PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #43

September 27, 1982

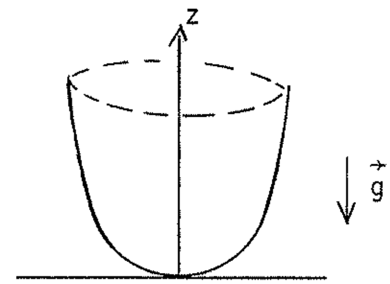
General Instructions

This Comprehensive Examination for Fall 1983 (#43) consists of six problems of equal weight (20 points each). Please check that you have all of them.

Work carefully, indicate your reasoning briefly, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you may find useful. Please return it at the end of the exam.

1. A classical particle of mass m slides without friction on the surface of a paraboloid of revolution given by $z = K\rho^2$ in cylindrical coordinates (ρ,ϕ,z). The axis of symmetry is vertical, and a uniform gravitational field $\vec{g}$ is directed downward. Define the potential energy V so that V = 0 at z = 0.



(8) (a) Find (but do not solve) a first-order differential equation for the time variation of the height z of the particle.

(4) (b) For nonzero values of the total energy E and angular momentum L (z-component), find the maximum and minimum values of z.

(8) (c) If the trajectory is a circle, and if the total energy is E, calculate the value of z and the period τ of the orbit.

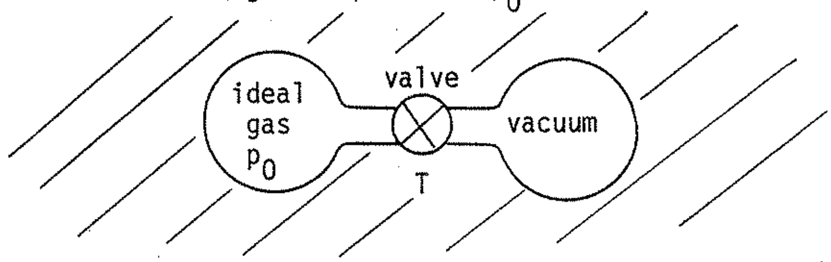
2. A uniform bar of length 2ℓ is drilled and threaded at its midpoint and can move without friction on a fixed thin vertical screw. The bar remains horizontal at all times.

(15) (a) If the screw pitch is such that the bar moves a vertical distance h in one turn, find the linear acceleration a of the bar after it is released from rest.

(5) (b) Find the torque exerted by the screw on the bar in terms of the acceleration a.

[Hint: Elementary methods are recommended.]

(20) 3. Two containers, connected by a small valve, are submerged in a liquid maintained at constant Kelvin temperature T. each container has volume V. Initially the container on the right is evacuated and the container on the left contains an ideal gas at pressure p_0, with mass m per molecule.



At time t = 0 the valve is opened so that its aperture area is A, and gas can flow slowly between the containers. Derive an expression for the pressure p in the left container, as a function of time. Assume that the temperature remains constant throughout the system.

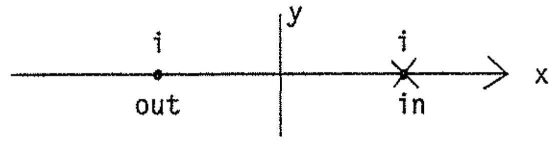
(20) 4. A hydrogen atom is placed in a crystal lattice environment which, in leading order, produces a weak quadrupole interaction

$$\Delta V = \frac{e^2 r^2}{a^3} P_2 \ (\cos\theta)$$

where a is a length constant characteristic of the atom-nearest-neighbor distance. Calculate the energy shifts of states n = 1, 2, 3 for this interaction. (You may leave your results in terms of nonzero integrals, and you may ignore spin.)

$$P_2(\cos\theta) = \frac{3}{2}\left(\cos^2\theta - \frac{1}{2}\right) = \sqrt{\frac{4\pi}{5}}\, y_{20}$$

5. Two parallel, infinitely long wires with equal currents i flowing along and opposite to the z-direction are fixed at $x = \pm a$, $y = 0$.



(7) (a) What is the magnitude and direction of the resultant magnetic field $\vec{B}$ at point x,y?

(8) (b) Determine the field at any point x,y and expand for $x \ll a$, $y \ll a$, keeping only terms through second order in (x/a), (y/a).

