PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #44

January 8, 1983

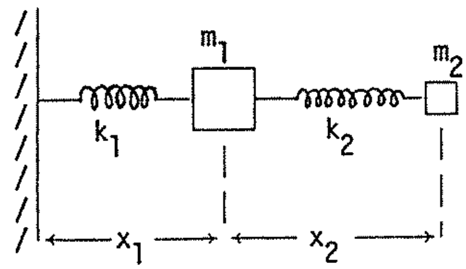
General Instructions

This Comprehensive Examination for Winter 1983 (#44) consists of six problems of equal weight (20 points each). Please check that you have all of them.

Work carefully, indicate your reasoning briefly, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you may find useful. Please return it at the end of the exam.

(20) 1. A double harmonic oscillator is shown in the diagram. Determine the normal frequencies (such that both masses vibrate with a single frequency). At equilibrium the lengths of the springs are ℓ_1 and ℓ_2, respectively.



(5) 2. (a) An otherwise free particle of mass μ is constrained to move on a circle of radius a. Determine the wavefunctions and energies.

(15) (b) A weak potential V is introduced, which is produced by two like sources at $\phi = 0$ and $\phi = \pi$, $V = f(\phi) + f(\phi-\pi)$, such that $\int_0^{2\pi} f(\phi)d\phi = 0$. Determine the energy shift of the states of part (a) and the corresponding lowest-order wave functions.

3. It is to be investigated whether the spherically symmetrical field,

$$\vec{E} = \begin{cases} \frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3} - \frac{r^2}{4R}\right)\hat{r} , & r < R , \quad \rho_0 = \text{constant charge density} \\ \frac{\rho_0 R^3}{12\varepsilon_0 r^2}\hat{r} , & r > R , \end{cases}$$

might, in fact, be an electrostatic field.

(4) (a) Could $\vec{E}$ be an electrostatic field? Explain.

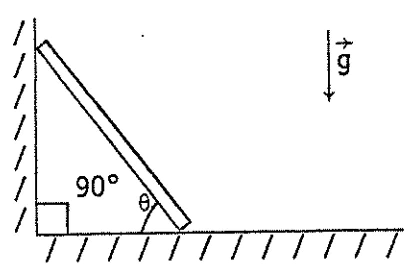
(6) (b) Determine the charge density $\rho(r)$ and the total charge Q which produce this field.

(10) (c) Determine the electrostatic potential energy associated with the charge distribution

$$\rho(r) = \begin{cases} \rho_0\left(1 - \frac{r}{R}\right) & r < R \\ 0 & r > R \end{cases}$$

by calculating the work necessary to build spherical shells of charge, bringing them in from ∞ and then smearing them over the surface, until the final radius is R.

(20) 4. A uniform board, having mass m and length L, leans against a vertical wall. Under the influence of gravity it slips downward, beginning from rest with an initial angle $\theta = \theta_0$. If there is no friction, what will be the value of θ when the top of the board loses contact with the wall?



(20) 5. A steel tank, well insulated on the outside, has interior volume V_1. The tank contains air at pressure P_1 and temperature T_1, and is attached to a compressed-air line which supplies air at steady conditions P' and T'. (Primed quantities refer to the air in the source line.) Initially the tank is shut off from the air line by a valve; the valve is then opened just long enough so that air flows into the tank until the final pressure is P_2. Find, for the data given below, how many moles, n_2, of air end up in the tank, and what its final temperature, T_2, will be.

Assume that:

(1) The tank wall exchanges heat with the air so rapidly that the wall and the air in it are always at the same temperature.

(2) Air may be considered a perfect gas whose specific heats $C_p = 7$ and $C_V = 5$ cal/mole K° remain constant during the process.

(3) In transport of this kind, the energy balance that applies (neglecting small changes in KE, PE_{grav}, and displaced pistons) is

$$n_2 \cdot (U_2 - H') - n_1 \cdot (U_1 - H') = Q,$$

where U = molar internal energy;

H' = molar enthalpy = $U' + P'V'$, where V' = molar volume occupied by air in line;

Q = heat added to the system fluid.

