\theta} = \frac{b\,(db/d\theta)}{\sin\theta}$$

$$\boxed{\sigma(\theta) = \frac{1}{\pi}\frac{k}{E}\frac{(1-\theta/\pi)}{\left[1-\left(1-\frac{\theta}{\pi}\right)^2\right]^2 \sin\theta}}$$

b)

$$\left.\frac{dN}{dt}\right|_{scatt} = I\,\sigma(\vartheta)\,d\Omega = 2\pi I\,\sigma(\vartheta)\sin\theta\,d\theta$$

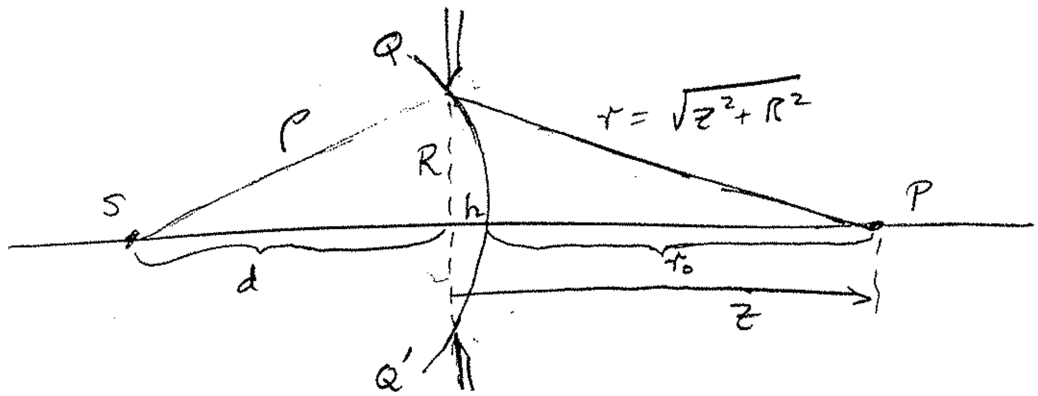
$$\frac{\Delta N}{N} = \frac{\Delta N}{IA} = \frac{2\pi}{A}\int_{\pi/2}^{\pi}\frac{(k/\pi E)(1-\theta/\pi)}{\left[1-\left(1-\frac{\theta}{\pi}\right)^2\right]^2}\,d\theta$$

$\alpha = 1 - \frac{\theta}{\pi}$, $d\theta = -\pi\,d\alpha$

$$\frac{\Delta N}{N} = \frac{\pi}{A}\int_{1/2}^{0}\frac{k}{\pi E}\frac{2\alpha(-\pi\,d\alpha)}{(1-\alpha^2)^2} = -\frac{\pi k}{EA}\int_{1/4}^{0}(1-y)^2\,dy$$

$$= \frac{\pi k}{EA}(1-y)\Big|_{0}^{1/4} = \frac{\pi k}{EA}\left\{\left(\frac{1}{3/4}\right) - \frac{1}{1}\right\} = \boxed{\frac{\pi k}{3EA}}$$

#5 By the Huygens-Fresnel principle, each exposed point on a wavefront such as QQ' is a source of secondary waves whose contributions add to give the wave at P:



a) The wavefront QQ' may be subdivided into annular regions, and if $r - r_0$ is an integer number of half-wavelengths, $r - r_0 = m\lambda/2$, then m Fresnel half-period zones will be exposed in the aperture, and $I(P)$ will be maximum (odd m) or minimum (even m).

$$\rho = \sqrt{d^2+R^2} \doteq d\left(1+\frac{R^2}{2d^2}\right) = d + \frac{R^2}{2d} = d+h, \quad h = \frac{R^2}{2d}$$

$$r_m - r_0 = \sqrt{z_m^2+R^2} - (z_m - h)$$

$$\doteq z_m\left(1+\frac{R^2}{2z_m^2}\right) - z + h = \frac{R^2}{2z_m} + \frac{R^2}{2d} = m\lambda/2$$

$$\frac{1}{z_m} + \frac{1}{d} = \frac{m\lambda}{R^2}, \quad \underline{z_m = \frac{R^2 d}{m\lambda d - R^2}}, \quad m = \begin{cases} 1,3,5\cdots \text{ for } I_{max} \\ 2,4,6\cdots \text{ for } I_{min} \end{cases}$$

b) For odd m, the contributions of the zones will approximately cancel in pairs, giving $I_{min}(P) \doteq 0$. For even m, pairwise cancellation will occur also, but the left-over odd zone will contribute amplitude approximately twice that of the unobstructed wave, so $I_{max}(P) \doteq 4I_0$.

6. $\nabla^2\psi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = \frac{1}{r^2}\left\{r^2\frac{d^2\psi}{dr^2} + 2r\frac{d\psi}{dr}\right\}$

$$= \frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr}$$

$r \geq a$ $\quad -\frac{\hbar^2}{2m}\left\{\frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr}\right\} = E\psi \qquad (1)$

$r < a$ $\quad -\frac{\hbar^2}{2m}\left\{ \quad " \quad \right\} = (E+V_0)\psi \qquad (2)$

let $\phi = r\psi \qquad \psi = \frac{\phi}{r}$

$$\begin{cases} \frac{2}{r}\cdot\frac{d\psi}{dr} = \left[\cancel{-\frac{1}{r^2}\phi} + \cancel{\frac{1}{r}\frac{d\phi}{dr}}\right]\cdot\frac{2}{r} \\ + \frac{d^2\psi}{dr^2} = \cancel{\frac{2}{r^2}\phi} - \cancel{\frac{1}{r^2}\frac{d\phi}{dr}} - \cancel{\frac{1}{r^2}\frac{d\phi}{dr}} + \frac{1}{r}\frac{d^2\phi}{dr^2} \end{cases}$$

$$= \frac{1}{r}\frac{d^2\phi}{dr^2}$$

so S' Eq's $\quad \frac{1}{\cancel{r}}\frac{d^2\phi}{dr^2} = E\frac{\phi}{\cancel{r}} \qquad (1)$

$$= (E+V_0)\phi \qquad (2)$$

(1) $\quad \frac{d^2\phi}{dr^2} - \alpha^2\phi = 0 \qquad \alpha = \sqrt{\frac{-2mE}{\hbar^2}}$ (exponential sol.)

(2) $\quad \frac{d^2\phi}{dr^2} + \beta^2\phi = 0 \qquad \beta = \sqrt{\frac{-2m(E+V_0)}{\hbar^2}}$ (oscillatory)

solutions: $\phi_1 = A e^{-\alpha r}$

$\phi_2 = B \sin \beta r$

match $\left.\begin{array}{l}\phi_1 = \phi_2 \\ \frac{d\phi_1}{dr} = \frac{d\phi_2}{dr}\end{array}\right\}$ at boundary

$-\alpha A e^{-\alpha a} = \beta B \cos \beta a$

$A e^{-\alpha a} = B \sin \beta a$

a) $-\alpha = \beta \cot \beta a$

b) Now let $E \to 0$ $\quad \alpha \to 0 \quad \beta \to \sqrt{\frac{2mV_0}{\hbar^2}}$

and $\beta a \to \pi/2$

$$\beta^2 a^2 = \frac{2mV_0 a^2}{\hbar^2} = \pi^2/4$$

$$V_0 = \frac{\pi^2 \hbar^2}{8ma^2}$$

c) $V_0 = \frac{\pi^2 \hbar^2 c^2}{\underbrace{8(m_0c^2)}_{940\,\mathrm{MeV}/2} a^2} = 5.5\ \mathrm{MeV}$