(5) (c) Determine whether the B field is locally uniform at $x = y = 0$. (This means that the field gradient is zero at the point in question.)

[Hint: $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$]

6. It is interesting to develop the effects of finite nuclear size on atomic spectra. A nice way to do this is to consider mu-mesic atoms (the μ^- interacts only very weakly with nuclei, so it often forms hydrogen-like atoms). Suppose a μ^- ($m_\mu = 207\ m_e$) is bound by the coulomb force to a nitrogen nucleus ${}^{A=14}_{Z=7}N$.

Remember that for a hydrogen-like atom (with infinitely massive nucleus of charge Z)

$$E_n = -\frac{m_e Z^2 e^4}{2\hbar^2 n^2} = -\frac{Z^2}{n^2} \times 13.6 \text{ eV} \qquad \psi_{1s} = \sqrt{\frac{Z^3}{\pi a_0^3}}\, e^{-\frac{Zr}{a_0}} \qquad \text{where } a_0 = \frac{\hbar^2}{m_e e^2} \approx 5.3 \times 10^{-9} \text{ cm}$$

$$\psi_{2s} = \sqrt{\frac{Z^3}{8\pi a_0^3}}\, e^{-\frac{Zr}{2a_0}} \left(1 - \frac{Zr}{a_0}\right).$$

(2) (a) As a starting point, assume the nucleus to be an infinitely massive point, and calculate numerical values for the two lowest energy levels of the mu-mesic atom. (Carry this, and all subsequent calculations, to the nearest 0.1 keV.)

(4) (b) Calculate the effect, on the first two levels, of the finite nuclear mass, still considered as a point, also to the nearest 0.1 keV.

(14) (c) Now consider the nucleus to be a sphere of radius $r_0 = 3.4 \times 10^{-13}$ cm, with a uniform charge density, so the electrical potential is

$$V(r) = \begin{cases} -V_0 + \frac{b}{2} r^2 & r < r_0 \\ -\frac{Ze^2}{r} & r > 0 \end{cases}$$

Determine V_0 and b, and calculate the correction (to the nearest 0.1 keV) on the lowest energy level due to this charge distribution, by treating the finite size as a perturbation in first order.

1.

a) $E = T + V$

$$= \frac{1}{2} m \left(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2 \right) + mgz$$

$$L = m\rho^2 \dot{\phi} = \text{constant} \longrightarrow \dot{\phi} = \frac{L}{m\rho^2}$$

$$\therefore E = \frac{m}{2} \left(\dot{\rho}^2 + \frac{L^2}{m^2 \rho^2} + \dot{z}^2 \right) + mgz$$

The constraint is $z = K\rho^2 \longrightarrow \frac{1}{\rho^2} = \frac{K}{z}$.

$$\dot{z} = 2K\rho\dot{\rho} \longrightarrow \dot{\rho}^2 = \frac{\dot{z}^2}{4K^2\rho^2} = \frac{\dot{z}^2}{4Kz}$$

$$\boxed{E = \frac{m}{2} \left(1 + \frac{1}{4Kz} \right) \dot{z}^2 + \frac{KL^2}{2mz} + mgz}$$

First-order (nonlinear) differential equation.

b) $\dot{z} = 0$ when $z = $ max or min.

Then $E = \frac{KL^2}{2mz} + mgz$ or $mgz^2 - Ez + \frac{KL^2}{2m} = 0$

$$z_{\substack{max \\ min}} = \frac{1}{2mg} \left[E \pm \sqrt{E^2 - 4(mg)\left(\frac{KL^2}{2m}\right)} \right]$$

$$\boxed{z_{\substack{max \\ min}} = \frac{1}{2mg} \left[E \pm \sqrt{E^2 - 2KL^2 g} \right]}$$

c) For a circular orbit, $z_{max} = z_{min} \Longrightarrow \sqrt{\ } = 0$.

$$\boxed{z = \frac{E}{2mg}}$$

This result can also be found by elementary methods, or by the virial theorem.