(4) Initially the air has tank volume $V_1 = 20\ m^3$, at $P_1 = 1$ bar and $T_1 = 22°C$.
The compressed-air line operates at $P' = 5$ bar and $T' = 30°C$.
The final air pressure in the tank is $P_2 = 3$ bar.
Mass of steel tank, $m_3 = 10^3$ kg.
Heat capacity of steel, $c_s = 100$ cal/kg K°.
Gas constant, $R = 8.3 \times 10^{-5}\ m^3$ bar/mole K°.

(20) 6. In the laboratory reference frame a relativistic particle having rest mass m_1 makes an inelastic collision with a stationary particle having rest mass m_2. Calculate the initial kinetic energy T_1 of the first particle if, following the collision, the particles have joined to form a composite system having excitation energy ΔE. The excitation energy is defined as follows: $\Delta E = E_{CM} - (m_1 + m_2)\, c^2$, where E_{CM} is the total energy in the center-of-mass (center-of-momentum) reference frame.

1. $L = T - V = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1 + \dot{x}_2)^2$
$- \frac{1}{2} k_1 (x_1 - l_1)^2 - \frac{1}{2} k_2 (x_2 - l_2)^2$

Let $\xi_1 = x_1 - l_1$, $\xi_2 = x_2 - l_2$

$\dot{\xi}_1 = \dot{x}_1$, $\dot{\xi}_2 = \dot{x}_2$

$$L = \frac{1}{2} (m_1 + m_2) \dot{\xi}_1^2 + m_2 \dot{\xi}_1 \dot{\xi}_2 + \frac{1}{2} m_2 \dot{\xi}_2^2 - \frac{1}{2} k_1 \xi_1^2 - \frac{1}{2} k_2 \xi_2^2$$

Euler Lagr eqs:

$$M \ddot{\xi}_1 + m_2 \ddot{\xi}_2 + k_1 \xi_1 = 0 , \quad M = m_1 + m_2$$
$$m_2 \ddot{\xi}_1 + m_2 \ddot{\xi}_2 + k_2 \xi_2 = 0$$

trial soln $\xi_1 = c_1 e^{i\omega t}$, $\xi_2 = c_2 e^{i\omega t}$

$$\left.\begin{array}{l} -M\omega^2 c_1 - m_2 \omega^2 c_2 + k_1 c_1 = 0 \\ -m_2 \omega^2 c_1 - m_2 \omega^2 c_2 + k_2 c_2 = 0 \end{array}\right\} \text{Let } \lambda = \omega^2$$

$$\begin{pmatrix} -\lambda M + k_1 & -m_2 \lambda \\ -m_2 \lambda & -m_2 \lambda + k_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$\underbrace{(m_2 M - m_2^2)}_{m_1 m_2} \lambda^2 - (M k_2 + m_2 k_1) \lambda + k_1 k_2 = 0$$

$$\lambda = \omega^2 = \frac{M k_2 + m_2 k_1 \pm \sqrt{(M k_2 + m_2 k_1)^2 - 4 m_1 m_2 k_1 k_2}}{2 m_1 m_2}$$

2. Soln

(a) Singlevaluedness $\Rightarrow$ $\varphi = \dfrac{e^{im\varphi}}{\sqrt{2\pi}}$

$$H_0 = -\frac{\hbar^2}{2\mu a^2}\frac{\partial^2}{\partial\varphi^2}, \quad \mu = \text{mass}$$

$$E_0 = \frac{\hbar^2 m^2}{2\mu a^2}$$

(b) there is degeneracy since states with $m_1 = -m_2$ have the same energy

the matrix elements between these states

$V \to f$ is $\frac{1}{2\pi}\int e^{-im_2\varphi}\left[f(\varphi) + f(\varphi - \pi)\right] e^{im_1\varphi}\, d\varphi$

$$= \frac{1}{2\pi}\int e^{i(m_1-m_2)\varphi}\left[f(\varphi) + \cancel{f(\varphi-\pi)}\right] d\varphi$$