$$\sqrt{\ } = 0 \Longrightarrow E^2 = 2KL^2g$$

$$L = \frac{E}{\sqrt{2Kg}}$$

$$L = m\rho^2\dot{\phi}$$

$$\dot{\phi} = \frac{L}{m\rho^2} = \frac{E}{\sqrt{2Kg}\cdot m\left(\frac{z}{K}\right)} = \frac{\cancel{E}K}{\sqrt{2Kg}\ \cancel{m}\left(\frac{\cancel{E}}{2\cancel{m}g}\right)}$$

$$= \sqrt{2Kg}$$

$$\dot{\phi} = \frac{2\pi}{\tau} \qquad \tau = \frac{2\pi}{\dot{\phi}} = \frac{2\pi}{\sqrt{2Kg}}$$

$$\boxed{\tau = \pi\sqrt{\frac{2}{Kg}}}$$

(20) A uniform bar of length $2l$ is fitted at its midpoint with a nut
2. which can move without friction on a vertical screw. The bar remains horizontal at all times, and turns with the nut. If the screw pitch is such that the nut moves a vertical distance h in one turn, find the acceleration of the nut when the system is released to move.

Although there is a gravitational force acting, it does no work on the frictionless system, so we know that from conservation of mechanical energy,

$$\Delta PE = \Delta KE = (\Delta KE)_{trans} + \Delta KE_{rot}$$

If the bar has mass m (density λ Kg/m) and moves downward a distance H

$$mgH = \tfrac{1}{2}m\dot{H}^2 + \tfrac{1}{2}I\omega^2$$

now $I = \int r^2 dm = 2\lambda\int_0^l x^2 dx = \frac{2}{3}\lambda l^3 = \frac{1}{3}ml^2$

and $\frac{2\pi}{h} = \frac{\Delta\phi}{\Delta H}$

so $\dot{\phi} = \frac{2\pi}{h}\dot{H}$

$$\cancel{m}gH = \tfrac{1}{2}\cancel{m}\dot{H}^2 + \tfrac{1}{2}(\tfrac{1}{3}\cancel{m}l^2)\dot{\phi}^2$$

$$gH = \tfrac{1}{2}\dot{H}^2 + \frac{1}{2}\frac{l^2}{3}\left(\frac{2\pi}{h}\right)^2\dot{H}^2.$$

differentiating WRT time

$$g\cancel{\dot{H}} = \cancel{\dot{H}}\ddot{H} + \frac{1}{3}l^2\left(\frac{2\pi}{h}\right)^2\cancel{\dot{H}}\ddot{H} = \ddot{H}\left(1+\frac{1}{3}\left(\frac{l\cdot 2\pi}{h}\right)^2\right)$$

or
$$\ddot{H} = \frac{g}{1+\frac{1}{3}\left(\frac{2\pi l}{h}\right)^2}$$

$$N = I\alpha$$

$$\alpha = \frac{2\pi}{h}a, \quad a = \ddot{H}$$

$$N = \frac{1}{3}ml^2\,\frac{2\pi}{h}\ddot{H}$$

3.

Let N_L and N_R denote the numbers of molecules in the left and right containers, respectively.

In the left container, at time t, $\quad pV = N_L kT$.

$$\frac{dp}{dt} = \frac{kT}{V}\frac{dN_L}{dt}$$

$$\frac{dN_L}{dt} = -\frac{1}{4}\frac{N_L}{V}\bar{v}A + \frac{1}{4}\frac{N_R}{V}\bar{v}A$$

where $\bar{v}$ = average speed $= \sqrt{\dfrac{8kT}{\pi m}}$.

The total number of molecules is constant: $N_L + N_R = N_0 = \dfrac{p_0 V}{kT}$

$N_R = N_0 - N_L$ $\qquad$ $(\ ) = N_0 - 2N_L$

$$\therefore \frac{dp}{dt} = \frac{kT}{V}\cdot\frac{1}{4}\frac{\bar{v}A}{V}\left(N_R - N_L\right)$$

$$= \frac{kT}{V}\cdot\frac{1}{4}\frac{\bar{v}A}{V}\left(\frac{p_0 V}{kT} - 2\frac{pV}{kT}\right)$$