$$+ \frac{1}{2\pi}\int e^{i(m_1-m_2)\varphi}\, \overset{f}{\cancel{V}}(\underbrace{\varphi - \pi}_{\varphi'})\, d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i(m_1-m_2)\varphi}\, \overset{f(\varphi)}{\cancel{V}}(\varphi)\, d\varphi \left[1 + e^{i(m_1-m_2)\pi}\right]$$

$$\underset{m_1=m_2}{=} \frac{\cancel{2}}{2\pi}\int_0^{2\pi} \overset{f}{\cancel{V}}(\varphi)\, d\varphi = 0$$

$$\underset{m_2=-m_1=-m}{=} \frac{2}{2\pi}\int_0^{2\pi} e^{2im\varphi}\, \overset{f}{\cancel{V}}(\varphi)\, d\varphi = 2\epsilon(m)$$

$$\psi_m = a e^{im\varphi} + b e^{-im\varphi}$$

$$\begin{pmatrix} -\lambda & 2\epsilon(m) \\ 2\epsilon(m) & -\lambda \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\lambda^2 = 4\epsilon^2 (m) \qquad \lambda = \pm 2\epsilon$$

$$\lambda = -2\epsilon, \begin{pmatrix} -2\epsilon & 2\epsilon \\ 2\epsilon & -2\epsilon \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow a = b$$

$$\psi = \frac{\cos m\varphi}{\sqrt{\pi/2}}$$

$$\lambda = +2\epsilon, \quad \begin{pmatrix} 2\epsilon & 2\epsilon \\ 2\epsilon & 2\epsilon \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad \Rightarrow a = -b$$

$$\psi = \frac{\sin m\varphi}{\sqrt{\pi/2}}$$

#3

a) The differential form of Gauss's Law is $\nabla\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$; in this problem, $E_\theta=E_\varphi=0$, so $\nabla\cdot\vec{E}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r)$

$$= \begin{cases} \frac{1}{r^2}\frac{\partial}{\partial r}\underbrace{\left(\frac{\rho_0 R^3}{12\varepsilon_0}\right)}_{=\text{const}} = 0 & \text{for } r>R \\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{\rho_0}{\varepsilon_0}\left(\frac{r^3}{3}-\frac{r^4}{4R}\right)\right)=\frac{1}{r^2}\left[\frac{\rho_0}{\varepsilon_0}\left(r^2-\frac{r^3}{R}\right)\right] = \frac{\rho_0}{\varepsilon_0}\left(1-\frac{r}{R}\right) & \text{for } r<R \end{cases}$$

making $\rho(r)=\varepsilon_0\nabla\cdot\vec{E}=\begin{cases} 0, & r>R \\ \rho_0\left(1-\frac{r}{R}\right), & r<R \end{cases}$

We could integrate $\rho(r)$ to find $Q=\int_0^R \rho(r)\,d\tau$ but an even simpler way is to use Gauss's Law for a spherical Gaussian surface at $r>R$:

$$Q_{enc}=\varepsilon_0\oint_S \vec{E}_{r>R}\cdot d\vec{a}=\frac{\rho_0 R^3}{12}\,2\pi\,\cancel{\frac{r^2}{r^2}}\underbrace{\int_0^\pi \sin\theta\,d\theta}_{=2}=\frac{1}{4}\,\rho_0\left(\frac{4}{3}\pi R^3\right)$$

b) Since $E_\theta=E_\varphi=\frac{\partial E_r}{\partial\theta}=\frac{\partial E_r}{\partial\varphi}=0$ for $r<R$ and $r>R$, then $\nabla\times\vec{E}=0$ everywhere

Since we have $\nabla\cdot\vec{E}\Big|_{\substack{r<R\\r>R}}$ and $\nabla\times\vec{E}=0$, the field described is a possible electrostatic $\vec{E}$.

Not $=0$ for $r<R$ and doesn't have to be for an electric field.

c) When the sphere is partly assembled, and has, say, radius r', the next spherical shell will contain $dq=(\rho\cdot 4\pi r'^2 dr')=\rho_0\left(1-\frac{r'}{R}\right)4\pi r'^2 dr'$.
This charge must be brought to a surface where the potential is already

$$V=\frac{q'}{4\pi\varepsilon_0 r'}=\frac{1}{4\pi\varepsilon_0 r'}\int\rho\,d\tau=\frac{\rho_0}{\cancel{4\pi}\varepsilon_0 r'}\int_0^{r'}\left(1-\frac{r'}{R}\right)\cancel{4\pi} r'^2 dr'=\frac{\rho_0}{\varepsilon_0}\left(\frac{r'^2}{3}-\frac{r'^3}{4R}\right)$$

So the work to bring dq to this surface (from ∞ to r') is just

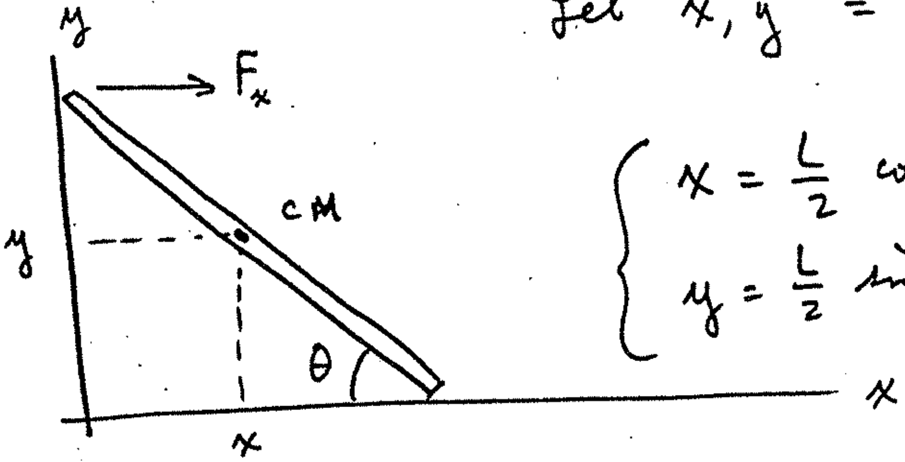
$$dW=V\,dq=\frac{\rho_0^2}{\varepsilon_0}\left(\frac{r'^2}{3}-\frac{r'^3}{4R}\right)\left(1-\frac{r'}{R}\right)4\pi r'^2 dr'=\frac{4\pi\rho_0^2}{\varepsilon_0}\left(\frac{r'^4}{3}-\frac{r'^5}{4R}-\frac{r'^5}{3R}+\frac{r'^6}{4R^2}\right)dr'$$

Since all points of the surface at r' are at the same potential, it costs no work to distribute dq over the sphere uniformly. This having been done,

$$PE=W=\int_0^R dW=\frac{4\pi\rho_0^2}{\varepsilon_0}\left(\frac{R^5}{15}-\frac{R^5}{24}-\frac{R^5}{18}+\frac{R^5}{28}\right)=\frac{4\pi\rho_0^2 R^5}{\varepsilon_0}\left[\frac{1}{3}\left(\frac{1}{5}-\frac{1}{6}\right)-\frac{1}{4}\left(\frac{1}{6}-\frac{1}{7}\right)\right]$$

$$\text{or } PE=\left(\frac{13\pi}{630}\right)\frac{\rho_0^2 R^5}{\varepsilon_0}$$

#4



Let x, y = coordinates of center of mass.

$$\begin{cases} x = \frac{L}{2}\cos\theta \; ; \; \dot{x} = -\frac{L}{2}\sin\theta\,\dot{\theta} \\ y = \frac{L}{2}\sin\theta \; ; \; \dot{y} = \frac{L}{2}\cos\theta\,\dot{\theta} \end{cases}$$

$$I_{CM} = \frac{1}{12} mL^2$$

Conservation of energy: $T + U = T_0 + U_0$

$$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_{CM}\dot{\theta}^2 + mg\,y = 0 + mg\,y_0$$

$$\frac{M}{2}\cdot\frac{L^2}{4}\underbrace{(\sin^2\theta + \cos^2\theta)}_{1}\dot{\theta}^2 + \frac{1}{2}\cdot\frac{1}{12}ML^2\dot{\theta}^2 + mg\frac{L}{2}\sin\theta = mg\frac{L}{2}\sin\theta_0$$

$$\frac{1}{6}L\dot{\theta}^2 + \frac{1}{2}g\sin\theta = \frac{1}{2}g\sin\theta_0$$

$$\underline{\dot{\theta}^2 = \frac{3g}{L}(\sin\theta_0 - \sin\theta)}$$ general relation between $\dot{\theta}$ and θ

The board loses contact with the wall when $F_x = 0$ or $\ddot{x} = 0$.