$$= \frac{1}{4}\frac{\bar{v}A}{V}\left(p_0 - 2p\right)$$

$$= -\frac{\bar{v}A}{2V}\left(p - \frac{p_0}{2}\right)$$

$$\int_{p_0}^{p}\frac{dp}{p - \frac{p_0}{2}} = -\frac{\bar{v}A}{2V}\int_0^t dt$$

$$\ln\left(\frac{p - \frac{p_0}{2}}{p_0/2}\right) = -\frac{\bar{v}A}{2V}t$$

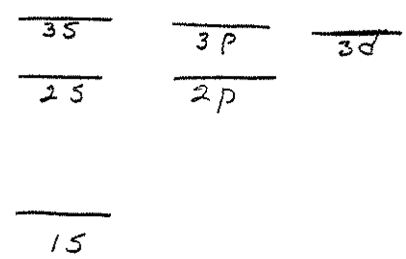
$$\frac{p - \frac{p_0}{2}}{p_0/2} = e^{-\frac{\bar{v}At}{2V}}$$

$$\boxed{p = \frac{p_0}{2}\left(1 + e^{-\frac{\bar{v}At}{2V}}\right)}$$

where $\bar{v} = \sqrt{\dfrac{8kT}{\pi m}}$.

4.

Spectrum



ΔV is not sph. symm., so l is not good, l_z and parity are good. ΔV is a rank 2 irr. tensor; T_{20}

<u>$N=1$</u>

1S state $\quad \Delta E = \int \psi_{100}(r)^2 \, \Delta V \, d^3r$

By the W.E. theorem this is $\langle 00 | T_{20} | 00 \rangle = \langle 0200 | 0$

$\langle \ \| \ \| \ \rangle = 0$

G.S. is not shifted

$N=2$. Since 2S and 2P have different parity, they are not mixed by the pert. as in $N=1$ the 2S state has zero shift. For 2p

$$\Delta E_{2p,m} = \langle 21m | T_{20} | 21m \rangle$$

$$= \underbrace{\langle 12m0 | 1m \rangle}_{\uparrow} \langle 21 \| T_{20} \| 21 \rangle$$

$$\Delta E_{2p,m=1} = \int |\psi_{2p,1}|^2 \frac{e^2 r^2}{a^3} P_2(\cos\theta) \, d^3r$$

$$\Delta E_{2p,m} = \frac{\langle 12m0 | 1m \rangle}{\langle 1210 | 11 \rangle} \Delta E_{2p,m=1}$$

$N = 3$ Table of states + conserved QN $\alpha = m, \pi$

state	α
3S	0, +
3p m=0	0, −
m=±1	±1, −
3d m=0	0, +
3d m=±1	±1, +
3d m=±2	±2, +

as we look through the table we see that two states have the Same α Q.N. 3S and 3d, m=0. these are mixed by the pert.

For all the rest, non-degen pert theory is applicable $\Delta E_{3S} = 0$ as before

$$\Delta E_{3p,m} = \frac{\langle 1 2 m 0 | 1 m \rangle}{\langle 1 2 1 0 | 1 1 \rangle} \Delta E_{3p, m=1}$$

$$\Delta_{3d, m \neq 0} = \frac{\langle 2 2 m 0 | 2 m \rangle}{\langle 2 2 2 0 | 2 2 \rangle} \Delta E_{3d, m=2}$$

$$\Delta E_{3d\ m=2} = \int |\psi_{3d}|^2 \frac{e^2 r^2}{a^3} P_2(\cos\theta)\, d^3r$$

For m=0 we project H on the 3d + 3S states. with m=0 and solve exactly

$$H = H_0 + \Delta V \quad , \quad \psi_1 \equiv \psi_{3S, m=0} \quad , \quad \psi_2 = \psi_{3d, m=0}$$

$$\langle i | H_0 + \Delta V | j \rangle = \epsilon_3 \cdot \delta_{ij} + \langle i | \Delta V | j \rangle$$