$\longrightarrow \frac{d}{dt}(\sin\theta\,\dot{\theta}) = 0$ or $\underline{\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2 = 0}$

$\longrightarrow 2\dot{\theta}\ddot{\theta} = -\frac{3g}{L}\cos\theta\,\dot{\theta}$ or $\underline{\ddot{\theta} = -\frac{3g}{2L}\cos\theta}$

$$\therefore \sin\theta\left(-\frac{3g}{2L}\cancel{\cos\theta}\right) + \cancel{\cos\theta}\cdot\frac{3g}{L}(\sin\theta_0 - \sin\theta) = 0$$

$-\frac{1}{2}\sin\theta + \sin\theta_0 - \sin\theta = 0$; $\sin\theta = \frac{2}{3}\sin\theta_0$ or $\boxed{\theta = \sin^{-1}\left(\frac{2}{3}\sin\theta_0\right)}$

#5 We have $n_2(U_2 - H') - n_1(U_1 - H') = Q$

Since $\underline{H' \equiv U' + P'V' = U' + RT'}$ for a perfect gas,

$$n_2(U_2 - U' - RT') - n_1(U_1 - U' - RT') = Q$$

But, for a perfect gas, $\underline{\Delta U = C_v \Delta T}$ and $\underline{R = C_p - C_v}$

So $n_2[C_v(T_2 - \cancel{T'}) - (C_p - \cancel{C_v})T'] - n_1[C_v(T_1 - \cancel{T'}) - (C_p - \cancel{C_v})T'] = Q$

$$\underline{n_2(C_v T_2 - C_p T') - n_1(C_v T_1 - C_p T') = Q}$$

Since $V_1 = V_2$, we can eliminate $n_1 = \frac{P_1 V_1}{RT_1}$ and $\boxed{n_2 = \frac{P_2 V_1}{RT_2}}$ ① to get an expression for T_2:

$$\frac{P_2 V_1}{R\cancel{T_2}} C_v \cancel{T_2} \pm \frac{P_2 V_1}{RT_2} C_p T' - \frac{P_1 V_1}{R\cancel{T_1}} C_v \cancel{T_1} + \frac{P_1 V_1}{RT_1} C_p T' = Q$$

Since both are always at the same temperature, the heat added to the air, Q, will be the heat lost by the tank wall:

$$-Q = M_s C_s (T_2 - T_1) \qquad \text{so } Q = M_s C_s T_1 - M_s C_s T_2$$

$$\frac{P_2 V_1}{R} C_v + \frac{P_2 V_1}{R} C_p \frac{T'}{T_2} - \frac{P_1 V_1}{R} C_v + \frac{P_1 V_1}{R} C_p \frac{T'}{T_1} = M_s C_s T_1 - M_s C_s T_2$$

multiplying by T_2 and collecting terms,

$$(M_s C_s) T_2^2 + \left\{ \frac{V_1}{R} \left[C_v (P_2 - P_1) + \frac{P_1}{} \frac{T'}{T_1} \right] - M_s C_s T_1 \right\} T_2 - \left(\frac{P_2 V_1}{R} C_p T' \right) = 0$$

Into which we put the data:

$$\cancel{10^5} T_2^2 + \left\{ \frac{20}{8.3} \times \cancel{10^5} \left[5(3-1) + (1)\frac{303}{295} \right] - \cancel{10^5}(295) \right\} T_2 - \left(\frac{3 \times 20}{8.3} \times \cancel{10^5} \times 7 \times 303 \right) = 0$$

$$T_2^2 + \{[24.1 + 2.5] - 295\} T_2 - 1.53 \times 10^4 = 0$$

$$\underline{T_2^2 - (268.4) T_2 - 1.53 \times 10^4 = 0}$$

So $\underline{T_2} = \frac{268.4 + \sqrt{(268.4)^2 + (4)(1.53 \times 10^4)}}{2} = \frac{268.4 + 364.7}{2} = \underline{316.6°K} = \underline{43.6°C}$