ϵ_3 = unperturbed energy of the $N=3$ state

$\Delta V_{11} \equiv \langle 1 | \Delta V | 1 \rangle = 0$ as for other S states

$$\Delta V_{12} \equiv \langle 1 | \Delta V | 2 \rangle = \int \psi_{3S} \frac{e^2 r^2}{a^3} P_2(\cos\theta)\, \psi_{3d}\, d^3r$$

$$\Delta V_{22} \equiv \langle 2 | \Delta V | 2 \rangle = \int |\psi_{3d}|^2 \frac{e^2 r^2}{a^3} P_2(\cos\theta)\, d^3r$$

the projection of H onto the $n=3$ state leads to the simultaneous equations

$$\begin{pmatrix} E-\epsilon_3 & -\Delta V_{12} \\ -\Delta V_{21} & E-\epsilon_3-\Delta V_{22} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

the determinental condition is

$$= D(D-\Delta V_{22}) - |\Delta V_{12}|^2 = 0$$

where $D = E - \epsilon_3$

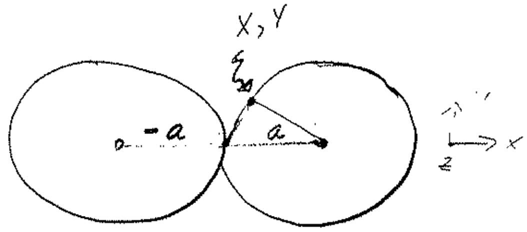
$$D^2 - \Delta V_{22} D - |\Delta V_{12}|^2 = 0$$

$$D = \frac{1}{2}\left[\Delta V_{22} \pm \sqrt{\Delta V_{22}^2 + 4|\Delta V_{12}|^2}\right]$$

$$E = \epsilon_3 + D$$

the two values $\pm$ give the energies of the two mixed $3s+3d$ $m=0$

5.



From a single wire $B_1 \equiv \frac{2i}{ca}$ or $\frac{\mu_0 i}{2\pi a}$

(b) $$\frac{B}{aB_1} = -\frac{\hat{z}\times\left[(x-a)\hat{x} + y\hat{y}\right]}{(x-a)^2+y^2} + \frac{\hat{z}\times\left[(x+a)\hat{x} + \hat{y}y\right]}{(x+a)^2+y^2}$$

$$= \frac{(a-x)\hat{y} + y\hat{x}}{a^2-2ax+x^2+y^2} - \frac{\left[-(a+x)\hat{y} + y\hat{x}\right]}{a^2+2ax+x^2+y^2}$$

$$\approx \left[\frac{(a-x)\hat{y} + y\hat{x}}{a^2}\right]\left[1+\frac{2x}{a} - \frac{x^2+y^2}{a^2} \overset{+}{-} \frac{1(-2)}{\cancel{2!}}\left(\frac{-2x}{a}\right)^2\right]$$

$$\pm \frac{-(a+x)\hat{y} + y\hat{x}}{a^2}\left[1-\frac{2x}{a} - \frac{x^2+y^2}{a^2} \overset{+}{-} \frac{1(-2)}{\cancel{2!}}\left(\frac{2x}{a}\right)^2\right]$$

$$\frac{1}{a}\left\{\hat{y}\left[2 - 2\frac{x^2+y^2}{a^2} - \frac{(4x^2)}{a^2} + \frac{8x^2}{a^2}\right] + \hat{x}\left[\frac{4xy}{a^2}\right]\right\} = \frac{2}{a}\left\{\hat{y}\left[1+\frac{2(x^2-y^2)}{a^2}\right] + \hat{x}\frac{2xy}{a^2}\right\}$$

(a) y dir

(c) Yes all derivatives of the above B are zero at $x, y = 0$ since there are no linear terms.

6 a) $E_n = -\dfrac{m_\mu Z^2 e^4}{2\hbar^2 n^2} = -\dfrac{m_\mu}{m_e}\dfrac{Z^2}{n^2}(13.6\,eV) = -207\,\dfrac{(7)^2}{n^2}(13.6\,eV) = -\dfrac{137.9}{n^2}$ keV

so $\underline{E_1 = -137.9 \text{ keV}}$ $\qquad$ $\underline{E_2 = -34.5 \text{ keV}}$

b) All that is needed is to replace m_μ (in E_n) by m_{reduced} with $M_N \cong (1840\, m_e) \times 7$

$$m_{\text{reduced}} = \frac{m_\mu M_N}{m_\mu + M_N} = \frac{\cancel{M_N}\, m_\mu}{\cancel{M_N}\left(1 + \frac{m_\mu}{M_N}\right)} \cong m_\mu\left(1 - \frac{m_\mu}{M_N}\right) = m_\mu(1 - 8\times10^{-3})$$

so $\underline{(\Delta E)_n} = +8\times10^{-3} E_n = \underline{+1.1 \text{ keV} \quad \text{for } E_1}$

$\qquad\qquad = \underline{+0.3 \text{ keV} \quad \text{for } E_2}$

c) Since the potential V and the field $\dfrac{\partial V}{\partial r}$ are single-valued everywhere,

at $r = r_0$, $\quad \left.\dfrac{\partial V_{in}}{\partial r}\right|_{r=r_0} = \left.\dfrac{\partial V_{out}}{\partial r}\right|_{r=r_0}$ $\qquad$ and $\quad V_{in}(r_0) = V_{out}(r_0)$.

$$b r_0 = \frac{Ze^2}{r_0^2} \qquad\qquad -V_0 + \frac{b}{2} r_0^2 = -\frac{Ze^2}{r_0}$$

$$\underline{b = \frac{Ze^2}{r_0^3}} \quad \longrightarrow \quad \underline{V_0} = \frac{Ze^2}{r_0} + \frac{b}{2} r_0^2 = \frac{Ze^2}{r_0} + \frac{1}{2}\frac{Ze^2}{r_0} = \underline{\frac{3}{2}\frac{Ze^2}{r_0}}$$

To compare with the point-nucleus case, we need to take the perturbation

$$H^{(1)} = \begin{cases} V(r) + \dfrac{Ze^2}{r} = -V_0 + \dfrac{b}{2} r^2 + \dfrac{Ze^2}{r} & r < r_0 \\ 0 & r > 0 \end{cases}$$

that is, we need to remove the coulomb potential $-\dfrac{Ze^2}{r}$ we had before.

then $(\Delta E)_{1S} = \langle 10 | H^{(1)} | 10 \rangle = \displaystyle\int_0^{r_0} |\psi_{1S}|^2 \left(-V_0 + \frac{b}{2} r^2 + \frac{Ze^2}{r}\right) d\vec{r}$

now $a_\mu = \dfrac{m_e}{m_\mu Z} a_0 = \dfrac{a_0}{(207)(7)} = \dfrac{a_0}{1449} \cong 3\times10^{-12}$ cm, about $10 r_0$

since the integration extends only to r_0, we will make negligible error

if we use $\quad |\psi_{1S}|^2 = |\psi_{1S}(0)|^2 = \dfrac{1}{\pi}\left(\dfrac{m_\mu}{m_e}\right)^3 \dfrac{Z^3}{a_0^3}$

6 So $\Delta E = \frac{1}{\cancel{\pi}} \frac{(1449)^3}{a_0^3} \int_0^{r_0} \left(-V_0 + \frac{b}{2} r^2 + \frac{Ze^2}{r}\right) 4\cancel{\pi} r^2\, dr = \frac{(1449)^3}{a_0^3}\, 4 \int_0^{r_0} \left(-V_0 r^2 + \frac{b}{2} r^4 + Ze^2 r\right) dr$

$= \frac{(1449)^3}{a_0^3}\, 4 \left(-V_0 \frac{r_0^3}{3} + \frac{b}{2}\frac{r_0^5}{5} + Ze^2 \frac{r_0^2}{2}\right)$ and, putting in V_0 and b,

$= \frac{(1449)^3}{a_0^3}\, 4 \left(\cancel{-\frac{3}{2} Ze^2 \frac{r_0^2}{3}} + \frac{Ze^2}{2}\frac{r_0^2}{5} + \cancel{\frac{Ze^2 r_0^2}{2}}\right) = (1449)^3\, Z \left(\frac{e^2}{a_0}\right) \frac{4}{10} \left(\frac{r_0}{a_0}\right)^2$

Now $\frac{e^2}{a_0} = 2\left|E_1\right|_{\text{hydrogen}} = 27.2\ \text{eV}$

so $\underline{(\Delta E)_1} = (1449)^3\, Z\, (27.2\ \text{eV})\, \frac{4}{10} \left(\frac{3.4\times10^{-13}}{5.3\times10^{-9}}\right)^2 = 949\ \text{eV} = \underline{0.9\ \text{keV}}$

About as big, and in the same sense, as the reduced mass correction.