From ①

$$\underline{n_2} = \frac{P_2 V_1}{RT_2} = \frac{3 \times 20}{8.3 \times 316.6} \times 10^5 = \underline{2283 \text{ moles}}$$

6

$E^2 - p^2c^2$ is invariant, and has the same value in the lab frame and in the CM frame.

$$(E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 c^2 = E_{cM}^2 \quad ; \quad \vec{p}_{cM} \equiv 0 .$$

$$p_1^2 c^2 = E_1^2 - m_1^2 c^4 \qquad \Delta E \equiv E_{cM} - (m_1 + m_2) c^2$$

$$p_2 = 0$$

$$\therefore \cancel{E_1^2} + 2E_1E_2 + E_2^2 - (\cancel{E_1^2} - m_1^2c^4) = \left[\Delta E + (m_1+m_2)c^2\right]^2$$

$$E_1 = T_1 + m_1c^2$$

$$E_2 = m_2c^2$$

$$\therefore 2(T_1 + m_1c^2)m_2c^2 + \cancel{m_2^2c^4} + \cancel{m_1^2c^4} = (\Delta E)^2 + 2\Delta E(m_1+m_2)c^2 + (\cancel{m_1^2} + 2m_1m_2 + \cancel{m_2^2})c^4$$

$$2T_1m_2c^2 + \cancel{2m_1m_2c^4} = (\Delta E)^2 + 2\Delta E(m_1+m_2)c^2 + \cancel{2m_1m_2c^4}$$

$$\boxed{T_1 = \frac{(\Delta E)^2 + 2\Delta E(m_1+m_2)c^2}{2m_2c^2}}$$

Alternative solution:

$$m_1 \gamma_1 c^2 + m_2 c^2 = M \gamma_V c^2 \quad \text{energy conservation}$$

$$m_1 \gamma_1 v_1 = M \gamma_V V \quad \text{momentum conservation}$$

where $\gamma_1 = \dfrac{1}{\sqrt{1-\frac{v_1^2}{c^2}}}$ and $\gamma_V = \dfrac{1}{\sqrt{1-\frac{V^2}{c^2}}}$.

Also $T_1 = m_1 c^2(\gamma_1 - 1)$ and $\Delta E = Mc^2 - (m_1+m_2)c^2$.

$$\frac{1}{\gamma_V^2} = 1 - \frac{V^2}{c^2} = 1 - \left(\frac{m_1 \gamma_1 v_1}{Mc}\right)^2 \cdot \frac{1}{\gamma_V^2}$$

$$\frac{1}{\gamma_V^2}\left[1 + (\ )^2\right] = 1 \implies \gamma_V = \sqrt{1 + \left(\frac{m_1 \gamma_1 v_1}{Mc}\right)^2}$$

$$m_1 \gamma_1 c^2 + m_2 c^2 = \sqrt{M^2 + \left(\frac{m_1 \gamma_1 v_1}{c}\right)^2}\ c^2$$

$$m_1^2 \gamma_1^2 c^4 + 2 m_1 m_2 \gamma_1 c^4 + m_2^2 c^4 = \underbrace{M^2 c^4}_{[\Delta E + (m_1+m_2)c^2]^2} + m_1 \gamma_1^2 c^2 v_1^2$$

$$\gamma_1^2 = \frac{1}{1 - v_1^2/c^2}$$

$$T_1 = m_1 c^2(\gamma_1 - 1) = \frac{[\Delta E + (m_1+m_2)c^2]^2 - (m_1^2+m_2^2)c^2}{2 m_2 c^2} - m_1 c$$

$$\boxed{T_1 = \frac{(\Delta E)^2 + 2(m_1+m_2)c^2 \Delta E}{2 m_2 c^2}}$$